Today we recall some formalism of categories, abelian categories, derived categories...

1. ADJOINT FUNCTORS

Much of the power of sheaf theory comes from the fact that very many of the functors which appear come in adjoint pairs.

Recall that an adjunction of functors \( F : A \cong B : G \) is a natural isomorphism

\[
\text{Hom}_A(Fa, b) \cong \text{Hom}_B(a, Gb)
\]

In this case, each of \( F, G \) determines the other, and there are natural maps \( a \to GFa \) and \( FGb \to b \).

The one which goes on the left in the hom (in this case \( F \)) is called the left adjoint, and the one which goes on the right (in this case \( G \)) is called the right adjoint. Sometimes one says that \( (F, G) \) is an adjoint pair.

A very useful fact about adjoint functors is their “continuity”, i.e., limit respecting properties: a functor which has a left adjoint (i.e., is a right adjoint) is “continuous” — it commutes with limits (e.g. products, kernels) — and a functor which has a right adjoint (i.e. is a left adjoint) is co-continuous, or commutes with colimits. This fact characterizes adjoint functors, up to set theoretical difficulties:

**Theorem 1. (Freyd’s adjoint functor theorem)** Say \( B \) is complete (has all limits) and \( G : B \to A \) is a continuous functor. Then it has a left adjoint if, and only if, for every object \( a \in A \), there’s some set of morphisms from \( a \) to objects in \( G(B) \) through which all morphisms from \( a \) to objects in \( G(B) \) factor.

Similarly if \( F : A \to B \) is cocontinuous from a cocomplete category, then it has a right adjoint if and only if, for each \( b \in B \) there’s some set of morphisms from objects in \( F(A) \) to \( b \) through which all such morphisms factor.

A prototypical example is the (free, forgetful) adjunction. That is, a “concrete category” (objects are sets and morphisms are functions) \( \mathcal{C} \) comes by definition with a “forgetful functor” to the category of sets. The formation of a “free” object in \( \mathcal{C} \) on a given set, if possible, gives a left adjoint to this functor. In particular: if \( \mathcal{C} \) has free objects, then the underlying set of a limit in \( \mathcal{C} \) is the limit of the corresponding underlying sets. The same isn’t true of colimits. This is why, e.g. the formula for products looks the same in most categories you can think of, but not the formula for coproducts.
2. **ABELIAN CATEGORIES**

Recall that a category is said to be:

- **pre-additive** if it’s “enriched over the monoidal category of abelian groups”, i.e., the hom spaces are abelian groups and composition is bilinear. Because finite products and coproducts agree for abelian groups, the same is true for a pre-additive category.
- **additive** if, in addition, all finite (co)products exist. The empty (co)product is denoted \(0\); it’s both initial and final.
- **pre-abelian** if, in addition, kernels and cokernels exist. (Recall kernels and cokernels are the equalizers and co-equalizers between a given morphism \(f : A \to B\) and the zero morphism. Or: the objects which represent \(C \mapsto \text{Ker}(\text{Hom}(C, A) \to \text{Hom}(C, B))\) or \(D \mapsto \text{Ker}(\text{Hom}(B, D) \to \text{Hom}(A, D))\).)
- **abelian** if, in addition, every epimorphism is a cokernel and every monomorphism is a kernel. Equivalently, if the “first isomorphism theorem” holds, i.e. the natural map from the coimage to the image is an isomorphism.

In abelian categories, exact sequences make good sense and behave as expected. We let \(A, B, C\) etc. denote abelian categories.

A functor \(f : A \to B\) is **left exact** if, equivalently,

- it carries the exact sequence
  \[
  0 \to a \to a' \to a'' \to 0
  \]
  to an exact sequence
  \[
  0 \to f(a) \to f(a') \to f(a'')
  \]
- it preserves kernels
- it commutes with finite limits

Similarly, a functor is **right exact** if, equivalently,

- it carries the exact sequence
  \[
  0 \to a \to a' \to a'' \to 0
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  to an exact sequence
  \[
  f(a) \to f(a') \to f(a'') \to 0
  \]
- it preserves cokernels
- it commutes with finite colimits

A functor is **exact** if it’s both left exact and right exact.

Note in particular that continuous functors (in particular, functors which have left adjoints) are left exact, and co-continuous functors (in particular, functors which have right adjoints) are right exact.
3. Categories of complexes

In an additive category \( \mathcal{A} \), it makes sense to ask that a sequence of maps

\[
\cdots \to a^{-1} \xrightarrow{d^{-1}} a^0 \xrightarrow{d^0} a^1 \to \cdots
\]

has the property that the composition of two consecutive maps is zero. Such is called a cochain complex; the maps \( d^i \) are called differentials. We’d indicate the above complex as \( a^\bullet \) or just \( a \).

These form a category; a morphism \( a^\bullet \to b^\bullet \) is a collection of morphisms \( a^i \to b^i \), commuting with the differentials. We write the category of complexes as \( C(\mathcal{A}) \).

We write \( a[k] \) for the complex \( a \) shifted \( k \) steps to the left and with the differential multiplied by \( (-1)^k \). That is, \( a[k]^n = a_{k+n} \).

Given a map \( f : a \to b \), the mapping cone \( \text{Cone}(f) \) has \( a[1] \oplus b \) as underlying graded vector space. Its differential is

\[
d_{\text{Cone}(f)} = \begin{bmatrix} d_{a[1]} & f \\ -f & d_b \end{bmatrix}
\]

(If we had not put the sign in the shift, we would have to put it into the mapping cone.) Note the triangularity of the above differential ensures there are maps \( b \to \text{Cone}(f) \to a[1] \).

Given two morphisms \( f, g : a^\bullet \to b^\bullet \), a homotopy between them is a morphism \( h : a^\bullet \to b^\bullet_{-1} \) with \( dh + hd = f - g \). We write \( K(\mathcal{A}) \) for the category formed from \( C(\mathcal{A}) \) by identifying homotopic morphisms. A morphism in \( C(\mathcal{A}) \) which becomes an isomorphism in \( K(\mathcal{A}) \) is called a homotopy equivalence.

Suppose given an exact sequence of complexes:

\[
0 \to a^\bullet \xrightarrow{f} b^\bullet \xrightarrow{g} c^\bullet \to 0
\]

Then the natural maps \( \text{Cone}(f) \to c^\bullet \) and \( a^\bullet \to \text{Cone}(g) \) are homotopy equivalences. That is, both kernels and cokernels are mapping cones, up to homotopy.

We write \( H^i(a^\bullet) := \ker(d^i)/\text{im}(d^{i-1}) \); these are called the cohomology groups of the complex. A morphism of complexes \( f : a^\bullet \to b^\bullet \) induces a morphism of cohomology groups. \( H^i(f) : H^i(a^\bullet) \to H^i(b^\bullet) \). Homotopy equivalences induce isomorphisms of cohomology. Any map which induces an isomorphism of cohomology is called a quasi-isomorphism. A complex with vanishing cohomology is said to be acyclic.

Example. A short exact sequence can be viewed as an acyclic complex. Note that a homotopy from the zero morphism to the identity morphism is the same as a splitting of the short exact sequence.

The maps

\[
\cdots \to H^i(a) \to H^i(b) \to H^i(\text{Cone}(f)) \to H^i(a[1]) = H^{i+1}(a) \to \cdots
\]

form a long exact sequence. Since cones are homotopy equivalent to kernels/cokernels, also exact sequences of complexes give rise to such long exact sequences of cohomology. More generally, anything homotopy equivalent to \( a \to b \to \text{Cone}(f) \to a[1] \) is called an exact triangle; they all
give rise to long exact sequences. Note in particular that a morphism is a quasi-isomorphism if and only if its cone is acyclic.

4. THE DERIVED CATEGORY

The derived category $D(A)$ is what you get by formally inverting quasi-isomorphisms. That is, it has the same objects as $K(A)$, and its morphism spaces $\text{Hom}_{D(A)}(a, b)$ would by definition be structures of the form $a \xrightarrow{\sim} c \rightarrow b$, where the left arrow is a quasi-isomorphism; possibly with longer chains, modulo some relations. There’s a set-theoretical issue: the possible objects $c$ to put in the middle needn’t form a set, so it’s not clear the morphism space does.

To deal with this problem, and in addition to be able to understand directly the derived category, we will look for a subcategory of $X \subset K(A)$ which, under the map $K(A) \rightarrow D(A)$, maps isomorphically to the image. Think for a moment about vector spaces: for $W \subset V$, one way to understand $V/W$ — or, as when we teach the undergraduates, a way to avoid discussing it — is to find some complementary $W^\perp \subset V$ which projects isomorphically to $V/W$.

The same idea helps get a handle on the derived category. Actually I’ll only explain here how to do this for the subcategory $K^+(A)$ of bounded-below complexes, and its corresponding image $D^+(A)$. It’s possible to do in general, but significantly more complicated.\footnote{see: Nicholas Spaltenstein, Resolutions of unbounded complexes, Compositio Mathematica 65.2 (1988): 121-154.}

In order that $X$ have any chance to map isomorphically to $D^+(A)$, it should

- be closed under shift and cones
- have the property that morphisms in $X$ are quasi-isomorphisms only if they’re homotopy equivalences — checking via looking at cones, it suffices to ask that acyclic objects in $X$ are null homotopic
- contain something quasi-isomorphic to every element of $K^+(A)$

Recall that an object $I$ is said to be injective if the following lifting property holds: given any map $f : a \rightarrow I$ and an inclusion $a \hookrightarrow b$, the map $f$ can be extended to $b$. That is, $\text{Hom}(\cdot, I)$ carries injections to surjections. Our desired $X$ will be the category of complexes of injectives, which is obviously closed under cones and shifts.

Let’s check the other properties above. We already saw that a basic difference between acyclicity and null homotopy is the splitting of short exact sequences. Given a short exact sequence of injectives $0 \rightarrow I \rightarrow J \rightarrow K \rightarrow 0$, the identity map $I \rightarrow I$ can be extended along the inclusion $i : I \rightarrow J$ to give a retraction $J \rightarrow I$. From this it readily follows that any acyclic complex of injectives is in fact null homotopic.

It remains to discuss when every object of $K^+(A)$ is quasi-isomorphic to a bounded-below complex of injectives. For elements of $A$ themselves, this amounts to finding “injective resolutions”, i.e. exact sequences of the form

$$0 \rightarrow a \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$
where all the $I^k$ are injective. For this to be possible, certainly each object should admit an inclusion into an injective object; the category is said to have \textit{enough injectives} when this is possible.

Having “enough injectives” is sufficient to construct injective resolutions: by definition one can find $a \hookrightarrow I^0$, and then $I^0/a \hookrightarrow I^1$, and so on.

For an arbitrary complex $a^* \in K^+(A)$, one first finds an injective resolution of each $a^i$, and then forms a double complex out of these resolutions. The basic idea is as follows: say one has $a^0 \hookrightarrow I^{0,0}$ and $a^1 \hookrightarrow I^{1,0}$. Then by definition of injective, the map $a^0 \rightarrow a^1 \rightarrow I^{1,0}$ lifts along the injective map $a^0 \hookrightarrow I^{0,0}$ to a map $I^{0,0} \rightarrow I^{1,0}$.

\textit{Because we began with a bounded below complex}, each term in the double complex is a \textit{finite} sum, and thus can be shown to be quasi-isomorphic to the original complex.

5. \textbf{Enough Injectives}

In the category of abelian groups, the obstruction to being injective is exemplified by the fact that the identity morphism of $n\mathbb{Z}$ can’t be lifted to a retraction of the inclusion $n\mathbb{Z} \rightarrow \mathbb{Z}$. In fact:

\textbf{Proposition 2.} An abelian group is injective (as an object in the category of abelian groups) if and only if every element can be divided by $n$, for all $n \in \mathbb{N}$.

\textit{Proof.} “Only if” can be seen by the same sort of counterexample as given before the proposition. Suppose given an abelian group $I$ satisfying the hypothesis (sometimes called being “divisible”), a morphism $f : A \rightarrow I$, and an inclusion $A \subset B$. We should lift $f : A \rightarrow I$ to a morphism $B \rightarrow I$.

It suffices to explain how to do this for a single element in $B \setminus A$, and then invoke Zorn’s lemma.

Anyway consider some element $b \in B \setminus A$. Either $\mathbb{Z}b \cap A = 0$ — in which case, the subgroup generated by $A$ and $b$ is the direct sum of $A$ and the cyclic group generated by $b$, and we can extend the map $A \rightarrow I$ to $\langle A, b \rangle \rightarrow I$ by sending $b \rightarrow 0$ — or, there’s some minimal $n$ for which $nb = a \in A$, in which case the map can be extended by sending $b \mapsto f(a)/n$. \hfill $\Box$

In particular, $\mathbb{Q}/\mathbb{Z}$ is injective. This module has the additional property of being “cogenerator”:

\textbf{Proposition 3.} $A = 0 \iff \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) = 0$

\textit{Proof.} If $A$’s nonzero, it has a cyclic subgroup, which admits a nonzero morphism to $\mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, this morphism lifts to a morphism from $A$. \hfill $\Box$

Given any injective cogenerator $I$, the “double dual” map $A \rightarrow \text{Hom}(\text{Hom}(A, I), I)$ is an inclusion. Indeed, its kernel $K$ consists of elements of $A$ which, under any map $A \rightarrow I$, are sent to zero. But if $K$ is nonzero, then, because $I$’s a cogenerator, there’s a nonzero map $f : K \rightarrow I$, and since $I$’s injective this map lifts to $\tilde{f} : A \rightarrow I$ — hence $K$ must be zero.

Finally, one constructs an embedding into an injective by taking generators $\mathbb{Z}^S \rightarrow \text{Hom}(A, I)$ and dualizing to get $A \hookrightarrow \text{Hom}(\text{Hom}(A, I), I) \hookrightarrow I^S$. 


For a continuous map $f : X \to Y$, recall there are morphisms

$$f^* : Shv(Y) \to Shv(X) : f_*$$

**Exercise.** Show these enjoy an adjunction

$$\text{Hom}_Y(f^* \cdot, \cdot) \cong \text{Hom}_X(\cdot, f_* \cdot)$$

In particular, $f^*$, as it has a left adjoint, is continuous; likewise $f_*$ is co-continuous. For sheaves $\mathcal{F}$ on $X$ and $\mathcal{G}$ on $Y$, there are natural maps $\mathcal{G} \to f_* f^* \mathcal{G}$, and likewise $f^* f_* \mathcal{F} \to \mathcal{F}$.

**Exercise.** Let $X^\text{disc}$ be the $X$ with the discrete topology; consider the (continuous!) map $\pi : X^\text{disc} \to X$. For a sheaf $\mathcal{F}$ on $X$, note that $(\pi_* \pi^* \mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x$. We call this sheaf $\mathcal{F}^{\text{disc}}$. Observe that the natural map $\mathcal{F} \to \pi_* \pi^* \mathcal{F}$ is an inclusion. Contemplate the stalks of $\pi_* \pi^* \mathcal{F}$.

Since questions of kernels, cokernels, etc. can be checked stalkwise, sheaves of abelian groups (or $R$-modules, or of modules over a sheaf of rings) are easily seen to form an abelian category.

The pushforward $f_*$, having a left adjoint, is left exact; $f^*$, having a right adjoint, is right exact.

We have already seen that $f_*$ is not right exact. However:

**Exercise.** By considering the inclusion of a point $x \hookrightarrow X \to Y$, observe that $(f^* \mathcal{G})_x \cong \mathcal{G}_{f(x)}$. Since questions of exactness can be checked at stalks, conclude that $f^*$ is exact.

Finally let us see that the category of sheaves of abelian groups has enough injectives.

**Proposition 4.** Let, for each point in $x$, $I(x)$ be the pushforward from the point $x$ of (the constant sheaf whose value is) some injective module. Consider the sheaf $I = \prod_{x \in X} I(x)$. Then the sheaf $I$ is an injective in the category of sheaves.

**Proof.** One has $\text{Hom}_X(\mathcal{F}, I) = \prod_{x \in X} \text{Hom}_X(\mathcal{F}, I(x)) = \prod_{x \in X} \text{Hom}_{\text{Ab}}(\mathcal{F}_x, I(x))$.

Given an inclusion of sheaves $\mathcal{F} \subset \mathcal{G}$ one has $\mathcal{F}_x \subset \mathcal{G}_x$ hence can lift maps $\mathcal{F}_x \to I(x)$ to $\mathcal{G}_x \to I(x)$, and thus by the above calculation, can lift maps $\mathcal{F} \to I$ to $\mathcal{G} \to I$. \hfill $\square$

Finally, for any sheaf $\mathcal{F}$, one can find for each stalk $\mathcal{F}_x$ some injective $I(x) \supset \mathcal{F}_x$. Checking at stalks, the natural map $\mathcal{F} \to \prod \mathcal{F}_x \to \prod I(x)$ is an injection into an injective. Thus the category of sheaves has enough injectives.