Invertible modules over $R$ are those with an inverse with respect to the tensor product.

**Lemma 1.** Say $M \otimes_R N \cong R$. Under this isomorphism, we have an expression of the form $1 = \sum m_i \otimes n_i$. In this case, the $m_i$ generate $M$ and the $n_i$ generate $N$. In particular, an invertible module is finitely generated.

**Proof.** Let $M'$ be the submodule generated by the $m_i$. Consider the composite $M' \otimes_R N \to M \otimes_R N \cong R$. It’s surjective, since 1 is in its image. Tensoring further with $M$ we find $M' = M' \otimes_R N \otimes_R M \to M \otimes_R N \otimes_R M \cong R \otimes_R M \cong M$ is surjective. \qed

We recall that $M$ is ‘projective’ means that for any surjection $N \twoheadrightarrow M$, every map to $M$ lifts to a map to $N$, i.e. $\text{Hom}(\cdot, N) \to \text{Hom}(\cdot, M)$ is surjective; that projectivity is equivalent to being a summand of a free module (consider any surjection from a free module; it has a section, hence splits), and that hence projectivity implies flatness.

**Lemma 2.** An invertible module is projective.

**Proof.** Consider as above $M \otimes_R N \cong R$, with corresponding $1 = \sum_{i=1}^k m_i \otimes n_i$. Consider the map $R^k \otimes N = N^k \to M \otimes_R N$ taking $(z_1, \ldots, z_k) \mapsto \sum m_i \otimes z_i$; compose it with the above isomorphism to get a surjective map $R^k \otimes N \to R$. As $R$ is free, this map splits; $R^k \otimes N \cong R \oplus Z$ for some module $Z$. Now tensoring with $M$, we find

$$R^k = R^k \otimes N \otimes M = (R \oplus Z) \otimes M = M \oplus (M \otimes Z)$$

Thus $M$ is a summand of a free module, hence projective. \qed

A finitely generated projective module is automatically finitely presented: say $M$ is finitely generated projective; then one has $R^n \to M$ which splits hence $R^n = M \oplus K$. Now $K$ is itself finitely generate projective; so there’s some $R^m \to K$, and we have an exact sequence $R^m \to R^n \to M \to 0$. (Conversely, one can show that a finitely presented flat module is projective.) In particular, invertible modules are finitely presented.

**Lemma 3.** A module over a local ring is invertible if and only if it is free.

**Proof.** We have already seen (in lecture 3) that for finitely generated modules over a local ring, flat implies free (and hence for such modules, projective, flat, and free are all equivalent). \qed
Lemma 4. The following are equivalent:

1. The natural map \( \text{Hom}(M, R) \otimes M \rightarrow R \) is an isomorphism.
2. There’s some \( R \)-module \( N \) with \( N \otimes M \cong R \).
3. \( M \) is finitely presented and \( M_p \cong R_p \) for all primes \( p \).
4. \( M \) is finitely presented and \( M_p \cong R_p \) for all maximal primes \( p \).

Proof. Obviously (1) implies (2); we’ve seen above (2) implies (3), which evidently implies (4).
Now (4) implies (1) because we can check that \( \text{Hom}(M, R) \otimes_R M \rightarrow R \) is an isomorphism after localizing at all primes, and by finite presentation \( \text{Hom}(M, R)_p = \text{Hom}(M_p, R_p) \).

Remark. It’s possible to show that also equivalent is the statement: \( M \) is finitely presented, and \( M_{a_i} = R_{a_i} \) for some collection of elements \( a_i \) generating the unit ideal.

A fractional ideal of \( R \) is a an \( R \)-submodule of the total ring of fractions, \( K \). We’ll restrict attention here to the case when \( R \) is a domain; see Eisenbud’s book for the general case.

For \( I \subset K \) a fractional ideal, we write \( I^{-1} = \{ x \in K \mid xI \subset R \} \). Note \( I^{-1}I \subset R \).

Lemma 5. Every invertible module is isomorphic to a fractional ideal.

Proof. Let \( M \) be invertible. We have \( M_p \cong R_p \) for all primes \( p \), in particular, for the prime 0. Thus \( M \otimes K \cong K \). Thus it suffices to show that \( M \rightarrow M \otimes K \) is an injection. We can check this after localizing at prime ideals, but at prime ideals \( M_p \cong R_p \).

Lemma 6. For invertible fractional ideals \( I, J \), one has \( I \otimes J \cong IJ \) and \( I^{-1}J \cong \text{Hom}(I, J) \).

Proof. There are natural maps \( I \otimes J \rightarrow IJ \) and \( I^{-1}J \rightarrow \text{Hom}(I, J) \). One can see these are isomorphisms by localizing at all primes.

Lemma 7. A finitely presented fractional ideal is invertible iff \( I^{-1}I = R \).

Proof. In case \( I \) is invertible, we have \( I^{-1} \cong \text{Hom}(I, R) \) and hence \( I^{-1}I \cong \text{Hom}(I, R) \otimes I \). Conversely, in case \( I^{-1}I = R \), we want to see that \( I_p \cong R_p \) at every prime \( p \). Since \( I^{-1}I = R \), there must be some \( x \in I^{-1} \) such that \( xI \not\subset p \); it follows \( xI_p = R_p \). This is the desired isomorphism.

Corollary 8. A finitely presented fractional ideal is invertible if and only if it is locally principal.

Proof. An ideal is isomorphic to the domain if and only if it is principal.

We will see that fractional ideals have good properties in Noetherian normal domains. First we need to develop some further properties of DVRs and normality. (We return to following Matsumura, chapter 11.) Recall we had already shown:

Lemma 9. Assume \( R \) isn’t Artin. The following are equivalent

1. \( R \) is a DVR
2. \( R \) is a local PID
(3) $R$ is a Noetherian local ring and the maximal ideal is principal.

Note we can reformulate (3) above as saying that the maximal ideal is invertible. Indeed, in a local ring invertible means isomorphic to the ring, and an ideal is isomorphic to the ring only if it’s principal: an isomorphism $R \cong I$ carries 1 to the generator. (In a domain, the converse also holds, since the surjective map $R \to I$ given by multiplication by the generator has no kernel.) A corollary:

**Corollary 10.** If $R$ is a Noetherian domain, and $\mathfrak{p}$ is invertible, then $R_{\mathfrak{p}}$ is a DVR. In particular, there’s no primes between $\mathfrak{p}$ and 0.

**Proof.** Thus an invertible ideal must be locally principal; in particular, in the above situation, $\mathfrak{p}_{\mathfrak{p}}$ is principal in $R_{\mathfrak{p}}$.

The condition that there’s no prime between $\mathfrak{p}$ and 0 has a name: we say such a $\mathfrak{p}$ has *height 1*. More generally, the height of a prime is the length of the longest chain of primes from it to zero (so the Krull dimension of a ring is the supremum of the heights of the primes).

Now we add one more equivalent condition to those in lemma 9:

**Lemma 11.** The above are equivalent to:

- $R$ is a normal Noetherian local ring, and the maximal ideal is associated to a principal ideal.
- $R$ is a 1 dimensional normal Noetherian local ring.

**Proof.** We’ve shown before that a DVR is a 1d normal noetherian local ring, and it’s obvious that if the maximal ideal is principal, then it’s associated to a principal ideal.

We’ll show the second statement above implies the first, which in turn implies the the maximal ideal is in fact principal. Indeed, consider an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$; there must be such an element since otherwise we could conclude $\mathfrak{m} = 0$ by Nakayama (and hence that $R$ is a field). Now $R/x$ has some associated prime; as the only primes are 0 and $\mathfrak{m}$, it can only be $\mathfrak{m}$, since 0 doesn’t contain the annihilator. Thus we see $\mathfrak{m}$ is associated to a principal ideal. We now proceed with only this hypothesis (forgetting that $x$ comes from $\mathfrak{m} \setminus \mathfrak{m}^2$.)

That $\mathfrak{m}$ is associated to $R/x$ means that $\mathfrak{m}$ is equal to the annihilator of some $\overline{y} \in R/x$, or in other words, $\mathfrak{m} = \{ z \in R \mid zy \in xR \}$.

Said differently, $\mathfrak{m} = \{ z \in R \mid z(y/x) \in R \}$, i.e. it is the inverse of the fractional ideal generated by $y/x$. It follows that $y/x \notin R$ (since otherwise the inverse of the fractional ideal it generated would contain $R$) and since $y \mathfrak{m} \subset xR$, we have $y/x \mathfrak{m} \subset R$, i.e. $y/x \in \mathfrak{m}^{-1}$. Thus we have $\mathfrak{m}^{-1} \supseteq R \supseteq \mathfrak{m}$.

By definition we have $\mathfrak{m}\mathfrak{m}^{-1} \subset R$; since $R \subset \mathfrak{m}^{-1}$ we have in fact $R \subset \mathfrak{m}^{-1}\mathfrak{m} \subset \mathfrak{m}$. As $\mathfrak{m}$ is an ideal, $\mathfrak{m}^{-1}\mathfrak{m}$ must also be an ideal, so either it’s $R$ or $\mathfrak{m}$. 

Now if \( mm^{-1} = m \), then in particular, \( (y/x)m \subset m \). As \( R \) is Noetherian, \( m \) is finitely generated, so we learn by Cayley-Hamilton that \( y/x \) satisfies a monic polynomial. But \( R \) was normal, so we would learn \( y/x \in R \); contradiction. Thus \( mm^{-1} = R \). \( \square \)

We now turn to another characterization of normality in the Noetherian setting. Recall that in general, we saw that a domain was normal if and only if it was the intersection of all valuation rings containing it. In the Noetherian case we will have a simpler criterion, but first we require the following:

**Lemma 12.** Assume \( R \) is a Noetherian domain such that all associated primes of principal ideals have height 1. Then \( R \) is the intersection of its localizations at height 1 primes.

**Proof.** We should show that if \( b/a \in R_p \) for all height 1 primes, then \( b/a \in R \). That is, if \( b \in aR_p \cap R \) for all height one primes, then \( b \in aR \). Take the primary decomposition of \( aR \); it is \( aR = q_1 \cap \cdots \cap q_n \) where \( q_i \) is \( p_i \)-primary. By hypothesis the \( p_i \) all have height one.

We have \( b \in aR_p \cap R = (q_1 \cap \cdots \cap q_n)R_p \cap R \). Check: because \( p \) is height one \((q_1 \cap \cdots \cap q_n)R_p = q_1R_p \). \( \square \)

Finally we arrive at:

**Theorem 13.** A Noetherian domain is normal if and only if the following conditions hold: (1) the localization at height one primes are DVRs and (2) associated primes of principal ideals have height 1.

**Proof.** Assume \( R \) is a Noetherian normal domain. Then the localization at a height one prime is a noetherian local normal domain of dimension 1, hence a DVR. Moreover, the localization at an associated prime to a principal ideal gives a noetherian local normal domain in which the maximal ideal is associated to a principal ideal; hence a DVR.

Conversely, suppose (2) holds above. Then by the lemma, the ring is the intersection of its localization at height one primes. if also (1) holds, these are all DVR, hence normal, hence their intersection is normal. \( \square \)

Compare this with the statement we learned last time: a Noetherian domain is has unique factorization if and only if the associated primes of principal ideals are principal.