Math 115. Basic Information on Matrices

This handout gives some basic information on matrices.

If you've already taken Math 54 (which is not a prerequisite for Math 115), then you probably don't need to read this.

2×2 Matrices

First, we note that a 2×2 matrix is an object

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

with $a, b, c, d \in \mathbb{R}$. The **entries** of the matrix are the numbers a, b, c, d. We define **matrix multiplication** by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} . \tag{2}$$

Lemma. *Multiplication of 2×2 matrices, as in (2), is associative.*

Proof. To each 2×2 matrix as in (1), we associate a function $\phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(x, y) = (ax + by, cx + dy)$. (Not all functions $\mathbb{R}^2 \to \mathbb{R}^2$ come from matrices in this way, but all 2×2 matrices determine functions $\mathbb{R}^2 \to \mathbb{R}^2$, so this gives an injection from the set of 2×2 matrices to the set of functions $\mathbb{R}^2 \to \mathbb{R}^2$.)

One can then check by direct computation that if $\psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ is similarly associated to the matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then the matrix product as in (2) is associated to the composition $\phi \circ \psi$. Since function composition is associative, it follows that matrix multiplication is also associative. $\square$

One also has the 2×2 **identity matrix** $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and it is easy to check by computation that $I_2 A = A I_2 = A$ for all 2×2 matrices A. (Also, I_2 is associated to the identity function $\mathbb{R}^2 \to \mathbb{R}^2$.)

A 2×2 matrix A has a **determinant**, denoted $\det A$ or $|A|$, given by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc .$$

(One should be careful not to confuse the notation for determinant with the notation for absolute value.)

We have $\det(AB) = (\det A)(\det B)$ for all 2×2 matrices. (This can be checked by direct computation.)

A 2×2 matrix A is said to be **nonsingular** if $\det A \ne 0$.

If A is nonsingular, then it has an inverse A^{-1}, given by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{\det A} & -\frac{b}{\det A} \\ -\frac{c}{\det A} & \frac{a}{\det A} \end{bmatrix} .$$

One can check (again by computation) that $AA^{-1} = A^{-1}A = I_2$ if A is nonsingular.

Matrices of Some Other Sizes

Similarly, we can define 2×1 matrices, 1×2 matrices, and 1×1 matrices to be matrices of the form

$$\begin{bmatrix} a \\ b \end{bmatrix}, \qquad \begin{bmatrix} a & b \end{bmatrix}, \qquad \text{and} \quad \begin{bmatrix} a \end{bmatrix},$$

respectively, and one can define multiplication of such matrices in certain cases:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae + bf \\ ce + df \end{bmatrix}, \qquad \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} ac + be & ad + bf \end{bmatrix}, \qquad \text{and}$$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac + bd \end{bmatrix}.$$

As in the earlier lemma, these multiplication operations are associative (when defined). Note that a product of an $m \times n$ matrix with an $m' \times n'$ matrix is defined if and only if $n = m'$ (and, if it is defined, then the product is an $m \times n'$ matrix). This corresponds to the fact that an $m \times n$ matrix can be associated to a certain type of function $\mathbb{R}^n \to \mathbb{R}^m$, and that a composition $\phi \circ \psi$ is defined if and only if the domain of ϕ equals the codomain of ψ.

Tranposes of Matrices

The **transpose** A^t of a matrix A is defined to be the matrix obtained by interchanging the rows and columns of the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \qquad \begin{bmatrix} a \\ b \end{bmatrix}^t = \begin{bmatrix} a & b \end{bmatrix}, \qquad \text{and} \quad \begin{bmatrix} a & b \end{bmatrix}^t = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Note that $(A^t)^t = A$ for all A, and that $A^t B^t = (BA)^t$ (whenever the product AB is defined).

Also, if A is a 2×2 matrix, then $\det(A^t) = \det A$ and $(A^t)^{-1} = (A^{-1})^t$.