Math 115. Proof of the Existence of the Greatest-Integer Function

This handout will prove the following lemma mentioned in class:

Lemma. *For each real number x there is a unique integer n such that*

$$n \le x < n + 1 .$$

Proof. We first show existence of n. This part will rely on the following axioms.

Archimedean Property of the Real Numbers. For each $x \in \mathbb{R}$ there is an integer n with $n > x$.

Well-Ordering Property of the Natural Numbers. Every nonempty subset of $\mathbb{N}$ has a smallest element.

Let x be any real number. By the Archimedean Property, there is an $m \in \mathbb{Z}$ with $m > x$. Let

$$S = \{k \in \mathbb{Z} : k \ge m - x\} .$$

Note that actually $S \subseteq \mathbb{N}$ because $m - x > 0$. Also, by the Archimedean Property applied to the real number $m - x$, S is nonempty.

Therefore we may apply the Well-Ordering Property to S. Let k be the smallest element of S. Since $k \in S$, $m - k \le x$.

Since k is the smallest element of S, we have $k-1 \notin S$. Therefore $k-1 < m-x$, so $m - k + 1 > x$.

Thus, letting $n = m - k$, we have

$$n \le x < n + 1 ,$$

as was to be shown.

Now consider the question of uniqueness. Suppose n, n' are integers such that $n \le x < n + 1$ and $n' \le x < n' + 1$. Then

$$\begin{aligned} n \le x &< n + 1 \qquad \text{and} \\ -n' - 1 < -x &\le -n' , \end{aligned}$$

where the second line comes from negating all terms of $n' \le x < n' + 1$.

Adding these two double-inequalities gives

$$n - n' - 1 < 0 < n - n' + 1 ,$$

so adding $n' - n$ to all three parts gives $-1 < n' - n < 1$. Since $n' - n$ is an integer in the range $(-1, 1)$, it must be zero. This gives $n' = n$, as was to be shown. $\square$