Math 115. Slides from the Lecture of November 19

This handout contains the slides from the lecture of November 19.

Arithmetic Functions

Definition.

(a). An *arithmetic function* is a function from $\mathbb{Z}_{>0}$ to $\mathbb{C}$.
(b). A *multiplicative function* is an arithmetic function f which is not the zero function, and which satisfies $f(mn) = f(m)f(n)$ for all relatively prime $m, n \in \mathbb{Z}_{>0}$.
(c). A *totally multiplicative function* is an arithmetic function f which is not the zero function, and which satisfies $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}_{>0}$.

Examples.

- The constant function 1 is totally multiplicative, and multiplicative
- So is $f(n) = n^r$, for any $r \in \mathbb{R}$
- If f is totally multiplicative, then it is multiplicative
- The Euler φ-function is multiplicative, but not totally multiplicative

Examples.

- The function $f(m)$ equal to the number of quadratic residues modulo m is multiplicative but not totally multiplicative
- The function $f(m)$ equal to the number of primitive roots modulo m is not multiplicative (why?)

Notation: When writing $\displaystyle\sum_{d|n}$ or $\displaystyle\prod_{d|n}$, we're tacitly requiring $d > 0$.

Definition. For $n \in \mathbb{Z}_{>0}$, define:

- $d(n) =$ the number of positive divisors of n: $\sum_{d|n} 1$
- $\sigma(n) =$ the sum of the positive divisors of n: $\sum_{d|n} d$
- $\sigma_k(n) = \sum_{d|n} d^k$ ($\sigma_0(n) = d(n)$, $\sigma_1(n) = \sigma(n)$)
- $\omega(n) =$ the number of primes dividing n
- $\Omega(n) =$ the number of primes dividing n, with multiplicity

Note that

$$\Omega\Big(\prod p^{\alpha(p)}\Big) = \sum \alpha(p)$$

Also $d\big(\prod p^{\alpha(p)}\big) = \prod(\alpha(p)+1)$. This is multiplicative.

And, $e^{\omega(n)}$ and $e^{\Omega(n)}$ are multiplicative and strongly multiplicative, respectively.

A multiplicative function is uniquely determined by its values on prime powers (as for $\phi(m)$).

A totally multiplicative function is uniquely determined by its values on prime numbers.