NOTES ON EQUIDISTRIBUTION AND BRILL-NOETHER ASYMPTOTICS

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1. What is this?

I review the how Brill-Noether loci for curves of fixed gonality arise naturally in the study of a certain equidistribution problem similar to that studied in [6, 5], and some corresponding predictions and geometric questions which arise. Most ideas in here I learned from Jacob Tsimerman.

One motivation for writing these notes is the recent interest and progress in understanding precisely the relevant Brill-Noether loci [4, 2, 3]. (These notes are likely to change over time.)

2. Counting on $\text{Bun}(\mathbb{P}^1)$.

We consider $\text{PGL}_k$ bundles over $\mathbb{P}^1$, which we regard as vector bundles defined up to tensor with a line bundle. These are in bijection with $(k-1)$-tuples of non-negative integers

$$a = (a_1, a_2, \ldots, a_{k-1}) \mapsto \mathcal{O}_a := \mathcal{O} \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_1 + a_2) \oplus \cdots \oplus \mathcal{O}(a_1 + \cdots + a_{k-1})$$

We write $\text{Bun}_k(\mathbb{P}^1)$ for the moduli stack of these bundles. It has $k$ components, corresponding to $\sum a_i$ modulo $k$. We denote these by $\text{Bun}_k(\mathbb{P}^1)_d$.

Fix a finite field $\mathbb{F}_q$. We write $[X]$ for the number of $\mathbb{F}_q$ points on $X$. If $X$ is a stack, we count a point by the inverse of the local automorphism group. E.g.,

$$[\text{Bun}_k(\mathbb{P}^1)] := \sum_a \frac{1}{[\text{Aut}(\mathcal{O}_a)]}$$

We write $P(a)$ for the subgroup of $\text{GL}(k)$ of block-upper triangular matrices, where a square block includes the $i$ and $i + 1$st rows if $a_i = 0$. Then one computes:

$$[\text{Aut}(\mathcal{O}_a)] = q^{\sum (k-i) a_i} \cdot [P(a)]$$

(Strictly speaking, we computed the automorphism as a bundle. If we computed the automorphism as a $\text{PGL}$ bundle, then there should be one less factor of $(q - 1)$ in the denominator. We will ignore this here.)

There is a formula for $[\text{Bun}_k(\mathbb{P}^1)]$, and more generally for point counts of moduli of $G$-bundles on arbitrary curves [1]. I’m going to work out some examples by hand first, in particular to make sure I get the normalizations right.
2.1. $k = 2$. In this case the tuple has a single element, and our notation is $\mathcal{O}_n = \mathcal{O} \oplus \mathcal{O}(n)$.
Let us count separately the even and odd components. On the even component,

$$[Bun_2(\mathbb{P}^1)_0] = \sum_{i=0}^{\infty} \frac{1}{\text{Aut}(\mathcal{O}_{2i})} = \frac{1}{\#_q \text{GL}(2)} + \frac{1}{q(q-1)^2} \sum_{i=1}^{\infty} \frac{1}{q^{2i}}$$

$$= \frac{1}{(q^2-1)(q^2-q)} + \frac{1}{q(q-1)^2} \frac{1}{q^2-1} = \frac{1}{(q-1)^2(q^2-1)}$$

On the odd component,

$$[Bun_2(\mathbb{P}^1)_0] = \sum_{i=1}^{\infty} \frac{1}{\text{Aut}(\mathcal{O}_{2i-1})} = \sum_{i=1}^{\infty} \frac{1}{(q-1)^2} \frac{1}{q^{2i}} = \frac{1}{(q-1)^2(q^2-1)}$$

These are equal, which is no accident.
Let us write $\overline{\mathcal{O}}_n$ for the closure, i.e., the locus of bundles to which the bundle $\mathcal{O}_n$ can degenerate.
We calculate that for all $a > 0$, one has the nice formula

$$(2) \quad \frac{[\mathcal{O}_a]}{[Bun_2(\mathbb{P}^1)_0]} = \sum_{i=0}^{\infty} \frac{\mathcal{O}_{a+2i}}{[Bun_2(\mathbb{P}^1)_0]} = \frac{1}{q^{a+1}} \sum_{i=0}^{\infty} \frac{1}{(q-1)^2} \frac{1}{q^{2i}} = \frac{1}{(q-1)^2(q^2-1)}$$

Note that knowing all these ratios is equivalent to knowing all the automorphism factors, up to a single scalar.

2.2. $k = 3$. Let us work out some formulas for the case $k = 3$. Now the tuples have two entries

$$\mathcal{O}(a_1, a_2) := \mathcal{O} \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$$

We count:

$$[Bun_3(\mathbb{P}^1)] = \frac{1}{[GL_3]} + \frac{2}{[GL_2][GL_1]} q^2 \sum_{i=1}^{\infty} \frac{1}{q^{2i}} + \frac{1}{[GL_1]^3} q^3 \sum_{i,j=1}^{\infty} \frac{1}{q^{2(i+j)}}$$

$$= \frac{1}{(q^3-1)(q^3-q)(q^3-q^2)} + \frac{2}{(q-1)^4} \frac{1}{q^2-1} + \frac{1}{(q-1)^3} \cdot \frac{1}{(q^2-1)^2}$$

$$= \frac{1}{q^3} \left( \frac{1}{(q^3-1)(q^2-1)(q-1)} + \frac{2}{(q^2-1)^2(q-1)^2} + \frac{1}{(q-1)^3(q^2-1)^2} \right)$$

$$= \frac{1}{q^3(q^2-1)^2} \left( \frac{q+1}{(q^3-1)} + \frac{2}{(q-1)^2} + \frac{1}{(q-1)^3} \right)$$

$$= \frac{1}{q^3(q^2-1)^2} \left( \frac{(q+1)(q-1)^3 + 2(q-1)^2(q-1) + q^3 - 1}{(q^3-1)(q-1)^3} \right)$$

$$= \frac{1}{q^3(q^2-1)^2} \left( \frac{3q^3(q-1)}{(q^3-1)(q-1)^3} \right)$$

$$= \frac{3}{(q-1)^2(q^2-1)^2(q^3-1)}$$
The components correspond to the different values of $a_1 + 2a_2$ modulo 3. As a sanity check we count the zero component.

$$[\text{Bun}_3(\mathbb{P}^1)_0] = \frac{1}{[\text{GL}_3]} + \frac{2}{[\text{GL}_2][\text{GL}_1]}q^2 \sum_{i=1}^{\infty} \frac{1}{q^{6i}} + \frac{1}{[\text{GL}_1]^3}q^3 \left( \sum_{j=1}^{\infty} \frac{1}{q^{4j}} + 2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{q^{2(3i+2j)}} \right)$$

Here, we taken the sum over the interior of the quadrant by splitting it into sums along the diagonals. Evaluating

$$= \frac{1}{[\text{GL}_3]} + \frac{2}{[\text{GL}_2][\text{GL}_1]}q^2 \cdot \frac{1}{q^6 - 1} + \frac{1}{[\text{GL}_1]^3}q^3 \left( \frac{1}{q^4 - 1} + \frac{2}{(q^4 - 1)(q^6 - 1)} \right)$$

$$= \frac{1}{q^3(q - 1)(q^2 - 1)(q^3 - 1)} + \frac{2}{q^3(q^2 - 1)(q^2 - 1)(q^3 - 1)} + \frac{q^6 + 1}{q^3(q - 1)^3(q^4 - 1)(q^6 - 1)}$$

$$= \frac{1}{(q - 1)^2(q^2 - 1)^2(q^3 - 1)}$$

The bundle $\mathcal{O}_{(a_1,a_2)}$ can specialize to $\mathcal{O}_{(b_1,b_2)}$ iff $2a_1 + a_2 \leq 2b_1 + b_2$ and $a_1 + 2a_2 \leq b_1 + 2b_2$. The locus of such $b$ is the complement in the positive quadrant of a boomerang shape with inner corner at $(a_1, a_2)$, which is a bit of a mess to sum over, and I have not tried to compute an explicit formula. Here is the simplest nontrivial calculation of the pointcount of a closure:

$$\frac{[\mathcal{O}_{(1,1)}]}{[\text{Bun}_3(\mathbb{P}^1)_0]} = \frac{[\text{Bun}_3(\mathbb{P}^1)_0] - [\mathcal{O}_{(0,0)}]}{[\text{Bun}_3(\mathbb{P}^1)_0]} = q^{-1} + q^{-2} - q^{-3}$$

2.3. General $k$.

Lemma 1.

$$[\text{Bun}_k(\mathbb{P}^1)_0] = \frac{q^{(k-1)^2}}{[\text{GL}_{k-1}][\text{GL}_k]}$$

Proof. One has in general from Harder’s formula:

$$[\text{Bun}_k(\mathbb{P}^1)_0] = \frac{q^{1-k^2}}{(q - 1)} \prod_{i=2}^{k} Z(\mathbb{P}^1, q^{-k})$$

Here, $Z$ is the zeta function $Z(\mathbb{P}^1, t) = \sum \text{Sym}^n(\mathbb{P}^1)t^n = \sum \mathbb{P}^n t^n = \frac{1}{(1-t)(1-qt)}$. Plugging in,

$$= \frac{q^{1-k^2}}{(q - 1)} \prod_{i=2}^{k} \frac{1}{(1-q^{-i})(1-q^{-1-i})} = \frac{1}{q - 1} \prod_{i=2}^{k} \frac{1}{(q^i - 1)(q^{i-1} - 1)} = \left( \prod_{i=1}^{k-1} \frac{1}{(q^i - 1)} \right) \left( \prod_{i=1}^{k} \frac{1}{(q^i - 1)} \right)$$

$$= \frac{q^{(k-1)^2}}{[\text{GL}_{k-1}][\text{GL}_k]} \quad \square
Question 2. Is there a simple conceptual explanation for the formula \([Bun_k(\mathbb{P}^1)_0] = q^{(k-1)^2}/[GL_{k-1}][GL_k] \)?

Note one could say even more appealingly that the count of rank \(k\) bundles with a fixed framing at one point is \(q^{(k-1)^2}/[GL_{k-1}]\), i.e., a quotient of all \((k-1) \times (k-1)\) matrices by the invertible ones.

In any case from the Lemma, we compute:

**Lemma 3.** One has

\[
\frac{[O_a]}{[Bun_k(\mathbb{P}^1)_0]} = q^{-(k-1)(k-1+\sum(k-1)a_i)}[GL_{k-1}][GL_k]/[P_a] = q^{-(k-1)(k-1+\sum(k-1)a_i)}[GL_{k-1}][Fl_a]
\]

Recall \(P_a\) is the block-upper-triangular matrices, where the blocks correspond to runs of zeros in \(a\). We write \([Fl_a]\) for the corresponding partial flag variety.

**Remark 4.** Note in particular that always this ratio is a polynomial (more precisely, in \(\mathbb{Z}[q^{-1}]\)), though not necessarily with positive coefficients. This polynomiality underwrites the reasonableness of the strategy discussed later.

**Remark 5.** It is instructive to consider what happened when \(k = 2\) and a nontrivial. In this case \([GL_{k-1}]/[P_a] = [GL_1][\mathbb{P}^1] = (q-1)(q+1) = q^2 - 1\). Passing to the closure gives a telescoping sum cancelling this term. Perhaps it is possible to more intelligently choose test functions to get better telescoping in general.

**Lemma 6.** The first nontrivial closure is counted by:

\[
\frac{[O_{(1,0,0,...,0,1)}]}{[Bun_k(\mathbb{P}^1)_0]} = \frac{[Bun_k(\mathbb{P}^1)_0] - [GL_k]}{[Bun_k(\mathbb{P}^1)_0]} = 1 - q^{-(k-1)^2}[GL_{k-1}]
\]

In other words, the probability of not being the generic rank \(k\) bundle is the same as the probability of not being an invertible \((k-1) \times (k-1)\) matrix.

Let’s check that this recovers our by hand calculations in the case \(k = 2, 3\).

\[
\frac{[O_{(1,1)}]}{[Bun_2(\mathbb{P}^1)_0]} = 1 - q^{-1}[GL_1] = 1 - q^{-1}(q-1) = q^{-1}
\]

\[
\frac{[O_{(1,1)}]}{[Bun_2(\mathbb{P}^1)_0]} = 1 - q^{-4}[GL_2] = 1 - q^{-4}(q^2-1)(q^2-q) = 1 - q^{-4}(q^4 - q^3 - q^2 + q) = q^{-1} + q^{-2} - q^{-3}
\]

**Remark 7.** I believe it is true that the closures \(O_a\) always have complement given by finitely many points. It follows from this and Lemma 3 that \(\frac{[O_a]}{[Bun_k(\mathbb{P}^1)_0]} \in \mathbb{Z}[q^{-1}]\). It is presumably straightforward to check from the formula that its leading (least negative degree) term is the same as \(\frac{[O_a]}{[Bun_k(\mathbb{P}^1)_0]}\).

**Question 8.** Is there a closed form formula for \([O_a]\) in general? (Presumably this is known and maybe even easy from the point of view of \(Bun_k(\mathbb{P}^1)\) being a symmetric space and these loci being orbit closures.)
3. An equidistribution question

Suppose given a sufficiently general collection of degree $k$ covers $\pi_i : C_i \to \mathbb{P}^1$. Here, sufficiently general means that the genus of $C_i$ goes to infinity, and possibly some other explicit list of conditions. (E.g. for hyperelliptic curves, no condition.)

Then $\pi_i$ induces a map $\pi_i^* : \text{Pic}(C_i) \to \text{Bun}_k(\mathbb{P}^1)$. Consider the measures $\mu_i$ on $\text{Bun}_k(\mathbb{P}^1)$ obtained by pushing forward the counting measure along this map, and the measure $\mu$ given by the (stacky) point counting measure on $\text{Bun}_k(\mathbb{P}^1)$. A natural conjecture on equidistribution in the context of maps between adelic symmetric spaces implies that $\hat{\mu}_i \to \hat{\mu}$, where by $\hat{\cdot}$ we mean the corresponding probability measure. The original conjecture and this translation are reviewed in the introduction and appendix of [6].

Let us translate this a bit. Fix now some specific degree $k$ map $\pi : C \to \mathbb{P}^1$. For a splitting type $a \in \text{Bun}_k(\mathbb{P}^1)$, we write (following the notation of [3], though our convention for denoting a splitting type is different)

$$\Sigma_a := \pi^{-1}(\mathcal{O}_a) \subset \text{Pic}(C)/\pi^*\mathcal{O}(1)$$

$$\overline{\Sigma}_a := \pi^{-1}(\overline{\mathcal{O}_a}) \subset \text{Pic}(C)/\pi^*\mathcal{O}(1)$$

Here the quotient $\text{Pic}(C)/\pi^*\mathcal{O}(1)$ corresponds to the fact that we have chosen to work with $\text{PGL}$ bundles.

Recall that the splitting type of a bundle $E$ on $\mathbb{P}^1$ is characterized by all $\dim H^0(\mathbb{P}^1, E(i))$. In case $E = \pi_*\mathcal{L}$, this is the same as asking for all $H^0(C, \mathcal{L} \otimes \pi^*\mathcal{O}(i))$. It follows that each $W_a$ is contained in exactly one of the classical Brill-Noether loci:

$$W_d^r := \{ \mathcal{L} \in \text{Pic}(C) \mid \text{for some } i : \mathcal{L} \otimes \pi^*\mathcal{O}(i) \in \text{Pic}_d(C) \text{ and } h^0(C, \mathcal{L} \otimes \mathcal{O}(i)) \geq r + 1 \}$$

(There is a formula for $d = d(a)$ in terms of the genus of $C$, the degree of the cover, and the splitting type $a$, which probably we should write here.)

Let us recall that there are polynomials in $\mathbb{Z}[q^{-1}]$

$$N_a(q^{-1}) := \frac{[\mathcal{O}_a]}{[\text{Bun}_k(\mathbb{P}^1)_{d(a)}]} \quad \overline{N}_a(q^{-1}) := \frac{[\overline{\mathcal{O}_a}]}{[\text{Bun}_k(\mathbb{P}^1)_{d(a)}]}$$

The equidistribution conjecture above is equivalent to (either of) the statements that, for all $a$, in any appropriate sequence of curves with genus going to infinity,

$$\frac{[\Sigma_a]}{[\text{Pic}_d(a)(C)]} \to N_a(q^{-1}) \quad \frac{[\Sigma_a]}{[\text{Pic}_d(a)(C)]} \to \overline{N}_a(q^{-1})$$

(Assuming the curves $C$ have at least one point, there is no actual dependence of the LHS denominator on $d(a)$.)
4. A cohomological approach

We recall that the Grothendieck-Lefschetz trace formula tells us how to count points on algebraic varieties:

\[ [X] = \text{trace}(\text{Frobenius}, H_c^*(X \otimes \mathbb{F}_q)) \]

Here, the cohomology is the \( \ell \)-adic cohomology with some \( \ell \) prime to \( q \), and the trace is taken with alternating signs according as the cohomological degree. From Deligne’s results on weights we have, for \( X \) compact, the estimate:

\[ |[X] - \text{trace}(\text{Frobenius}, H_{\geq d}^c(X \otimes \mathbb{F}_q))| \leq (\dim H^*(X)) q^{d/2} \]

**Definition 9.** Let \( \pi : C \to \mathbb{P}^1 \) be a \( k \)-gonal curve, and \( a \) a splitting type. For \( 0 < \eta < 1 \), we say that the \( a \)-splitting locus of \( C \) is \( \eta \)-standard if:

1. as Frobenius modules,
   \[ H_{\geq 2ng-2 deg(N_a)}(\Sigma_a) = N_a(\mathbb{T}) \otimes H_{\geq 2ng}(Pic_d(a)(C)) \]
   where \( \mathbb{T} \) is the “étale realization of the Tate motive”, i.e. \( \otimes \mathbb{T} \) shifts things down in cohomological degree by two and Tate twists them by one. Here \( deg(N_a) \) means the lowest degree in \( q^{-1} \), i.e. \( deg(q^{-4} + q^{-5}) = 4 \).
2. (The exponential bound)
   \[ \dim H^*(\Sigma_a) \leq q^{ng/2} \]

The definition is saying that the top degree cohomology groups behave as they should, and there is not enough total Betti number for the lower degree ones to spoil this. That is, if the \( a \)-splitting locus of \( C \) is \( \eta \)-standard then by the above estimate

\[ \frac{[\Sigma_a] - N_a(q^{-1})[Pic_d(a)(C)]}{[Pic_d(a)(C)]} \leq q^{(\eta-1)g} \]

which goes to zero as \( g \to \infty \).

In practice, establishing the exponential bound is a seriously nontrivial problem. The reason we need to have something like it is that one expects to have essentially no fine control of what appears in the lower cohomology groups of \( \Sigma_a \).

So if one hopes to approach the equidistribution problem cohomologically, one should believe something like the following:

**Conjecture 10.** (Cohomological equidistribution) Fix \( k \). Then for all sufficiently large \( \mathbb{F}_q \), for any splitting type \( a \) there exists \( \eta \) so that for all \( g \gg 0 \), all \( 1 \)-gonal curves of genus \( g \) have \( \eta \)-standard \( a \)-splitting locus.

\(^1\)Maybe some additional hypotheses are required, the nature of which can be in principle be extracted from a discussion along the lines of what is in the appendix of [6].
In this conjecture, \( F_q \) is chosen in terms of the gonality \( k \), but not in terms of \( g \). It implies the originally desired equidistribution statement, for this \( F_q \) and \( k \), by Equation 3.

**Remark** 11. Suppose, for a given curve, we had a numerical statement like Equation 3, and that we knew the exponential bound. We could reverse the logic to deduce in particular that since \( N_a \) has leading term \( q^{-2} \), the top degree cohomology of \( \Sigma_a \) must be one dimensional; we would also know in which degree \( (\cdot) \) it occurred. We could deduce that \( \Sigma_a \) is of the expected codimension \( (\cdot) \) and has only one component. The \( (\cdot) \) can be read off of Lemma 3.

The fact that this piece has the expected dimension for a generic curve is in fact known [3]; it is not yet known that there is only one component.

### 4.1. The hyperelliptic case.

As a warmup to the actual problem of interest in [6], we showed:

**Proposition 12.** [6] For \( k = 2 \), i.e. hyperelliptic curves, Conjecture 10 holds with \( \eta \sim 1/2 \). In fact, the statement holds for all \( g \), not just those sufficiently large.

The nature of the proof was as follows. The relevant loci are the images of the symmetric powers under the Abel-Jacobi map. The polynomials \( N_a \) are just \( q^{-2} \). This means that our job is to show that the upper cohomologies of these loci are the same as the corresponding ones of the Jacobian (appropriately shifted down). One does this as follows. First, the loci are set-theoretically cut out by sections of an ample line bundle, hence by Lefschetz hyperplane theorem one knows their lower cohomologies. Then one shows that these loci are homology manifolds (i.e. the constant sheaf is the same as the intersection cohomology sheaf), hence they satisfy Poincaré duality, from which we learn their upper cohomology groups. Finally, the total cohomology can be bounded by that of the symmetric product (by the decomposition theorem).

### 4.2. Trigonal thoughts.

None presently!

### 4.3. General questions.

In the argument in [6] for the hyperelliptic case, a key role was played by the fact that we understood resolutions of all splitting loci \( \Sigma \). In that case we could just use the symmetric powers, but in general we need more spaces:

**Problem 13.** Describe resolutions of the \( \Sigma_a \). (Also good: any dominant maps from smooth proper varieties, or for that matter from varieties whose cohomology can be computed. Ideally the degeneracy loci of the maps should be described inductively in terms more special splitting types.)

The role of the resolutions was to understand how the intersection cohomology complex and the constant sheaf on \( \Sigma_a \) were related (the relation having been: equality). One could also just try and do this directly:

**Problem 14.** Expand the constant sheaf on \( \Sigma_a \) in intersection cohomology sheaves. (And determine the extent to which these satisfy Lefschetz hyperplane).
Remark 15. Possibly one can try and reverse-engineer this by first understanding how best to collect \( N_a \) into telescoping sums giving monomials. I.e., one would hope that such a sum has a geometric counterpart which is a sheaf satisfying Lefschetz hyperplane theorem.

This will surely require:

**Problem 16.** Describe the singularities of \( \Sigma_a \).

Remark 17. We note that in [3] these loci are identified as degeneracy loci; one could approach this problem by ‘universally’ solving this problem for the corresponding universal degeneracy loci, then (in the spirit of [6]) bounding the locus where the singularities fail to be given by the universal calculation.

Also crucial to the approach of [6] is:

**Problem 18.** Bound the cohomology of \( \Sigma_a \).

Remark 19. One might approach this by first trying to do this for a general curve, and then using characteristic cycle methods etc. to bound how much worse it can be on a particular curve, similar to what is done in [6, 5].

Another tantalizing possibility is raised by the formula

\[
\frac{\mathcal{O}_{(1,0,0,...,0,1)}}{[\text{Bun}_k(\mathbb{P}^1)_0]} = 1 - q^{-(k-1)^2}[GL_{k-1}]
\]

Since we want to learn correspondingly that

\[
\frac{\Sigma_{(1,0,0,...,0,1)}}{[\text{Pic}(C)\sim]} \sim 1 - q^{-(k-1)^2}[GL_{k-1}]
\]

one might hope it is useful to do something like:

**Problem 20.** For a \( k \)-gonal curve, describe the locus \( \Sigma_{(1,0,0,...,0,1)} \) (I believe this is just the classical \( \Theta \) divisor) as a degeneracy locus of an endomorphism of a rank \( k - 1 \) bundle.

I am not exactly sure how the answer to the above question would be useful, but it seems like it must be. As noted there are some descriptions as degeneracy loci already in [3], but I have not yet internalized what they mean; possibly already answering the above question.

If this has a positive answer, then of course one would like to know the same for the other \( \Sigma_a \), i.e. if they have degeneracy loci descriptions which correspond to the formulas (which I do not presently know but is surely not actually hard to compute) for \( \overline{N}_a \).

So far we have been thinking about how studying Brill-Noether loci may help solve an equidistribution problem. But one could also go the other way:

**Problem 21.** To draw the conclusions on the connectedness and dimension of the \( \overline{\Sigma}_a \) in Remark 11, it was not necessary to have fixed a choice of finite field before choosing the curve. Moreover, to
draw a conclusion for a general curve, it would be enough to know the equidistribution statement “on average”. Both of these facts should make the equidistribution question much easier; perhaps in this form it can be solved by other methods, and the result transported to give connectedness and of-expected-dimensionality of the $\Sigma_a$. (Note one would still have to establish the exponential bound.)

REFERENCES