LECTURE 1: SOME GENERALITIES; 1 DIMENSIONAL EXAMPLES

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Historically, sheaves come from topology and analysis; subsequently they have played a fundamental role in algebraic geometry and certain branches of representation theory. Our motivations here – and consequently the sorts of examples we will consider – are rather different: we are ultimately interested in applications to symplectic geometry. Today we dive right in, and discuss constructible sheaves on $\mathbb{R}$.

1. SOME SHEAF GENERALITIES

Let $X$ be a topological space. We write $\text{Opens}(X)$ for the category whose objects are open sets of $X$, and in which a morphism $U \to V$ is just an inclusion $U \subset V$. We write $\text{Opens}(X)^{\text{op}}$ for the opposite category, so we do not have to use the word “contravariant functor”.

Suppose $C$ is some category, and we have a functor $\mathcal{F} : \text{Opens}(X)^{\text{op}} \to C$.

Spelling it out very explicitly: for each open set $U \subset X$, there should be an element $\mathcal{F}(U) \in C$. For each inclusion $U \subset V$, there should be a “restriction” morphism $\mathcal{F}(V) \to \mathcal{F}(U)$. Given a chain of inclusions $U \subset V \subset W$, the composition $\mathcal{F}(W) \to \mathcal{F}(V) \to \mathcal{F}(U)$ should be equal to the given morphism $\mathcal{F}(W) \to \mathcal{F}(U)$.

We call such a functor a presheaf $\mathcal{F}$ on $X$ with coefficients in $C$. Morphisms of presheaves are (by definition) morphisms of these functors. Often, one calls $\mathcal{F}(U)$ the “sections over $U$”, and calls $\mathcal{F}(X)$ the “global sections”. For $V \subset U$ and $s \in \mathcal{F}(U)$, the image of the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is often indicated as $s|_V$ or $s_V$.

Example. For some $c \in C$, the constant presheaf $\mathcal{L}_X^{\text{pre}}$ is the functor taking $U \mapsto c$, and in which all restriction morphisms are the identity $1_c$. One can view the elements of $\mathcal{L}_X^{\text{pre}}(U)$ as constant functions $U \to c$, and the restriction maps as restriction of said functions.

Example. One could also take

$$U \mapsto \begin{cases} 0 & U = X \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

again with all morphisms either given by the identity when possible and the zero morphism otherwise. Note this presheaf differs from $\mathbb{Z}_X$ only in what it assigns to the largest open set, i.e. the space itself. This example is meant to illustrate that what a presheaf assigns to larger open sets is not determined by what it assigns to smaller ones.
Example. Recall that for any category has the “Yoneda embedding” into its category of set-valued contravariant functors via $x \mapsto \text{Hom}(\cdot, x)$. In particular, $\text{Opens}(X)$ embeds into the category of set-valued presheaves on $X$; the open set $U$ going to the functor defined by

$$h_U(V) = \text{Hom}(V, U) = \begin{cases} \text{a one element set} & V \subset U \\ \emptyset & \text{otherwise} \end{cases}$$

Consider an open set $U \subset X$, and a covering $U = \bigcup U_i$. In the category of open sets, there are compatible morphisms $U_i \cap U_j \to U_i \to U$. Correspondingly there are compatible morphisms $F(U) \to F(U_i) \to F(U_i \cap U_j)$. We say $F$ is a sheaf if the corresponding induced morphism

$$F(U) \to \lim \left(F(U_i) \Rightarrow F(U_i \cap U_j)\right)$$

is an isomorphism.

Example. When the $U_i$ are disjoint, one is asking that $F(U) \to \prod F(U_i)$ is an isomorphism. In particular, if $X$ contains any two disjoint open sets $U, V$, and the diagonal map $c \to c \times c$ is not an isomorphism (generally it is not), then the constant presheaf $\mathcal{C}^{pre}_X$ is not a sheaf.

The categories $C$ of interest to us (sets, abelian groups, $R$-modules) contain all (small) limits; one says that such $C$ are complete. We will also tend to work with continuously concrete categories; “concrete” meaning it carries a faithful “forgetful” functor to the category of sets, and “continuous” meaning that this functor preserves limits. (When, as is often the case, the forgetful functor is right adjoint to a free construction, it is automatically continuous due to being a right adjoint.) In these cases, recall that the limit above is described explicitly as tuples

$$\lim \left(F(U_i) \Rightarrow F(U_i \cap U_j)\right) = \left\{ (s_i \in F(U_i)) \mid s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \right\}$$

Thus the condition that $F$ is a sheaf can be written as:

Given a cover of an open $U$ by a collection of opens $U_i$
And any tuple $(s_i \in F(U_i))$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$
There exists a unique $s \in F(U)$ with $s|_{U_i} = s_i$.

Example. The constant presheaf $\mathcal{C}^{pre}_X$ is not a sheaf. Indeed, consider a disconnected open set $U = U_1 \coprod U_2$. By definition $\mathcal{C}^{pre}_X(U) = \mathcal{C}^{pre}_X(U_1) = \mathcal{C}^{pre}_X(U_2)$. On the other hand, for a sheaf $F$ we would have $F(U) = F(U_1) \times F(U_2)$.

Exercise. Define the constant sheaf $\mathcal{C}_X$ by taking $\mathcal{C}_X(U)$ to be the locally constant functions $U \to c$, with the restriction being restriction of functions.

We will see more examples next class when we take up the generalities in earnest.

Remark. Note that in the category of open sets, the colimit of any diagram is just the union of open sets appearing. The limit of a diagram exists and is equal to intersection of open sets appearing, if
and only if this is an open set. So the condition that \( \mathcal{F} \) is a sheaf is the assertion that the colimit \( U = \text{colim} (U_i \longleftarrow U_i \cap U_j) \) is carried to the limit \( \mathcal{F}(U) \cong \lim (F(U_i) \rightarrow F(U_i \cap U_j)) \).

It is however certainly \textbf{not true} that sheaves carry all colimits of open sets to limits. Indeed, there would be very few such things. Consider e.g. two open sets \( U, V \) which overlap. Then we have both \( U \cup V = \text{colim} \{U, V\} \) and \( U \cup V = \text{colim} (U \leftarrow U \cap V \rightarrow V) \). Asking a functor to carry these both to limits means asking that \( \mathcal{F}(U) \times \mathcal{F}(V) = \mathcal{F}(U) \times_{\mathcal{F}(U \cap V)} \mathcal{F}(V) \), i.e., that every section of \( U \) and every section of \( V \) have the same restriction in \( U \cap V \). Evidently this is not true, e.g. for a constant sheaf.

Suppose given a continuous map of topological spaces, \( f : X \rightarrow Y \). Sheaves can be pushed forward and pulled back according to the following formulas.

If \( \mathcal{F} \) is a presheaf on \( X \), then one has a presheaf \( f_* \mathcal{F} \) on \( Y \) defined as follows:

\[
(f_* \mathcal{F})(U) := \mathcal{F}(f^{-1}(U))
\]

This makes sense because \( f \) is continuous, so \( f^{-1}(U) \) is an open set.

\textit{Exercise.} If \( \mathcal{F} \) is a sheaf, then so is \( f_* \mathcal{F} \).

Pulling sheaves back is harder. For \( \mathcal{G} \) a sheaf on \( Y \), one might try to write \( (f^{-1} \mathcal{G})(U) = \mathcal{G}(f(U)) \), but \( f(U) \) needn’t be an open set. Instead one writes:

\[
(f^{-1} \mathcal{G})(U) = \text{colim}_{V \supset f(U)} \mathcal{G}(V)
\]

For this to always make sense, our coefficient category must have filtered colimits. (This is true for the category of sets, which has all colimits.)

\textit{Example.} Even if \( \mathcal{G} \) is a sheaf, it needn’t be the case that \( f^{-1} \mathcal{G} \) is a sheaf. Indeed, consider a map \( X \rightarrow \text{point} \). Let us consider the constant sheaf on the point (there are no other kinds anyway). Its pullback according to the above formula is the constant presheaf on \( X \), which we know is not a sheaf.

We will see next time how to fix this problem and define a pullback for sheaves. There is however one obvious case when the pullback is already a sheaf: namely when \( i : U \rightarrow Y \) is the inclusion of an open subset. In this case we write \( i^{-1} \mathcal{G} \) as \( \mathcal{G}|_U \).

\textit{Exercise.} More generally, is it true that when \( f : X \rightarrow Y \) is an injection, and \( \mathcal{G} \) is a sheaf on \( Y \), then \( f^{-1} \mathcal{G} \) is already a sheaf on \( X \)?

One particularly important case of the pullback is the inclusion of a point, \( i : y \rightarrow Y \). In this case, for a sheaf \( \mathcal{G} \) on \( Y \), the (sections over \( y \) of) the restriction \( i^* \mathcal{G} \) is generally denoted by \( \mathcal{G}_y \), and called the \textit{stalk} at \( y \).
2. CONSTRUCTIBLE SHEAVES ON $\mathbb{R}$

Consider a sheaf on the real line $\mathbb{R}$ with coefficients in some category $C$. The restriction morphisms of this sheaf give rise to a diagram

$$\mathcal{F}((-\infty, 0)) \leftarrow \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}((0, \infty))$$

Conversely, from a diagram $a \leftarrow b \rightarrow c$, one can define a sheaf $\mathcal{F}_{a \leftarrow b \rightarrow c}$ as follows.

- For an open interval $L$ to the left of zero, we set $\mathcal{F}_{a \leftarrow b \rightarrow c}(L) = a$.
- For an open interval $M$ including zero we set $\mathcal{F}_{a \leftarrow b \rightarrow c}(M) = b$.
- For an open interval $R$ to the right of zero we set $\mathcal{F}_{a \leftarrow b \rightarrow c}(R) = c$.
- The restriction maps between open intervals are either the identity or the map in the diagram.
- Any open set $U \subset \mathbb{R}$ is a disjoint union of open intervals; the value of a sheaf on such a union is just the product of its values, and likewise for the restriction maps.

Note there are isomorphisms $\mathcal{F}_{a \leftarrow b \rightarrow c}|_{(-\infty, 0)} \cong a|_{(-\infty, 0)}$ and $\mathcal{F}_{a \leftarrow b \rightarrow c}|_{(0, \infty)} \cong c|_{(0, \infty)}$. More generally, we say that a sheaf $\mathcal{G}$ on $\mathbb{R}$ whose restrictions $\mathcal{G}|_{(-\infty, 0)}$ and $\mathcal{G}|_{(0, \infty)}$ are constant sheaves is constructible with respect to the stratification $\mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, \infty)$.

**Exercise.** Show that the map $a \leftarrow b \rightarrow c \mapsto \mathcal{F}_{a \leftarrow b \rightarrow c}$ is an equivalence of categories between diagrams of the shape $\bullet \leftarrow \bullet \rightarrow \bullet$ (i.e., representations of an $A_3$ quiver) and the category of sheaves constructible with respect to the above stratification.

Hint: the main point is to show essential surjectivity of the above morphism, and the main point here is to establish that for any sheaf constructible with respect to the above stratification, the restriction map between two intervals containing 0 is always an isomorphism. If $J \subset I$ are two such intervals, consider a cover of $I$ by $J$ and two end pieces $L, R$, and then use the fact that $\mathcal{F}(L) \rightarrow \mathcal{F}(L \cap J)$ and $\mathcal{F}(R) \rightarrow \mathcal{F}(R \cap J)$ must be isomorphisms.

More generally, suppose given a decomposition of $\mathbb{R}$ into a disjoint union of (finitely many) open intervals and points. We say that a sheaf is constructible with respect to this stratification if its restriction to each open interval is a constant sheaf. By the same argument and construction as for the above exercise, one can show:

**Proposition 1.** Consider the stratification $\mathbb{R} = (-\infty, t_1) \cup \{t_1\} \cup (t_1, t_2) \cup \{t_2\} \cup \cdots (t_n, \infty)$. There is a functor from sheaves on $\mathbb{R}$ to diagrams given by taking a sheaf $\mathcal{F}$ to

$$\mathcal{F}(-\infty, t_1) \leftarrow \mathcal{F}(-\infty, t_2) \rightarrow \mathcal{F}(t_1, t_2) \leftarrow \mathcal{F}(t_1, t_3) \rightarrow \mathcal{F}(t_2, t_3) \leftarrow \cdots \leftarrow \mathcal{F}(t_{n-1}, \infty) \rightarrow \mathcal{F}(t_n, \infty)$$

This functor restricts to an equivalence of categories between sheaves constructible with respect to this stratification and diagrams of this form.
3. A TASTE OF THE MICROSUPPORT

In general, when $\mathcal{F}$ is a sheaf of abelian groups on $X$, then we say the support of $\mathcal{F}$ is the closure of the locus of points at which the stalk $\mathcal{F}_x$ is nonzero.

When $X$ is a manifold, one can ask a more sophisticated question: in which directions is the stalk of $\mathcal{F}$ changing? The answer is given by the microsupport, which is in general a conical locus in the cotangent bundle.

We will give precise definitions later, but already we can draw some pictures. Consider a point $(x, \xi)$ in $T^*\mathbb{R}$; here $x$ is the position and $\xi$ is the cotangent vector. Suppose also given a sheaf $\mathcal{F}$ on $\mathbb{R}$.

(Pseudo-definition)

For $\xi > 0$, the point $(x, \xi)$ is in the microsupport if, for small $\epsilon$, the restriction morphism $\mathcal{F}(x - \epsilon, x + \epsilon) \to \mathcal{F}(x - \epsilon, x)$ is an isomorphism.

Likewise for $\xi < 0$, the point $(x, \xi)$ is in the microsupport if $\mathcal{F}(x - \epsilon, x + \epsilon) \to \mathcal{F}(x, x + \epsilon)$ is an isomorphism.

A point $(x, 0)$ is in the microsupport if it’s in the support.

This definition doesn’t make sense in higher dimensions, and we will ultimately want use derived sections rather than sections to define the microsupport. But, in the case at hand, this agrees with the correct definition.

Exercise. (I did this in class) Draw the microsupports of the sheaves $\mathcal{F}_{\alpha \leftarrow \beta \rightarrow \gamma}$. Of especial interest are the following six cases:

\[
\begin{align*}
\mathbb{Z} & \leftarrow \mathbb{Z} \rightarrow \mathbb{Z} & \mathbb{Z} & \leftarrow \mathbb{Z} \rightarrow 0 & \mathbb{Z} & \leftarrow 0 \rightarrow 0 \\
0 & \leftarrow \mathbb{Z} \rightarrow \mathbb{Z} & 0 & \leftarrow 0 \rightarrow \mathbb{Z} & 0 & \leftarrow \mathbb{Z} \rightarrow 0
\end{align*}
\]