Spectral statistics of non-selfadjoint operators subject to small random perturbations

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What is this talk about

For $0 < h \ll 1$, we consider a *non-selfadjoint* P_h on $L^2(S^1)$

$$P_h := hD_x + \mathrm{e}^{-ix}, \quad D_x := \frac{1}{i}\frac{d}{dx}.$$

$\rightarrow \mathrm{Spec}(P_h) = h\mathbb{Z}$

What is the spectrum of a small perturbation of P_h*?*

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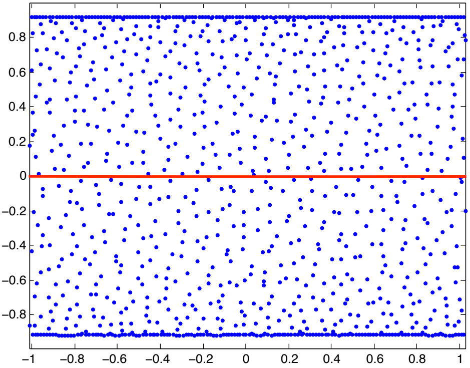


Numerical illustration:

$\rightarrow \mathrm{Spec}(P_h + \delta Q_\omega)$,

$\rightarrow \|\delta Q_\omega\| \approx 10^{-9}$

What is this talk about

Model for numerical errors

- $A_N + \delta Q_N$ [von Neumann-Goldstine '47], [Spielmann-Teng '02, Edelman-Rao '05, Trefethen-Embree '05]

Random matrix

- $Q_N + \delta A_N$, $\delta \asymp 1$, A_N small rank [Tao-Vu '10, Tao '13]
- $A_N + \delta Q_N$, $\delta \asymp 1$ [Bordenave-Capitaine '16]
- $J_N + \delta Q_N$, $\delta = o(1)$ [Davies-Hager '08], [Guionnet-Matchett-Wood-Zeitouni '14]
- $T_N + \delta Q_N$, $\delta = o(1)$ [Sjöstrand-V '14-'16]

Non-selfadjoint spectral problems

- Quantum Resonances,
- Kramers-Fokker-Planck type operators, damped wave equation
- Evolution equations (also in the non-linear case), ...

Non-selfadjoint (Pseudo-)differential operators

- $P_h + \delta Q_\omega$, $\delta = o(1)$ [Hager '06,'08, Sjöstrand '08-'14, Bordeaux-Montrieux '08,'10]
- $T + \delta Q_\omega$, $\delta = o(1)$ [Christiansen-Zworski '10]

Non-selfadjoint operators and Spectral Instability

If $P : \mathscr{H} \to \mathscr{H}$ is not normal, $(P-z)^{-1}$ may be very large even far away from $\mathrm{Spec}(P)$:

$$\|(P-z)^{-1}\| \gg \mathrm{dist}(z, \mathrm{Spec}(P))^{-1}.$$

Pseudospectral effect: The spectrum can be very unstable under small perturbations of the operator.

ε-pseudospectrum [Trefethen-Embree '05], defined by

$$\mathrm{Spec}_\varepsilon(P) := \mathrm{Spec}(P) \cup \left\{ z \in \mathbb{C};\ \|(P-z)^{-1}\| > \varepsilon^{-1} \right\};$$

Equivalently:

$$\begin{aligned} z \in \mathrm{Spec}_\varepsilon(P) &\Longleftrightarrow \exists Q \in \mathscr{L}(\mathscr{H}),\ \|Q\| < 1,\ z \in \mathrm{Spec}(P + \varepsilon Q) \\ &\qquad \text{(instability of spectrum w.r.t. perturbations)} \\ &\Longleftrightarrow z \in \mathrm{Spec}(P) \text{ or } \exists u_z \in \mathscr{D}(P),\ \|(P-z)u_z\| < \varepsilon \|u_z\| \\ &\qquad \text{(existence of quasimodes)} \end{aligned}$$

Example

For $0 < h \ll 1$, we consider P_h on $L^2(S^1)$ [Hager '06]

$$P_h := hD_x + g(x), \quad D_x := \frac{1}{i}\frac{d}{dx}, \quad g \in \mathcal{C}^\infty(S^1;\mathbb{C})$$
$$p(x,\xi) = \xi + g(x), \quad (x,\xi) \in T^*S^1.$$

Zone of spectral instability:

$$\Sigma := \overline{p(T^*S^1)}$$

Outside Σ: *Spectral stability*

- $\rightarrow$ $z \in \mathbb{C}\backslash\Sigma \implies \|(P_h - z)^{-1}\| = \mathcal{O}(1)$ uniformly as $h \to 0$, as $(P_h - z)$ is elliptic,
- $\rightarrow$ If $0 < \delta \ll h^\kappa$, $\kappa > 0$, then $\mathrm{Spec}(P_h + \delta Q) \subset \Sigma + o(1)$.

Energy shell: for any $z \in \Omega \Subset \mathring{\Sigma}$ the energy shell

$$p^{-1}(z) := \{\rho_+(z), \rho_-(z)\} \subset T^*S^1, \quad \text{s.t.: } \pm\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho_\pm) < 0.$$

Quasimodes and Pseudospectrum

Energy shell: for any $z \in \Omega \Subset \mathring{\Sigma}$ the energy shell

$$p^{-1}(z) := \{\rho_+(z), \rho_-(z)\} \subset T^*S^1, \quad \text{s.t.: } \pm\{\operatorname{Re} p, \operatorname{Im} p\}(\rho_\pm) < 0.$$

Quasimodes [Davies '99, Dencker-Sjöstrand-Zworski '04]

- $\forall\ \rho_+(z)$: $\exists$ a quasimode $e_+(z;h)$ microlocalized in $\rho_+(z)$ with

$$\|(P_h - z)e_+(z;h)\| = \mathcal{O}(h^\infty)\|e_+(z;h)\|$$

- $\forall\ \rho_-(z)$: $\exists$ a quasimode $e_-(z;h)$ microlocalized in $\rho_-(z)$ with

$$\|(P_h - z)^* e_-(z;h)\| = \mathcal{O}(h^\infty)\|e_+(z;h)\|$$

$\Longrightarrow$ every $z \in \Omega$ is in the h^∞-pseudospectrum of P_h:

i.e. for $\delta = h^M$, $M \gg 1$, $\exists$ a bounded operator Q such that $z \in \operatorname{Spec}(P_h + \delta Q)$.

Question: *What does the spectrum of $P_h + \delta Q$ look like for a generic perturbation?*

Adding a small random perturbation

Basis: $\{e_k\}_{k\in\mathbb{N}}$ be an ONB of L^2

- E.g.: the Fourier modes $e_k(x) = \mathrm{e}^{ikx}$ for $L^2(S^1)$
- Take $N(h)$ so that $\{e_k\}_{k<N(h)}$ covers a neighbourhood of $p^{-1}(\Omega)$.

Define the **random operators**

$$(RM) \quad Q_\omega = \sum_{j,k<N(h)} \alpha_{j,k} e_j \otimes e_k^*, \qquad (RP) \quad V_\omega = \sum_{j<N(h)} \alpha_j e_j.$$

where $\alpha_\bullet$ are complex valued iid random variables satisfying

$$\mathbb{E}[\alpha_\bullet] = 0, \quad \mathbb{E}[\alpha_\bullet^2] = 0, \quad \mathbb{E}[|\alpha_\bullet|^2] = 1, \quad \mathbb{E}[|\alpha_\bullet|^{4+\varepsilon_0}] < \infty.$$

Bounded perturbation

- (RM) $\|Q_\omega\|_{HS} \leq Ch^{-2}$ with probability $\geq 1 - \mathcal{O}(h^3)$.
- (RP) $\|V_\omega\|_\infty \leq Ch^{-2}$ with probability $\geq 1 - \mathcal{O}(h^3)$.

Macroscopic spectral distribution

Theorem (Hager '06, Hager-Sjöstrand '08)

$\mathrm{Spec}(P_h + \delta Q_\omega)$ satisfies a probabilistic Weyl's law. For $\Gamma \subset \Omega \Subset \mathring{\Sigma}$ a domain with smooth boundary $\partial\Gamma$, then, with probability $\geq 1 - h^\kappa$,

$$\#(\mathrm{Spec}(P_h^\delta) \cap \Gamma) = \frac{1}{2\pi h}\left(\iint_{p^{-1}(\Gamma)} dx d\xi + o(1)\right), \quad \text{as } h \to 0^+.$$

(RM) [Hager '06] for $P_h = hD_x + g(x)$, with $p^{-1}(z) = \{\rho_+, \rho_-\}$.

$\rightarrow$ [Hager-Sjöstrand '08] for $P_h = Op_h(p)$ on $\mathbb{R}^d$.

(RP) [Hager '06b] $P_h = Op_h(p)$ on $\mathbb{R}^1$ with

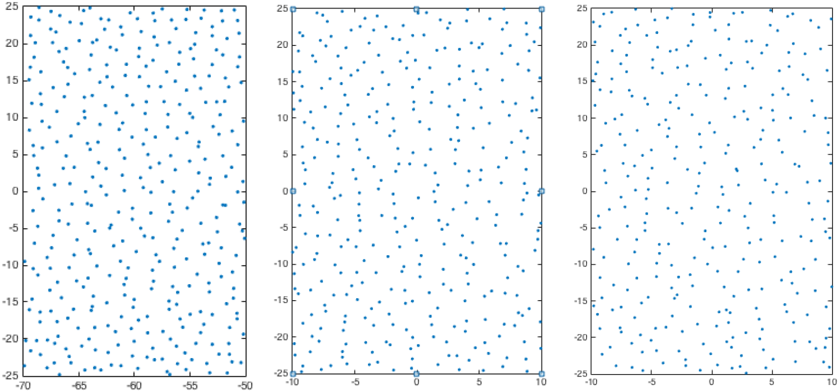
- $p(x,\xi) = p(x,-\xi)$
- $p^{-1}(z) = \{\rho_\pm^j; j = 1, \ldots, J\}$

$\rightarrow$ [Sjöstrand '08, '09] $P_h = Op_h(p)$ on $\mathbb{R}^d$ or compact manifold M.

[Bordeaux-Montrieux '08]

Numerical Experiments

Numerical Experiments

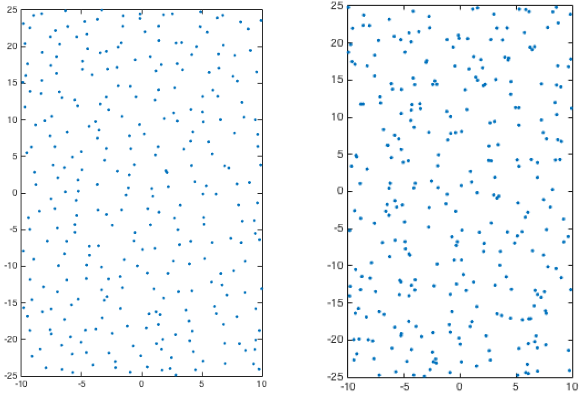


$$hD_x + \mathrm{e}^{-ix} + \delta Q_\omega \quad \longleftrightarrow \quad (hD_x)^2 + \mathrm{e}^{-ix} + \delta Q_\omega \quad \longleftrightarrow \quad (hD_x)^2 + \mathrm{e}^{-3ix} + \delta Q_\omega$$

Spectrum for **3 different operators** on S^1 perturbed by the **same** δQ_ω.

What is the difference ?

Numerical Experiments

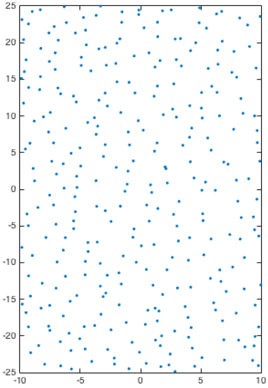


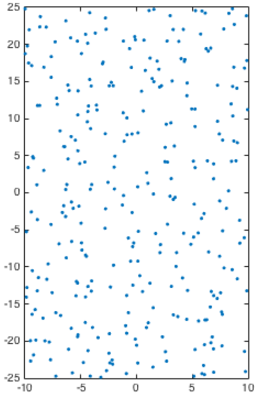
random matrix δQ_ω vs random potential δV_ω

For the **same operator** $P_3 = (hD_x)^2 + \mathrm{e}^{-3ix}$ on S^1 we compare different types of random perturbations.

What is the difference ?

Q_ω vs V_ω





random matrix δQ_ω VS random potential δV_ω

Differences in the fine structure of eigenvalues $\rightarrow$ **spectral correlations**

$\rightarrow$ δQ_ω: eigenvalues show repulsion on the scale of the mean level spacing

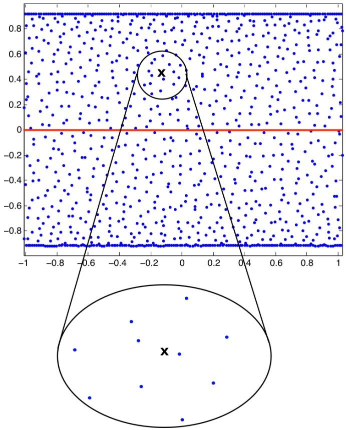
$\rightarrow$ δV_ω: eigenvalues can be clustered

Point process of eigenvalues – Microscale

Local Statistics

Weyl law $\implies$ average spacing of the eigenvalues of P_h^δ at $z_0 \in \mathring{\Sigma}$ is $d_h(z_0)^{1/2} \asymp h^{-1/2}$.

$$\mathcal{Z}_h^\delta = \sum_{\lambda \in \sigma(P_h^\delta)} \delta_\lambda \quad \xrightarrow{\text{RESCALE}} \quad \widetilde{\mathcal{Z}}_h^\delta = \sum_{\lambda \in \sigma(P_h^\delta)} \delta_{(\lambda - z_0)\sqrt{d_h(z_0)}}$$



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Correlation functions

The k-point density of $\widetilde{\mathcal{Z}}_h^\delta$ is defined outside $\Delta = \{z \in \mathbb{C}^k; z_i = z_j \text{ for } i \neq j\}$, to avoid trivial self-correlations, by:

$$\mathbb{E}\left[(\widetilde{\mathcal{Z}}_h^\delta)^{\otimes k}(\varphi)\right] = \mathbb{E}\Bigg[\sum_{\lambda_1,\ldots,\lambda_k \in \sigma(P_h^\delta)} \varphi(\lambda_1,\ldots,\lambda_k)\Bigg]$$

$$= \int_{\Omega^k} \varphi(z) d_h^k(z_1,\ldots,z_k) L(dz_1 \cdots dz_k), \quad \varphi \in \mathcal{C}_0(\mathbb{C}^k \backslash \Delta)$$

k-point correlation function:

$$K_h^k(z_1,\ldots,z_k) := \frac{d_h^k(z_1,\ldots,z_k)}{d_h^1(z_1)\cdots d_h^1(z_k)} \qquad (z_1,\ldots,z_k) \in \mathbb{C}^k \backslash \Delta.$$

Local Statistics: 2-point correlation function

$$K_h^2(z_1, z_2) = \frac{d_h^2(z_1, z_2)}{d_h^1(z_1) d_h^1(z_2)}$$

For $P_h^\delta = hD_x + g(x) + \delta Q_\omega$, with $p^{-1}(z) = \{\rho_+(z), \rho_-(z)\}$ for each $z \in \Omega$
[Hager '06] $\rightarrow$ universal limiting behaviour (scaling limit):

Theorem (V '14)

For any $z_0 \in \Omega$ and any $w_1 \neq w_2 \in \mathbb{C}$, we have

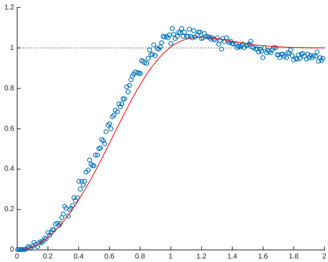
$$K_h^2(z_0 + d_h(z_0)^{-1/2} w_1, z_0 + d_h(z_0)^{-1/2} w_2) \longrightarrow \widetilde{K}^2(w_1, w_2) = \kappa\Big(\frac{\pi}{2} |w_1 - w_2|^2\Big),$$

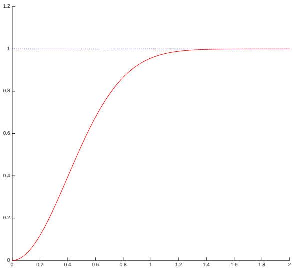
as $h \rightarrow 0$, with

$$\kappa(t) = \frac{(\sinh^2 t + t^2) \cosh t - 2t \sinh t}{\sinh^3 t}.$$

- $\rightarrow$ the scaling limit is **independent of** z_0 and a function of the **distance**;
- $\rightarrow$ quadratic repulsion at short distances : $\kappa(t) = t(1 + \mathcal{O}(t^2))$, $t \rightarrow 0$;
- $\rightarrow$ decorrelation at long distances : $\kappa(t) = 1 + \mathcal{O}(t^2 \mathrm{e}^{-2t})$, $t \rightarrow +\infty$.

Scaling limit is not Ginibre





LHS Red line: $r \mapsto \kappa(r^2)$, with

$$\widetilde{K}^2(w_1, w_2) = \kappa\Big(\frac{\pi}{2}|w_1 - w_2|^2\Big).$$

Blue circles: Numerically obtained histogram data of K_h^2, $h = 10^{-3}$, averaged over 200 realisations of Gaussian random matrices.

RHS $\widetilde{K}^2$ differs from the 2-point function of the Ginibre ensemble (Q_ω alone, in the Gaussian case) :

$$\widetilde{K}^2_{Ginibre}(w_1, w_2) = 1 - \mathrm{e}^{-\pi|w_1 - w_2|^2}.$$

The Gaussian analytic function

Random analytic function (RAF)

$$g : \text{Proba space} \longrightarrow \mathcal{H}(O)$$

Gaussian analytic function (GAF)

$$(g(z_1), \dots g(z_n)) \sim \mathcal{N}_{\mathbb{C}}(0, \Sigma), \quad \text{for all } z_1, \dots, z_n \in O, n \in \mathbb{N}$$

$$\Sigma_{i,j} = \mathbb{E}[g(z_i)\overline{g(z_j)}] =: C(z_i, \overline{z}_j),$$

where C is called the covariance kernel $\to$ distribution of g.

Example: $\alpha_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ *iid*

$$g(z) = \sum_{n \geq 0} \alpha_n \frac{\pi^{n/2} z^n}{\sqrt{n!}}, \quad C(z, \overline{w}) = \mathrm{e}^{\pi z \bar{w}}, \qquad \xi_{GAF} = \sum_{\lambda \in g^{-1}(0)} \delta_\lambda$$

- GAF $\Rightarrow$ covariance kernel determines all k-point correlation functions of ξ_{GAF} (Kac-Rice formula)!
- $\widetilde{K}^2$ of Hager's model $= K^2_{GAF}$,

GAF and Universality

- [Hannay '95] studied the statistics of random spin states $\to K^2_{GAF}$ as a scaling limit of the 2-point correlation function.
- [Bleher-Shiffman-Zelditch '00] zeros of random holomorphic sections of $L^{\otimes N}$, where L is a positive Hermitian line bundle over a compact Kähler manifold M, in the limit $N \to \infty$.
  $\to$ for $\dim_{\mathbb{C}} M = 1$, they obtain $K^k_{GAF}(z)$ as the scaling limit k-point correlation function.

Theorem (Nonnenmacher-V '16)

Assume that $p^{-1}(z) = \{\rho(z)_+, \rho(z)_-\}$ for any $z \in \Omega \Subset \mathring{\Sigma}$. Then,

$$\widetilde{\mathcal{Z}}^{\delta}_h \xrightarrow{d} \xi_{GAF}, \quad h \to 0.$$

Moreover, for any $k \geq 1$ and any $z_0 \in \mathring{\Sigma}$ the k-point correlation function of $\widetilde{\mathcal{Z}}^{\delta}_h$ satisfies the scaling limit

$$\forall w \in \mathbb{C}^k \backslash \Delta : \quad K^k_h(w) \longrightarrow K^k_{GAF}(w), \text{ as } h \to 0,$$

where K^k_{GAF} is the k-point correlation function of ξ_{GAF}.

Sketch of the Proof

From Eigenvalues to Zeros of a RAF

z-(anti)-holomorphic quasimodes (WKB)

$$\widetilde{e}_\pm(x,z;h) = \chi_\pm(x)\mathrm{e}^{\frac{i}{h}\varphi_\pm(x,z)}, \quad \|\widetilde{e}_\pm\| = \mathrm{e}^{\frac{1}{h}\Phi_\pm(z;h)}$$

$$e_\pm = \widetilde{e}_\pm \mathrm{e}^{-\frac{1}{h}\Phi_\pm} \implies \|(P_h - z)e_+\|, \|(P_h - z)^* e_-\| = \mathcal{O}(h^\infty)$$

$$\mathrm{WF}_h(e_\pm) = \{\rho_\pm\}$$

Grushin problem for $P_h^\delta - z$

→ Set

$$R_+ u = (u|e_+),\ u \in H^1(S^1), \qquad R_- u_- = u_- e_-,\ u_- \in \mathbb{C}.$$

Then

$$\begin{pmatrix} P_h - z & R_- \\ R_+ & 0 \end{pmatrix} :\ H^1(S^1)\times\mathbb{C} \to L^2(S^1)\times\mathbb{C} \text{ is of inverse } \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix};$$

→ Schur's complement formula $\implies z \in \sigma(P_h) \iff E_{-+}(z) = 0$

→ $e_\pm$ are h^∞-quasimodes for $P_h \implies E_{-+}(z) = \mathcal{O}(h^\infty)$

Use same Grushin Problem for $P_h^\delta - z$ to obtain:

$$E_{-+}^\delta(z) = E_{-+}(z) - \delta(Q_\omega e_+(z)|e_-(z)) + \mathcal{O}(\delta^2 h^{5/2}),$$

By Shur's formula: **study the zeros of** $E_{-+}^\delta(z)$.

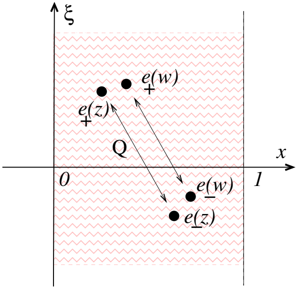
Zeros of a RAF

E^{δ}_{-+} only smooth in z, but can be made holomorphic as it satisfies a $\overline{\partial}$-equation:

$$\partial_{\bar z} E^{\delta}_{-+}(z) + \partial_{\bar z} f^{\delta}(z)\, E^{\delta}_{-+}(z) = 0.$$

We are then left to study the zeros of the RAF

$$F^{\delta}(z) = (Q_{\omega}\widetilde{e}_{+}(\underbrace{z_0 + h^{1/2}z}_{=\widetilde{z}})|\widetilde{e}_{-}(z_0 + h^{1/2}z)) + small$$



1) Q_{ω} **can couple** $\widetilde{e}_{-}(z)$ to $\widetilde{e}_{+}(z)$

2) $g_h(z) = (Q_{\omega}\widetilde{e}_{+}(\tilde z)|\widetilde{e}_{-}(\tilde z)) \xrightarrow{d} GAF,\ h \to 0$

3) For any $\varepsilon > 0$ we have $\mathbb{P}[|small| > \varepsilon] \to 0$, as $h \to 0$.

$$\Longrightarrow \qquad F^{\delta}(z) \xrightarrow{d} GAF$$

4) [Shirai '12] observed that this implies that

$$\xi_{F^{\delta}(z)} \xrightarrow{d} \xi_{\mathrm{GAF}}, \quad h \to 0, \qquad \xi_f = \sum_{\lambda \in f^{-1}(0)} \delta_{\lambda}$$

Covariance and CLT

$$g_h(z) = (Q_\omega \widetilde{e}_+(\tilde{z})|\widetilde{e}_-(\tilde{z})) = \sum_{i,j<N(h)} \alpha_{i,j}(\widetilde{e}_+(\tilde{z})|e_i)(e_j|\widetilde{e}_-(\tilde{z})) \xrightarrow{d} GAF, \ h \to 0.$$

Covariance

$$\mathbb{E}[g_h(z)\overline{g_h(w)}] = (\widetilde{e}_+(z)|\widetilde{e}_+(w))(\widetilde{e}_-(w)|\widetilde{e}_-(z)) + \mathcal{O}(h^\infty)$$

$\widetilde{e}_\pm(z)$ microlocalized in a $\sqrt{h}$-neighbourhood of $\rho_\pm(z)$, thus

$$(\widetilde{e}_\pm(z_0 + h^{1/2}z)|\widetilde{e}_\pm(z_0 + h^{1/2}w)) = \mathrm{e}^{\sigma_\pm(z_0)z\overline{w} + \mathcal{O}(\sqrt{h})}$$

where $\sigma_+ + \sigma_- = d(z_0)/2 \implies$ rescaling z, w by $\sqrt{d(z_0)/(2\pi)}$ and performing the limit $h \to 0^+$, yields the covariance kernel

$$\mathbb{E}[(Q_\omega \widetilde{e}_+(z)|\widetilde{e}_-(z))\overline{(Q_\omega \widetilde{e}_+(w)|\widetilde{e}_-(w))}] \longrightarrow C(z, \bar{w}) = \mathrm{e}^{\pi z \bar{w}}, \quad h \to 0.$$

Central Limit Theorem under the Lyapunov condition we need to check that as $h \to 0$

$$\sum_{i,j<N(h)} |(\widetilde{e}_+(\tilde{z})|e_i)|^4 \cdot |(e_j|\widetilde{e}_-(\tilde{w}))|^4 \longrightarrow 0.$$

More general 1-D Pseudos

Operators with J quasimodes

The operators we consider: P_h be the Weyl quantization of $p \in S(\mathbb{R}^2; m)$

$$(P_h u)(x) = \frac{1}{2\pi h} \iint \mathrm{e}^{\frac{i}{h}(x-y)\xi} p\left(\frac{x+y}{2}, \xi\right) u(y)dyd\xi.$$

- $\Omega \Subset \mathring{\Sigma}$ and $\sigma(P_h) \cap \Omega$ purely discrete

The energy shell: for every $z \in \Omega$

$$p^{-1}(z) = \{\rho_\pm^j(z); j = 1, \ldots, J\} \text{ with } \pm\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho_\pm^j(z)) < 0.$$

$\Longrightarrow$ $(P_h - z)$, $(P_h - z)^*$ have J quasimodes $e_\pm^j(z; h)$ microlocalized in $\rho_\pm^j(z)$.

Grushin Problem for P_h:

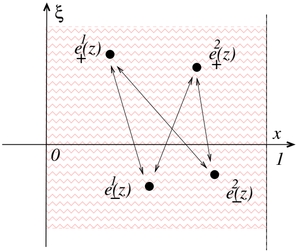
$$\begin{pmatrix} P_h - z & R_- \\ R_+ & 0 \end{pmatrix} : H(m) \times \mathbb{C}^J \to L^2(S^1) \times \mathbb{C}^J \text{ is of inverse } \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$$

with

$$(R_+ u)_k = (u|e_+^k), \ u \in H(m), \quad R_- u_- = \sum_k u_-^k e_-^k, \ u_- \in \mathbb{C}^J.$$

Operators with J quasimodes - Random Matrix

$$z \in \sigma(P_h^\delta) \Longleftrightarrow \det E_{-+}^\delta(z) = (-\delta)^J \det[(Q_\omega e_+^i(z)|e_-^j(z))_{ij}] + small = 0.$$



1) Rescale: $\tilde z = z_0 + h^{1/2} z$
2) $(Q_\omega e_+^i(\tilde z)|e_-^j(\tilde z)) \xrightarrow{d} \mathrm{GAF}_{ij}$, $h \to 0$ with covariance $\mathrm{e}^{(\sigma_-^j + \sigma_+^j) z\bar w}$, $h \to 0$
3) GAF_{ij} are independent
4) $\sum_{j=1}^J (\sigma_+^j(z) + \sigma_-^j(z)) L(dz) = p_*(d\xi \wedge dx)$.

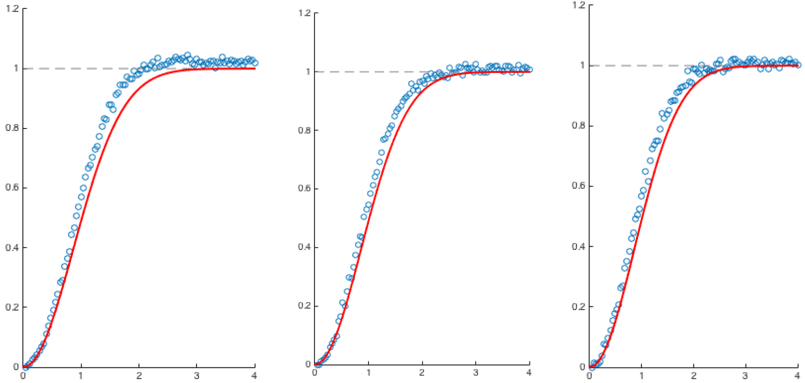
Theorem (Nonnenmacher-V '16)

For any $z_0 \in \mathring{\Sigma}$

$$\sum_{\lambda \in \mathrm{Spec}(P+\delta Q_\omega)} \delta_{(\lambda - z_0)h^{-1/2}} \xrightarrow{d} \sum_{\lambda \in F^{-1}(0)} \delta_\lambda, \quad F(z) = \det(GAF_{i,j}(z))_{1\le i,j\le J}$$

Question: What is the statistics of the zeros of $\det(\mathrm{GAF}_{ij})$?

Operators with J quasimodes - Random Matrix



Blue circles: numerically obtained histogram data of the rescaled 2-point correlation function of the operators ($h = 10^{-3},\ \delta = 10^{-12}$)

$$P_J^\delta = (hD_x)^2 + \mathrm{e}^{-i(J/2)x} + \delta Q_\omega, \quad J = 2, 6, 10 \text{ (\# of quasimodes)}$$

Red line: for comparison, the scaling limit 2-point correlation function of the Ginibre ensemble (as a function of the distance).

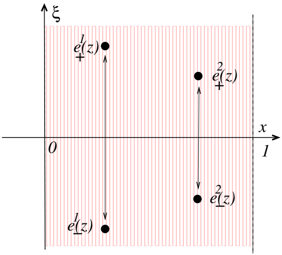
Conjecture

For $J \geq 1$, two eigenvalues repel each other quadratically (at the scale of $\sqrt{h}$).

Operators with J quasimodes - Random Potential

- Now: perturbation by (RP) δV_ω, with $V_\omega(x) = \sum_{k<C/h} \alpha_k e_k(x)$.
- Symmetric symbol $p(x,\xi) = p(x,-\xi)$
  $\implies \rho^j_\pm(z) = (x^j, \pm\xi^j)$, with $x^i \neq x^j$ for $i \neq j$.

Effective Hamiltonian: $\det E^\delta_{-+}(z) = (-\delta)^J \det[(V_\omega e^i_+(z)|e^j_-(z))_{ij}] + small$



1) The effect of V_ω is local:
   $(V_\omega e^i_+(z)|e^j_-(z)) = \int V e^i_+(z)\overline{e^j_-(z)}dx$,
   $\implies$ V_ω can couple e^j_- and e^j_+ $\iff$ $x^i = x^j$.
2) $(V_\omega e^i_+(z)|e^j_-(z)) = \mathcal{O}(h^\infty)$ for $i \neq j$
3) Rescale: $\tilde{z} = z_0 + h^{1/2} z$
4) $(V_\omega e^i_+(\tilde{z})|e^i_-(\tilde{z})) \xrightarrow{d} \mathrm{GAF}_i$, with covariance
   $\mathrm{e}^{2\sigma_i z\bar{w}}$ and $2\sum_i \sigma_i(z) L(dz) = p_*(d\xi \wedge dx)$

Moreover, the GAF_i are **independent** !

$$\implies (-\delta\sqrt{h})^{-J} \det E^\delta_{-+}(z) \xrightarrow{d} \prod_{i=1}^{J} GAF_i(z), \quad \text{as } h \to 0.$$

Operators with J quasimodes - Random Potential

Theorem (Nonnenmacher-V '16)

For any $z_0 \in \mathring{\Sigma}$

$$\sum_{\lambda \in \mathrm{Spec}(P+\delta V_\omega)} \delta_{(\lambda - z_0)h^{-1/2}} \xrightarrow{d} \sum_{\lambda \in \bigcup_{j=1}^{J} GAF_i^{-1}(0)} \delta_\lambda, \quad h \to 0$$

$\Longrightarrow$ Around z_0 the local rescaled limiting point process of eigenvalues of P_h^δ is given by the superposition of J independent GAF-processes with covariance kernel $\mathrm{e}^{2\sigma_i u\bar{v}}$.

$\Longrightarrow$ The global k-point densities can be obtained from the k-point density of each of these J GAF-processes.

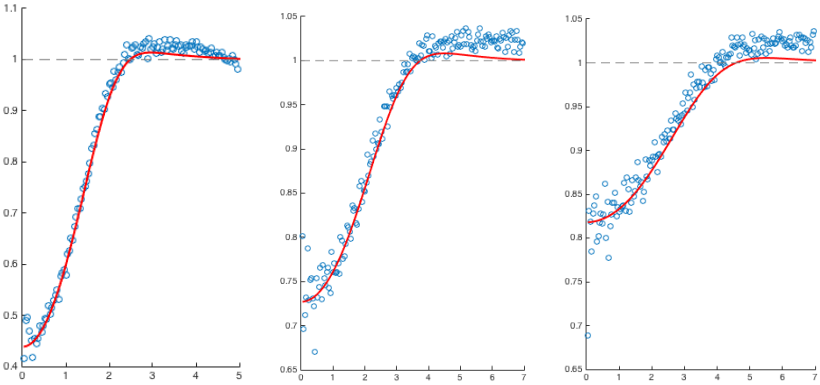
Absence of close range repulsion: For $|z_1 - z_2| \ll 1$

$$K^2(z_1,z_2) = 1 - \sum_{j=1}^{J} \frac{(\sigma_+^j(z_0))^2}{\left(\sum_{j=1}^{J} \sigma_+^j(z_0)\right)^2} \left[1 - \frac{\sigma_+^j(z_0)}{4}|z_1 - z_2|^2 (1 + \mathcal{O}\left((|z_1 - z_2|^2)^2\right)\right]$$

Long range decorrelation: For $|z_1 - z_2| \gg 1$

$$K^2(z_1,z_2) = 1 + \mathcal{O}\left(\mathrm{e}^{-\min_j \sigma_+^j(z_0)|z_1 - z_2|^2/2}\right).$$

Operators with J quasimodes - Random Potential



Blue circles: numerically obtained histogram data of the rescaled 2-point correlation function of the operators ($h = 10^{-3}$, $\delta = 10^{-12}$)

$$P_J^\delta = (hD_x)^2 + \mathrm{e}^{-i(J/2)x} + \delta V_\omega, \quad J = 2, 4, 6 \text{ (\# of quasimodes)}$$

→ Absence of quadratic repulsion at the origin! The presence of J independent processes allows for clusters of size $\leq J$ eigenvalues.

Conclusions and Perspectives

1) **Macroscopic distribution** is given by a Weyl law (with good probability).

2) **Microscopic distribution** is universal but depends on the structure of the energy shell and type of random perturbation.
    - Case of J quasimodes and perturbation by a random matrix ?
    - Eigenvalue correlations for P_h^δ close to the pseudospectral boundary;

3) **NSA operators in dimension** $d > 1$: the energy shell is a codimension 2 submanifold $\Longrightarrow$ number of quasimodes is $\sim h^{1-d} \Longrightarrow E_{-+}^\delta$ is a large "random matrix".

4) **Weaker non-selfadjointness**
    - In our case: the non-normality comes from the principal symbol p.
    - In the case of the damped wave equation (Sjöstrand '00, Anantharaman '10) the principal symbol is real-valued and the non-normality comes from the subprincipal symbol.
    - The effects of random perturbations in this case are as of yet unknown.

Merci de votre attention