Quantum topology from symplectic geometry

Vivek Shende

March 11, 2019
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\[ \theta \mapsto (x(\theta), y(\theta), z(\theta)) \]
Knots

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Two knots are the same
if there is a 1-parameter family of such embeddings interpolating between them.
Reidemeister moves

Any two diagrams of the same knot are related by a sequence of these:

This is great for showing knots are the same, but not for showing they are different.

(How do you know when to give up?)
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How do you tell two knots apart?

Knot invariants

Knot invariants are rules for assigning some quantity to each knot, so that the quantity stays constant in 1-parameter families. Two knots assigned different values must be different knots! A knot invariant is most useful when:
- It is computable from a knot presentation,
- has a geometric meaning,
- and takes different values on different knots. These desiderata are in considerable tension.

One way to show some rule defines an invariant:
Check it doesn't change under the Reidemeister moves. This method does not usually clarify the geometric meaning of the invariant.
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Gauss’s linking number (1833)

For two knots with disjoint image, \( f(\theta), g(\phi) \), consider the work done by simultaneously moving a magnetic charge along \( g \), and an electric charge along \( f \).

\[
W \sim \frac{1}{4} \pi \int \int |f(\theta) - g(\phi)|^3 \cdot (df \times dg)
\]

By Stokes theorem, this quantity is constant as \( f \) or \( g \) is varied without its image meeting the other. In fact, it is an integer, which measures the signed number of times the two knots have to cross before becoming totally separated.
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Poincaré’s fundamental group (1895)

For a knot $K$, 

$$\pi_1(R^3 \setminus K, x) = \{\text{paths in } R^3 \setminus K \text{ which begin and end at } x\}$$

Paths can be composed: do one, then the other. They form a group.

This invariant is very strong — Waldhausen showed it essentially classifies knots — and has a clear geometric meaning, but is very hard to compute in general.
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To each knot is associated a Laurent polynomial $t \mapsto 1 - t + t^2$. In general, to $K$ we associate the (suitably normalized) generator of the ideal in $\mathbb{Z}[\pi_1/\pi_1,\pi_1]$ which annihilates the module $\mathbb{Z}[\pi_1,\pi_1]/\left[\pi_1,\pi_1\right]$, where $\pi_1 = \pi_1(R - K)$. (There are many other descriptions.) Many knots other than $\mathcal{A}$ have Alexander polynomial 1. By Freedman's disk theorem, any such knot bounds a locally flat disk in the 4-ball.
Alexander’s polynomial (1923)

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\begin{align*}
\bigcirc & \quad \mapsto \quad 1 \\
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The skein relation

After fixing the value of the unknot, the Alexander polynomial is the unique \( \mathbb{Z}[t, t^{-1}] \)-valued knot invariant such that:

\[
A(\begin{array}{c} \circ \end{array}) - A(\begin{array}{c} \bullet \end{array}) = (t^{1/2} - t^{-1/2})A(\begin{array}{c} \bigcirc \end{array})
\]
While braid representations arising from certain operator algebras coming from statistical mechanics, Jones found a knot invariant satisfying a similar relation:

$$t - 1 \cdot J(\not) - t \cdot J(\not) = (t^{1/2} - t^{-1/2}) \cdot J(H)$$

The existence (and uniqueness) of this invariant are proven by induction on Reidemeister moves. No "three-dimensional" definition is known. No-one knows if any knot other than the trivial knot has Jones polynomial 1.
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No-one knows if any knot other than \( \bigcirc \) has Jones polynomial 1.
The HOMFLYPT polynomial (1985)

Not long after, many people realized that in fact, there’s a unique \( \mathbb{Q}(a^{\pm}, z^{\pm}) \)-valued knot invariant satisfying the relation

\[
a P(\leftrightarrow) - a^{-1} P(\leftrightarrow) = z P(\langle\rangle)
\]

We normalize it by

\[
a - a^{-1} = z P(\bigcirc)
\]
The skein module

In any 3-manifold $M$, we define the skein module $Sk(M)$ to be formal $\mathbb{Q}(a^\pm, z^\pm)$-linear combinations of links, modulo the relations.

An equivalent way to assert the existence of the HOMFLYPT polynomial is to assert that $Sk(S^3)$ is 1-dimensional, since then $[K] = P(K) \cdot [\emptyset]$.
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Witten’s Chern-Simons theory (1987)

\[ \langle K \rangle \sim \int DA \ e^{\frac{ik}{4\pi}} \int_{S^3} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \ Hol_K(A) \]
One way we learn geometry from physics

Physical systems may depend on parameters, yet have some quantities which are conserved as parameters vary. Different limits may have different geometric descriptions — e.g., some of the geometry may collapse or decouple. In each limit, some quantities may have geometric interpretations. If these quantities are conserved, then we discover they have two different geometric interpretations. The physical arguments may have not been mathematically rigorous (e.g. because we do not know how to define the Feynman integral), the resulting statement is often mathematically precise – though still conjectural.
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I.e., the physicists tell us:
And our job is:

one limit → system → another limit

one geometry → ? → another geometry
Three paths to the skein relation

2d CFT

Witten

Chern-Simons

Witten

Ooguri, Vafa

Topological string

Gromov, Floer, Kontsevich, Fukaya ...

Quantum groups

Drinfeld, Jimbo

Reshetikhin, Turaev

HOMFLYPT


Holomorphic curve counting
Three paths to the skein relation

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Topological open string

Gromov, Floer, Kontsevich, Fukaya, Ekholm, Shende

Holomorphic open curve counting

Witten

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If \((X, J)\) is an almost complex manifold, and \((C, j)\) is a Riemann surface, we can ask for a map \(f : C \to X\) satisfying \(df \circ j = J \circ df\). This is a nonlinear elliptic PDE. Its linearization is Fredholm with index computed by Riemann-Roch to be \(2(\int_C f^* c_1(M) + (3-d)(1-g))\).

In an ideal world, the index is the dimension of the space of deformations of such maps. In the real world, we perturb the equation to make this true. The index vanishes when \(M\) is a Calabi-Yau manifold \((c_1(M) = 0)\) of complex dimension \(d = 3\).
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J-holomorphic maps: local theory

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The index vanishes when \(M\) is a Calabi-Yau manifold \((c_1(M) = 0)\) of complex dimension \(d = 3\).
When there's a symplectic form $\omega$ so that $g = \omega(\cdot, J \cdot)$ is a metric, then the quantity $I = \int_C f^* \omega$ is both topological (because $\omega$ is closed) and is the $g$-area of $f(C)$. Using this control, Gromov classified how families of such maps may degenerate, and showed that after allowing certain explicit bubbling behaviors, the space of solutions becomes compact (after fixing topological data).
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Gromov-Witten invariants of Calabi-Yau 3-folds

Since the moduli of holomorphic maps to a Calabi-Yau 3-fold is compact (for fixed topological data), and zero dimensional (after perturbation), one can define a number by counting the points of this set.
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To see this did not depend on the perturbation $\xi$, one considers a 1-parameter family of perturbations $\xi_t$ connecting two generic choices $\xi_0$ and $\xi_1$. 
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The moduli space of maps which are holomorphic for some perturbation in the path $M(\xi_t)$ gives a 1-dimensional cobordism from $M(\xi_0)$ to $M(\xi_1)$. 
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Degenerations have complex codimension 2, so are not encountered in this path, so

$$0 = [\partial M(\xi_t)] = [M(\xi_0)] - [M(\xi_1)]$$
Open Gromov-Witten invariants?

A similar story holds for studying maps from a curve with boundary \((C, \partial C)\) to a symplectic manifold with Lagrangian \((X, L)\)...
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This is because there is a 1-real parameter space of choices to smooth a boundary degeneration, as compared to a 1-complex parameter space of choices to smooth an interior one.
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Naively, this seems to mean the corresponding count is not well defined.
The boundary term

\[ a \, P(\times) - a^{-1} \, P(\times) = z \, P(\langle \rangle) \]

\[ a - a^{-1} = z \, P(\bigcirc) \]
Open Gromov-Witten invariants

**Theorem.** (Ekholm-Shende) There exists a space of parameters $\lambda$ such that

$$
\Psi_{X,L,\lambda} := 1 + \sum_{u \text{ primitive}} \sum_{\chi} \#M_{u,\chi}(X, L, \lambda) \cdot z^{-\chi} \cdot Q^{u^*[\Sigma]} \cdot a^{u \phi L} \cdot \langle \partial u \rangle \in Sk(L)[[Q]]
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is independent of the choice of $\lambda$, and of deformations of $X, L$. 
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In this partition function, $z$ counts the Euler characteristic of the source curve, $a$ counts its linking number with the Lagrangian, and $Q$ the degree of the map.
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is independent of the choice of $\lambda$, and of deformations of $X, L$.

In this partition function, $z$ counts the Euler characteristic of the source curve, $a$ counts its linking number with the Lagrangian, and $Q$ the degree of the map.

The term $\partial u$ is the image of the boundary, which is evaluated into $Sk(L)$ is the skein module of the Lagrangian itself.
Geometric meaning of the HOMFLYPT polynomial

**Theorem.** (Ekholm-Shende)
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Let $K \subset S^3$ be a knot, and $L \subset T^*S^3 \setminus S^3$ a push-off of the conormal bundle to $K$. 

This (makes sense of and) confirms a prediction of Ooguri and Vafa.
Theorem. (Ekholm-Shende)
Let $K \subset S^3$ be a knot, and $L \subset T^*S^3 \setminus S^3$ a push-off of the conormal bundle to $K$. Then the degree 1 term of $\Psi_{T^*S^3 \setminus S^3, L}$ is the HOMFLYPT polynomial of $K$.

This (makes sense of and) confirms a prediction of Ooguri and Vafa.
Geometric meaning of the HOMFLYPT polynomial