Math 115. Slides from the Lecture of October 31 (corrected)

This handout contains the slides from the lecture of October 31, revised to correct errors in the definitions of positive definite and negative definite.

Binary Quadratic Forms (Cont'd)

Throughout today's class, "form" means a binary quadratic form $ax^2 + bxy + cy^2$ *, with* $a, b, c \in \mathbb{Z}$ *, excluding* $a = b = c = 0$ *.*

Definition. A binary quadratic form $f(x, y)$ is –

- **indefinite** if it takes on both positive and negative values;
- **positive semidefinite** if $f(x_0, y_0) \ge 0$ for all x_0, y_0 ;
- **positive definite** if $f(x_0, y_0) > 0$ for all x_0, y_0 with $(x_0, y_0) \ne (0, 0)$;
- **negative semidefinite** if $f(x_0, y_0) \le 0$ for all x_0, y_0 ;
- **negative definite** if $f(x_0, y_0) < 0$ for all x_0, y_0 with $(x_0, y_0) \ne (0, 0)$;
- **definite** if it is positive definite or negative definite; and
- **semidefinite** if it is positive semidefinite or negative semidefinite.

Here x_0 and y_0 can be in $\mathbb{R}$, or in $\mathbb{Q}$, or in $\mathbb{Z}$ (all give the same answers).

Examples.

- $x^2 - 2y^2$ $d = 8$ indefinite
- $x^2 - y^2$ $d = 4$ indefinite
- $x^2 + y^2$ $d = -4$ positive definite
- $-x^2 - y^2$ $d = -4$ negative definite
- x^2 or $(x - y)^2$ $d = 0$ positive semidefinite

Theorem. *Let* $f(x, y)$ *be a binary quadratic form and let* d *be its discriminant.*

(a). *If* $d > 0$ *then* f *is indefinite.*
(b). *If* $d = 0$ *then* f *is semidefinite but not definite.*
(c). *If* $d < 0$ *then* f *is definite.*

Proof. First note that

$$4af(x, y) = (2ax + by)^2 - dy^2 \qquad \text{and} \qquad 4cf(x, y) = (bx + 2cy)^2 - dx^2 \ .$$

(a). Assume that $d > 0$.
If $a \ne 0$ then $f(1, 0) = a$ and $f(b, -2a) = -ad$ have opposite signs.
If $c \ne 0$ then a similar situation occurs.
If $a = c = 0$ then $f(x, y) = bxy$, which is easy.

(b). Assume that $d = 0$.
Then a and c can't both be zero, so $f(x, y) = \frac{1}{4a}(2ax + by)^2$ or $f(x, y) = \frac{1}{4c}(bx + 2cy)^2$.

(c). Assume that $d < 0$.
Then $a \ne 0$ (and $c \ne 0$), and $4af$ is a sum of two positive multiples of squares, which can't both be zero unless $x = y = 0$. □