Example: Let \( \sigma = \left( \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{array} \right) \in S_6 \).

Then \( \sigma(1) = 5 \), \( \sigma(5) = 4 \), and \( \sigma(4) = 1 \).

\[ 1 \to 5 \to 4 \to 1 \to 5 \to \ldots \]

This says what \( \sigma \) does to 1, 5, and 4, but not 2, 3, or 6.

Similarly, \( \sigma(2) = 6 \) and \( \sigma(6) = 2 \).

And finally \( \sigma(3) = 3 \).

So this is a pictorial way of what \( \sigma \) does to each element of the set \( A \) that it acts on.

In fact, given a set \( A \) and a permutation \( \sigma \in S_A \), we can define a relation on \( A \) by \( a \sim b \) if \( \sigma^n(a) = b \) for some \( n \in \mathbb{Z} \).

This is an equivalence relation on \( A \):

1. Reflexive: \( \sigma^0(a) = a \quad \forall a \in A \)
2. Symmetric: \( \sigma^k(a) = b \Rightarrow \sigma^{-k}(b) = a \).
3. Transitive: \( \sigma^k(a) = b \) and \( \sigma^m(b) = c \Rightarrow \sigma^{k+m}(a) = c \).

This gives a partition of \( A \).

Definition: The cells in this partition are called the orbit of \( a \) under \( \sigma \).

The orbit containing some element \( a \in A \) is called the orbit of \( a \) under \( \sigma \); it is denoted \( O_{\sigma,a} \).

A more concise way to write a permutation of a (finite) set is cycle notation:
IF \( a_1, \ldots, a_n \) are distinct elements of a set \( A \),
then \((a_1, a_2, \ldots, a_n)\) is the permutation \( \sigma \) of \( A \) defined by
\[
\sigma(a_i) = a_{i+1} \quad \text{for all } i = 1, \ldots, n-1; \quad \sigma(a_n) = a_1 \quad \text{and}
\]
\[
\sigma(a) = a \quad \text{if } a \notin \{a_1, \ldots, a_n\}.
\]
Then \( \sigma \) in the example can be written
\[
\sigma = (1, 5, 4)(2, 6) (3)
\]
or just \((1, 5, 4)(2, 6)\).

Here juxtaposition means composition of functions (i.e., the group operation in \( S_A \)).

\[ \sigma = (1, 5, 4) o (2, 6). \]

**Note:** Cycles can be written with any starting point.

\[
(1, 5, 4) = (5, 4, 1) = (4, 1, 5) \neq (1, 4, 5).
\]

**Thm:** Every permutation of a finite set is a product of disjoint cycles (as in the example).

**Def:** A cycle is a permutation having at most one orbit with \( > 1 \) element.

Two cycles \( \mu_1 \) and \( \mu_2 \) are disjoint if
\[
\{ a \in A : \mu_1(a) \neq a \} \cap \{ a \in A : \mu_2(a) \neq a \} = \emptyset.
\]

**Note:** If two cycles \( \mu_1 \) and \( \mu_2 \) are disjoint, then they commute: \( \mu_1 \mu_2 = \mu_2 \mu_1 \).

**Note:** One can have cycles as permutations of an infinite set; for example \( \sigma(n) = n+1 \) is a cycle in \( S_\mathbb{Z} \).
It has one orbit, namely \( \mathbb{Z} \) itself, but you could write it in cycle notation:
\[
(\ldots, -2, -1, 0, 1, 2, \ldots)
\]
but this is rarely done.
To recap: To write a permutation of a finite set $A$ as a product of disjoint cycles, take an element of $A$, and find its orbit by repeatedly applying $\sigma$ until you get the element back again:

$$\sigma \in A; \quad \text{let } \sigma_1 = \sigma, \quad \sigma_2 = \sigma(\sigma_1), \quad \sigma_3 = \sigma(\sigma_2), \ldots$$

$$\sigma^n = \sigma(\sigma^{n-1}) = \sigma(\sigma^{n-2}) = \cdots = \sigma(\sigma_1) = \sigma_1$$

Thus the cycle is $(\sigma_1, \sigma_2, \ldots, \sigma^n)$.

If we've encountered all elements of $A$, then $\sigma^n = 1$; otherwise go back and start with some element that hasn't been encountered yet.

Another example: Write $(1, 2, 3)(2, 3)$ as a product of disjoint cycles:

$$(1, 2, 3)(2, 3) = (1, 2)(3) = (1, 2)$$

$S_3 = \{ (1), (1, 2, 3), (3, 2, 1), (2, 3), (1, 3), (1, 2) \}$

This says: $\rho_0\rho_1\rho_2\rho_3\rho_2\rho_3 \rho_1 \rho_0 \rho_3 \rho_1 = \rho_3$.

Yet another example: $(1, 2)(2, 3)(3, 4) = (1, 2, 3, 4)$. (in $S_4$)

This leads to...

Prop: Any permutation of a finite set can be written as a (finite) product of transpositions.

Proof: Take any permutation, write it as a product of (disjoint) cycles, and then apply the identity

$$(a_1, a_2, \ldots, a_n) = (a_1, a_2)(a_2, a_3)\cdots(a_{n-1}, a_n)$$

to each of them.

(This works also in $S_1$ and $S_0$, because both are the trivial group, and so you just need to note that the identity element is the empty product.)
\[(1,2,3)(2,3) = (1 \ 2 \ 3)(1 \ 2 \ 3)\]

\[\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \]

\[\sigma(2) = 3, \quad \sigma(3) = 1, \quad \sigma(1) = 2. \]

\[\sigma \circ \tau(2) = 1. \]

\(\text{Thm: No permutation in } S_n \ (n \geq 0) \text{ can be written both as a product of an even number of transpositions and as a product of an odd number of transpositions.}\)

\[\text{Proof #1: By linear algebra (see book).} \]

\[\text{Proof #2: If } n < 2, \text{ then no permutation can be written as an odd a product of an odd number of transpositions, because there are no transpositions.} \]

\[\text{If } n \geq 2, \text{ again see book. We use} \]

\[(i, a_1, \ldots, a_r) (j, b_1, \ldots, b_s) (i, j) = (i, b_1, \ldots, b_s, j, a_1, \ldots, a_r) \]

\[\text{(this works also if } r > 0 \text{ or } s > 0 \text{ or both.)}\]

\[\text{See book.}\]

\[\text{Def: The length of a cycle is the number of elements in its orbit with } \geq 1 \text{ element, or } 1 \text{ if there is no such orbit.}\]

\[\text{A transposition is a cycle of length } 2 \]

\[= (i, j) \text{ with } i \neq j.\]