Math 115. Slides from the Lecture of November 5

This handout contains the slides from the lecture of November 5.

Equivalent Forms have the Same Discriminant

Theorem. *If f and g are equivalent forms, then they have the same discriminant.*

Proof. Write $f(x,y)=ax^2+bxy+cy^2$, and let $d=b^2-4ac$. Also let $F=\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$; then

$$\begin{aligned} [\,x \quad y\,]\,F\begin{bmatrix} x \\ y \end{bmatrix} &= [\,x \quad y\,]\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [\,x \quad y\,]\begin{bmatrix} ax+by/2 \\ bx/2+cy \end{bmatrix} \\ &= x(ax+by/2)+y(bx/2+cy) = ax^2+bxy+cy^2 \\ &= f(x,y)\ , \end{aligned}$$

and

$$\det F = ac - b^2/4 = -d/4\ .$$

Also let $g(x,y)=a'x^2+b'xy+c'y^2$, let $d'=b'^2-4a'c'$, and let $G=\begin{bmatrix} a' & b'/2 \\ b'/2 & c' \end{bmatrix}$. Let $M\in\Gamma$ be a matrix that takes f to g , so that $g=f\circ T_M$. Then

$$\begin{aligned} g(x,y)=f(T_M(x,y)) &= \left(M\begin{bmatrix} x \\ y \end{bmatrix}\right)^t F\left(M\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= \begin{bmatrix} x \\ y \end{bmatrix}^t M^tFM\begin{bmatrix} x \\ y \end{bmatrix} \\ &= [\,x \quad y\,]\left(M^tFM\right)\begin{bmatrix} x \\ y \end{bmatrix}\ , \end{aligned}$$

Recalling that $g(x,y)=[\,x \quad y\,]\,G\begin{bmatrix} x \\ y \end{bmatrix}$, we then have

$$G=M^TFM\ .$$

However, this takes a proof.
First, we note (from the identity $(AB)^t=B^tA^t$) that

$$(M^tFM)^t=M^tF^t(M^t)^t=M^tFM\ ,$$

and therefore M^tFM is a symmetric matrix.

Write $M^tFM = \begin{bmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{bmatrix}$. Then

$$a'x^2 + b'xy + c'y^2 = g(x,y) = [\,x \quad y\,] \begin{bmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \alpha x^2 + \beta xy + \gamma y^2$$

for all $x, y \in \mathbb{Z}$.

This implies $a' = \alpha$, $b' = \beta$, and $c' = \gamma$. Thus

$$M^tFM = G\ .$$

Taking determinants of both sides of $M^tFM = G$ then gives

$$\begin{aligned} -\frac{d'}{4} = \det G &= \det\bigl(M^tFM\bigr) \\ &= \bigl(\det\bigl(M^t\bigr)\bigr)\bigl(\det F\bigr)\bigl(\det M\bigr) \\ &= 1 \cdot \det F \cdot 1 \\ &= \det F = -\frac{d}{4}\ , \end{aligned}$$

and therefore $d' = d$. □

Our Next Goal

Our next goal will be to show that, for each integer d, there are only finitely many equivalence classes of forms of discriminant d.

Definition. Let $f(x,y) = ax^2 + bxy + cy^2$ be a form whose discriminant is not a perfect square. Then f is **reduced** if:

$$\begin{aligned} -|a| < b \le |a| < |c|\ , &\qquad \text{or} \\ 0 \le b \le |a| = |c|\ . & \end{aligned}$$

Theorem. *Let d be an integer, not a perfect square. Then every form of discriminant d is equivalent to a reduced form.*

[Watch for: Where in the proof is the fact that d is not a perfect square used?]

Proof. Let $f(x,y) = ax^2 + bxy + cy^2$ be a form of discriminant d (with d not a perfect square). We want to show that it's equivalent to a reduced form. Note that $a \ne 0$ and $c \ne 0$. [Why?]

We carry out the following procedure:

Step 1. If $|c| < |a|$, then $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \Gamma$ takes f to $f(y, -x) = cx^2 - bxy + ay^2$. So, after doing this if necessary, we may assume that $|a| \le |c|$.

Step 2. Notice that, for any $m \in \mathbb{Z}$, the matrix $M = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \in \Gamma$ takes $f(x, y)$ to $f(x + my, y) = ax^2 + (2am + b)xy + (am^2 + bm + c)y^2$. Choose m such that $-|a| < 2am + b \le |a|$.

Step 3. If $|c| < |a|$, go back to Step 1 (otherwise continue).

Step 4.

- If $|c| > |a|$, congratulations! You have a reduced form.
- If $|c| = |a|$ and $b \ge 0$, congratulations! You have a reduced form.
- If $|c| = |a|$ and $b < 0$, then use $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ again. Your form is now reduced, because $0 < b < |a| = |c|$.

Can this procedure go on forever?

Only if you do Step 1 infinitely many times.

However, Step 1 strictly decreases $|a|$, and all other steps leave $|a|$ unchanged. So the procedure must eventually stop. $\square$

Where did we use the assumption that d is not a perfect square?

In Step 2, we needed $|a| \ne 0$.

An Example

Reduce

$$f(x, y) = 17x^2 - 26xy + 10y^2 .$$

Step 1: $|c| < |a|$, so apply $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to get $f \sim 10x^2 + 26xy + 17y^2$.

Step 2: $b > |a|$, so choose m such that $-|a| < 2am + b \le |a|$; $-10 < 20m + 26 \le 10$; $m = -1$.

So apply $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ to get $f \sim 10x^2 + 6xy + y^2$.

Step 3: $|c| < |a|$, so go back to Step 1.

Step 1: $|c| < |a|$, so apply $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to get $f \sim x^2 - 6xy + 10y^2$.

Step 2: $b \le -|a|$, so choose m such that $-|a| < 2am + b \le |a|$; $-1 < 2m - 6 \le 1$; $m = 3$. Apply $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ to get $f \sim x^2 + 0xy + y^2$.

Step 3: $|c| \ge |a|$, so don't go back.

Step 4: Congratulations! ($|c| = |a|$ and $b \ge 0$). This is reduced.

Note: $x^2 + y^2$ has discriminant -4, and so does f, because

$$26^2 - 4 \cdot 17 \cdot 10 = 676 - 680 = -4 .$$