Math 115. Slides from the Lecture of November 21

This handout contains the slides from the lecture of November 21.

Finite Simple Continued Fractions

Definition. Let $n \in \mathbb{N}$ and $x_0, \dots, x_n \in \mathbb{R}$ with $x_i > 0$ for all $i > 0$. Then

$$\langle x_0, \dots, x_n \rangle = \begin{cases} x_0 & \text{if } n = 0\,; \\ x_0 + \frac{1}{\langle x_1, \dots, x_n \rangle} & \text{if } n > 0\,. \end{cases}$$

Some basic properties:

(1). $\langle x_0, x_1, \dots, x_n \rangle = x_0 + \dfrac{1}{\langle x_1, \dots, x_n \rangle}$ if $n > 0$ (by definition).

(2). $\langle x_0, \dots, x_n \rangle = \langle x_0, \dots, x_{n-2}, x_{n-1} + \frac{1}{x_n} \rangle$ if $n > 0$ (by induction)

(3). $\langle x_0, \dots, x_n \rangle \ge x_0$, with equality $\iff$ $n = 0$.

Proof of (3). This is proved by induction on $n \ge 0$.

Base case: If $n = 0$ then it's easy.

Inductive step: If $n > 0$ and it's true for $n - 1$, then

$$\langle x_1, \dots, x_n \rangle \ge x_1 > 0\,,$$

so $1/\langle x_1, \dots, x_n \rangle > 0$. Therefore

$$\langle x_0, x_1, \dots, x_n \rangle = x_0 + \frac{1}{\langle x_1, \dots, x_n \rangle} > x_0\,.$$

□

Definition. A finite continued fraction is **simple** if $x_0, \dots, x_n \in \mathbb{Z}$.

Every finite simple continued fraction evaluates to a rational number.

We'll show: Every rational number can be written as a finite simple continued fraction in exactly two ways.

(**Later we'll see:** Every (real) irrational number can be written uniquely as an infinite simple continued fraction.)

Theorem (Existence). *For all $x \in \mathbb{Q}$ there is a finite sequence $a_0, \dots, a_n \in \mathbb{Z}$ with $n \in \mathbb{N}$ such that $a_i > 0$ for all $i > 0$ and $x = \langle a_0, \dots, a_n \rangle$.*

Proof. …