LECTURE 2: SHEAFIFICATION, STRATIFICATIONS, AND ČECH COHOMOLOGY

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A presheaf is a huge amount of data: one abelian group (or other appropriate object) for each open set. Because sheaves are determined locally, they are in some sense slightly less data. The sheaves of interest to us – the constructible sheaves, which become constant after restriction to the strata of a stratification – are still less data, and can be defined, manipulated, and computed with in an essentially combinatorial manner. In this lecture, we begin explaining how this is done.

1. SHEAFIFICATION

The usual prototypical examples of sheaves are sheaves of functions.

Example. If $X$ is a topological space and $S$ is a set, then there’s a sheaf

$$\text{Fun} : U \mapsto \text{Functions}(U \to S)$$

The restriction maps are just restriction of functions. Given a cover $U = \bigcup U_i$ and functions $f_i : U_i \to S$ which agree on the overlaps, there’s a unique function $f : U \to S$ defined by asserting that it’s equal to $f_i$ on $U_i$.

Let $X$ be a topological space and $\mathcal{F}$ be a sheaf. The following is a tautological restatement of what it means to be a subsheaf, which may nonetheless be enlightening. Let $P$ be a property of sections, i.e., a true-false statement about each $s \in \mathcal{F}(U)$, which is inherited by restrictions — $P(s) \implies P(s|_V)$ — and can be checked locally — if $U = \bigcup U_i$ and $P(s|_{U_i})$ for all $i$, then $P(s)$. Then $\mathcal{F}_P(U) := \{s \in \mathcal{F}(U) \mid P(s)\}$ is a subsheaf.

From the above description, it’s clear that the following are subsheaves of the sheaf of functions (when $X$ and $S$ have appropriate structures such that they make sense): continuous functions; $C^k$ functions; analytic functions; algebraic functions; locally constant functions; locally integrable functions.

By contrast, the following are presheaves of functions which needn’t be sheaves (because they’re defined by a property that can’t be checked locally): integrable functions, bounded functions, $L^p$ functions, constant functions.

Example. More generally, if $\phi : T \to X$ is a map from any set $T$ to $X$, there’s a sheaf given by the local sections of this: $\text{Sect} : U \mapsto \{f : U \to T \mid \phi \circ f = \text{id}_U\}$.

As with functions, there are subsheaves given by continuous, smooth, flat, etc. sections, when $T$ has appropriate structures for this to make sense.
A basic fact about functions is that they are determined by their values at each point. For an arbitrary presheaf, sections \( s \in \mathcal{F}(U) \) don’t have “values” at points, but recall that for a point \( p \), the stalk \( \mathcal{F}_p \) is defined by

\[
\mathcal{F}_p = \text{colim}_{V \ni p} \mathcal{F}(V) = \bigcup_{p \in V} \mathcal{F}(V)/\text{restriction}
\]

Thus for \( p \in U \), a section \( s \in \mathcal{F}(U) \) has germs \( s_p \in \mathcal{F}_p \) given by the image of \( s \) under the natural map \( \mathcal{F}(U) \hookrightarrow \mathcal{F}_p \).

**Exercise.** Describe the stalk at zero of various sorts of sheaves of functions on \( \mathbb{R} \).

Sheaves are “determined locally”; one sense in which that is true is given in the following exercise:

**Exercise.** Let \( \mathcal{F} \) be a sheaf, say of abelian groups. Given \( s \in \mathcal{F}(U) \), show that \( s = 0 \) if and only if \( s_p = 0 \) for all \( p \in U \). Show that \( \mathcal{F} = 0 \) if and only if \( \mathcal{F}_p = 0 \) for all \( p \). Show that a map \( \mathcal{F} \to \mathcal{G} \) of sheaves is an isomorphism if and only if the induced maps on stalks are isomorphisms. Show the same for monomorphisms and epimorphisms of sheaves.

Note: epimorphisms in the category of sheaves are not the same as epimorphisms in the category of presheaves. In particular, they needn’t induce epimorphisms on global sections.

One can also use the stalks to produce sheaves from presheaves.

**Theorem 1.** (Sheafification) Let \( \mathcal{F}^{\text{pre}} \) be a presheaf on the space \( X \). Then there’s a sheaf \( \mathcal{F} \) and a morphism \( \mathcal{F}^{\text{pre}} \to \mathcal{F} \) which is universal among morphisms from \( \mathcal{F}^{\text{pre}} \) to sheaves.

In particular, the morphism \( \mathcal{F}^{\text{pre}} \to \mathcal{F} \) induces isomorphisms on stalks.

**Proof.** We’ll just describe how to make \( \mathcal{F} \). Begin by making \( \coprod_p \mathcal{F}^{\text{pre}}_p \). This has a map to \( X \) given by taking \( \mathcal{F}^{\text{pre}}_p \to p \). A section of this map is an assignment, to each point \( p \), of an element of the stalk \( \mathcal{F}^{\text{pre}}_p \). The sheaf \( \mathcal{F} \) will be the subsheaf of the sheaf of all sections of this map consisting of those sections which locally come from \( \mathcal{F}^{\text{pre}}_p \):

\[
\mathcal{F}(U) = \{ p \mapsto s_p \in \mathcal{F}_p \mid \text{there is a cover } U = \bigcup U_i \& s_i \in \mathcal{F}(U_i) \text{ such that } p \in U_i \implies s_i|_p = s_p \}
\]

(It’s possible to say this in a funny way by putting a topology on \( \coprod_p \mathcal{F}^{\text{pre}}_p \) so that the above are the continuous sections of \( \coprod_p \mathcal{F}^{\text{pre}}_p \to X \).

The fancy way of phrasing the above theorem is “the forgetful functor from the category of sheaves in the category of presheaves admits a left adjoint”. In particular, the forgetful functor to presheaves it itself a right adjoint; hence in particular, preserves colimits and epimorphisms.

It’s obvious that presheaves of abelian groups form an abelian category; once we know the same for sheaves, it follows from this right adjointness an exact sequence of sheaves is a left exact sequence of presheaves. Exactness of a sequence of presheaves is just the same as exactness of the sequences of sections; so:
**Corollary 2.** For an open set $U$, the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ from sheaves of abelian groups to abelian groups is left exact.

**Exercise.** Show that this functor need not be exact.

Recall that a base of a topological space $X$ is a collection of open sets $U_\alpha$ such that any other open set can be written as the union of these sets. A local base at a point $x \in X$ is a collection of open sets $V_\alpha \ni x$ so that for any open set $U \ni x$, there is some $V_\alpha \subset U$. It is useful to note that collection of open sets is a base of the topology if and only if it contains a local base at every point.

Let $\beta$ be such a collection; it determines a subcategory $\beta(X)$ of $\text{Opens}(X)$. A $\beta$-presheaf is a contravariant functor from $\beta(X)^{\text{op}}$. Note there’s a forgetful functor from presheaves to $\beta$-presheaves.

**Exercise.** Since $\beta$ contains a local base around every point, observe that the formula for stalks makes sense when restricted to $\beta$ and gives the same result. Observe that the formula for the sheafification makes sense for a $\beta$-presheaf. Show that this gives a left adjoint to the forgetful functor from sheaves to $\beta$-presheaves. In particular, the following are the same: beginning with a presheaf and sheafifying, and beginning with a presheaf, forgetting to a $\beta$-presheaf, and then “$\beta$-sheafifying”.

We will very frequently use the result of the above exercise in order to specify sheaves, by describing their restriction to a particularly convenient choice of base of the topology.

## 2. Čech cohomology

In this section, we discuss only sheaves of abelian groups. For an exact sequence of sheaves,

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

we saw already that taking sections over an open set $U$ gives an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

but the last map needn’t be surjective. Sheaf cohomology allows us to continue the above to a long exact sequence.

Today we’ll discuss the Čech cohomology groups; perhaps later we’ll say something about how these fit into the general “derived functor” and “derived category” formalisms. The point is that we’d like to get to making computations as soon as possible, and these will in any case be done with the Čech theory.

From any presheaf $\mathcal{F}$ and any cover $U = \bigsqcup U_i$, by using the restriction maps and choosing appropriate signs, one can form a cochain complex

$$\bigoplus_{i} \mathcal{F}(U_i) \to \bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \to \bigoplus_{i,j,k} \mathcal{F}(U_i \cap U_j \cap U_k) \to \cdots$$
The fact that it is a complex follows from the fact that the $\mathcal{F}$ is a functor; more precisely, from the fact that the restrictions $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j) \to \mathcal{F}(U_i \cap U_j \cap U_k)$ and $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_k) \to \mathcal{F}(U_i \cap U_j \cap U_k)$ agree. If $\mathcal{F}$ is a sheaf, then in addition the restriction map $\mathcal{F}(U) \to \bigoplus_i \mathcal{F}(U_i)$ induces an isomorphism between $\mathcal{F}(U)$ and the zeroth cohomology of this complex.

Thus one defines, for a sheaf $\mathcal{F}$ and a cover $\mathfrak{U}: U = \bigcup U_i$, the Cech cohomology for this cover $H^*_\text{Cech}(\mathfrak{U}; \mathcal{F})$ to be the cohomology of this complex. We will also write $R\Gamma_{\text{Cech}}(\mathfrak{U}; \mathcal{F})$ for this complex itself, considered up to quasi-isomorphism.

Given a refinement of the cover to $\mathfrak{V}: U = \bigcup V_i$, there is a natural map induced by restrictions $R\Gamma_{\text{Cech}}(\mathfrak{U}; \mathcal{F}) \to R\Gamma_{\text{Cech}}(\mathfrak{V}; \mathcal{F})$. The Cech cohomology $R\Gamma_{\text{Cech}}(U, \mathcal{F})$ is by definition the colimit over such covers.

Exercise. Show that, in the limit, the map associating sheaves to their Cech complexes is exact. (Hint: we’re taking a colimit over increasingly small open sets; it’s similar to the reason that associating sheaves to their stalks is exact.) Conclude that taking Cech cohomology gives a long exact sequence, etc.

3. Stratifications

From the definitions, sheaves involve data of all open sets, and computing their Cech cohomology involves a limit over all covers. However, in the setting of interest to us, all the relevant information and manipulations can be performed with a finite amount of data. We saw how to do this for certain sheaves on $\mathbb{R}$ in the previous section; we now begin to formulate what to do in the general case.

Definition 3. A stratification of a space $X$ is a decomposition into a collection of subsets $X = \bigsqcup X_i$ called “strata”, such that each $X_i$ is open in its closure, and $\partial X_i = \overline{X_i} \setminus X_i$ is again a union of strata.

Usually we will be interested in so-called Whitney stratifications, i.e. they must additionally satisfy various other conditions, to be introduced later. In particular, we will generally be interested in the case when all the strata are smooth manifolds.

Exercise. Make sure the following discussion holds without imposing such conditions.

There is a category associated to a stratification, with objects $X_i$ and morphisms $X_i \to X_j$ when $X_i \subset \partial X_j$. The star of a stratum is the union of strata in whose boundary it is, i.e. the union of strata to which it maps in the above category. In these terms, the category can also be described by saying $X_i \to X_j$ when $\text{Star}(X_i) \supset \text{Star}(X_j)$. By definition, this category is a poset, i.e. there is at most one morphism between any two objects.

Exercise. Let $\mathcal{G}: X = \bigsqcup X_i$ be a stratification; we write also $\mathcal{G}$ for the category associated to the stratification. Show that a sheaf $\mathcal{F}$ on $X$ induces a (covariant) functor from $\mathcal{G}$. That is, there’s a functor:
|Ś| : Sheaves with C coefficients ↦ Functors(Ś → C)

(Note: we will generally use stratifications in which the stars are open sets, but even if this is not true, one can define the above functor by sending $X_i \mapsto \colim_{U \supset \text{Star}(X_i)} \mathcal{F}(U).$

**Definition 4.** A sheaf $\mathcal{F}$ is **constructible** with respect to the stratification $X = \bigsqcup X_i$ if the restrictions $\mathcal{F}|_{X_i}$ are constant sheaves.

**Remark.** Often the word “constructible” is used to mean that, in addition to the above, a finiteness condition on stalks is imposed. We do not impose this condition in the current discussion, but may later. More seriously, often the word “constant” in the above is replaced by “locally constant”. We really mean constant here, especially in the following statements; note however that if all the strata are contractible (as can always be ensured by refinement for real manifolds), then there is no difference between these notions.

Consider a stratification $\mathcal{S} : X = \bigsqcup X_i$. For a point $p \in X$, write $X(p)$ for the stratum containing $p$.

Assume there exists, for each point $p$, a local base of neighborhoods $U_i(p)$, **without repetition**, i.e., no $U_i(p)$ agrees with any $U_i(q)$ for $p \neq q$. (The existence of such will be guaranteed by the Whitney conditions, but can also be seen easily in any example one draws.) Let $\beta$ be the base of the topology of $X$ given by the union of these local bases.

Observe that if ever $U_i(p) \supset U_j(q)$, then it must be the case that $X(p) \supset X(q)$, hence in the category $\mathcal{S}$, $X(p) \mapsto X(q)$.

Given a functor $\phi : \mathcal{S} \rightarrow C$, show there’s a $\beta$-presheaf defined on open sets $\mathcal{F}(U_i(p)) = \phi(X(p))$, and with restriction maps determined by this functor via the observation above. Sheafification determines a functor

$$\text{Functors}(\mathcal{S} \rightarrow C) \rightarrow \mathcal{S} – \text{Constructible sheaves with C coefficients}$$

**Exercise.** This functor is (left or right?) adjoint to the functor $|\mathcal{S}$ described above.

We will see later that for sufficiently good stratifications (all strata are contractible, and the stars of all strata are contractible), that the above determines an equivalence of categories between constructible sheaves and functors from stratifications. Relatedly, we will also see that in this situation, Čech cohomology of constructible sheaves can already be computed from the open cover given by stars of strata.