Math 115. More Slides from the Lecture of November 21 (revised)

Infinite Simple Continued Fractions

For the rest of today's class: Let $a_0, a_1, \dots,$ be an infinite sequence of integers, with $a_i > 0$ for all $i > 0$.

How might we define (the value of) $\langle a_0, a_1, \dots \rangle$?

Answer: As a limit:

$$\langle a_0, a_1, \dots \rangle = \lim_{n\to\infty} \langle a_0, \dots, a_n \rangle .$$

This leads to looking at functions $\langle a_0, \dots, a_{n-1}, x \rangle$ with $x \in \mathbb{R}$, $x > 0$.

Note that $\langle a_0, x \rangle = a_0 + \frac{1}{x}$ is a decreasing function of x, but $\langle a_0, a_1, x \rangle = a_0 + \frac{1}{a_1 + \frac{1}{x}}$ is increasing.

We'll use sequences (h_n) and (k_n) with $n = -2, -1, 0, 1, \dots$ defined inductively by

$$\begin{aligned} h_{-2} &= 0\,, \quad & h_{-1} &= 1\,, \quad & h_n &= a_n h_{n-1} + h_{n-2} \quad \text{for all } n \ge 0 \\ k_{-2} &= 1\,, \quad & k_{-1} &= 0\,, \quad & k_n &= a_n k_{n-1} + k_{n-2} \end{aligned}$$

Theorem. *For any sequence (a_n) as above,*

$$\langle a_0, \dots, a_{n-1}, x \rangle = \frac{h_{n-1}x + h_{n-2}}{k_{n-1}x + k_{n-2}} \qquad \textit{for all } n \ge 0 \textit{ and all } x \in \mathbb{R}_{>0} .$$

Proof. Induction on $n \in \mathbb{N}$.

Base case: When $n = 0$, the left-hand side is $\langle x \rangle = x$, and the right-hand side is

$$\frac{h_{-1}x + h_{-2}}{k_{-1}x + k_{-2}} = \frac{1x + 0}{0x + 1} = x .$$

Inductive step: Given $n \in \mathbb{N}$, we'll assume it's true for n and show it's true for $n + 1$:

$$\begin{aligned} \langle a_0, \dots, a_n, x \rangle = \langle a_0, \dots, a_{n-1}, a_n + \tfrac{1}{x} \rangle &= \frac{h_{n-1}\big(a_n + \frac{1}{x}\big) + h_{n-2}}{k_{n-1}\big(a_n + \frac{1}{x}\big) + k_{n-2}} \\ = \frac{a_n h_{n-1} x + h_{n-1} + h_{n-2} x}{a_n k_{n-1} x + k_{n-1} + k_{n-2} x} &= \frac{(a_n h_{n-1} + h_{n-2})x + h_{n-1}}{(a_n k_{n-1} + k_{n-2})x + k_{n-1}} \\ = \frac{h_n x + h_{n-1}}{k_n x + k_{n-1}} . \end{aligned}$$

□

Proposition. *Let* $n \in \mathbb{N}$ *and let* $r_n = \langle a_0, \dots, a_n \rangle$. *Then* $r_n = h_n/k_n$.

Proof. Let $x = a_n$ in the above theorem. Then

$$r_n = \langle a_0, \dots, a_n \rangle = \frac{h_{n-1}a_n + h_{n-2}}{k_{n-1}a_n + k_{n-2}} = \frac{h_n}{k_n} \ .$$

□

Lemma.

(a). $h_n k_{n-1} - h_{n-1}k_n = (-1)^{n-1}$ *for all* $n \geq -1$

(b). $r_n - r_{n-1} = (-1)^{n-1}/k_n k_{n-1}$ *for all* $n \geq 1$.

Proof. (a). Use induction on n .

First note that

$$h_n k_{n-1} - h_{n-1}k_n = \det \begin{bmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{bmatrix} \qquad (*)$$

and that

$$\begin{bmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{bmatrix} = \begin{bmatrix} h_{n-1} & h_{n-2} \\ k_{n-1} & k_{n-2} \end{bmatrix} \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \ .$$

Base case: Assume $n = -1$. Then, by (*),

$$h_n k_{n-1} - h_{n-1}k_n = \begin{vmatrix} h_{-1} & h_{-2} \\ k_{-1} & k_{-2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 = (-1)^{-2} \ .$$

Inductive step: If $n > 0$ and the result is true for $n-1$, then

$$\begin{aligned} h_n k_{n-1} - h_{n-1}k_n &= \begin{vmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{vmatrix} = \det\left(\begin{bmatrix} h_{n-1} & h_{n-2} \\ k_{n-1} & k_{n-2} \end{bmatrix} \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \begin{vmatrix} h_{n-1} & h_{n-2} \\ k_{n-1} & k_{n-2} \end{vmatrix} \cdot \begin{vmatrix} a_n & 1 \\ 1 & 0 \end{vmatrix} = (-1)^{n-2} \cdot (-1) = (-1)^{n-1} \ . \end{aligned}$$

(b). For all $n \geq 1$ we have

$$r_n - r_{n-1} = \frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}} = \frac{h_n k_{n-1} - h_{n-1}k_n}{k_n k_{n-1}} = \frac{(-1)^{n-1}}{k_n k_{n-1}} \ .$$

□

Corollary. $\gcd(h_n, k_n) = 1$ *for all* $n \geq 0$.

Proof. Indeed, $h_n k_{n-1} - h_{n-1}k_n = \pm 1$ for all n . □

So the fraction $r_n = h_n/k_n$ is in lowest terms for all n .