A ring is a set $R$ with two binary operations, an addition $+$ and a multiplication $\cdot$. Always there should be an identity $0$ for addition, an identity $1$ for multiplication, $(R, +, 0)$ should be a commutative group, and $(R, \cdot, 1)$ should be associative. Multiplication distributes over the addition: $a(b + c) = ab + ac$. In these lectures we restrict attention to rings in which the multiplication is also commutative, unless specifically mentioned otherwise. Maps of rings should respect $\cdot$, $+$, $0$, $1$.

Here are some rings you may recognize:

- The zero ring $\{0\}$
- The integers $\mathbb{Z}$
- Finite fields $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})[x]/(x^2 + x + 1)$
- Other fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$
- Polynomial rings $\mathbb{Z}[x], \mathbb{Q}[x], \mathbb{C}[x], \mathbb{Q}[x, y]
- Quotients of polynomial rings $\mathbb{Q}[x, y]/(xy - 1), \mathbb{Q}[x, y]/(x^2 - y^3), \mathbb{Q}[x, y, z]/(xy - z^2)$
- Power series rings $\mathbb{C}[[t]]$
- Rings of continuous, differentiable, holomorphic functions

An $R$-module is a commutative group $M$ ‘on which $R$ acts’, i.e. the data of a map of (noncommutative) rings $R \to \text{Hom}_\mathbb{Z}(M, M)$. When $R$ is a field, then an $R$-module is exactly the same as a vector space over $R$.

A subset preserved by the $R$-action is called a submodule. A map of $R$-modules is a map of abelian groups respecting the $R$-action. I.e., a map $\phi : M \to N$ such that $r\phi(m) = \phi(rm)$.

We denote the set of such by $\text{Hom}_R(M, N)$. The hom set is naturally an commutative group, and because $R$ is commutative, it carries the structure of an $R$-module.\(^1\)

It’s difficult to say anything clearly without a little bit of category theory. A category $C$ has objects, and morphisms. Each morphism has a source object and a target object. We write $x \xrightarrow{f} y$ for a morphism $f$ with source $x$ and target $y$. Given morphisms $x \xrightarrow{f} y \xrightarrow{g} z$, there’s a morphism $x \xrightarrow{gf} z$. We ask that the morphisms from $x$ to $y$ form a set, which we denote $\text{Hom}_C(x, y)$.

The category set: objects are sets, morphisms are functions. Many other categories are subcategories of this category, like the category ring whose objects are rings and whose morphisms

\(^1\)As we will see later (maybe in the exercises), one need not even write formulas to give the structure of abelian group or $R$-module to the hom sets. The naked category of $R$-modules already knows it.
are ring maps; the category $R \mod$ whose objects are $R$-modules and whose morphisms are $R$-module maps.

Example. Not every category is like this: for any group $G$, there’s a category with one object, whose morphisms are $G$. Let’s call it $BG$.

A functor $C \to D$ is a map of objects and a map of morphisms, compatible with all structures (i.e. it respects the source and target, carries identities to identities, and compositions to compositions). For instance, forgetting that a ring is a ring gives a forgetful functor from the category of rings to the category of sets.

We’ll write $\text{Hom}(C, D)$ for the functors from $C$ to $D$. Functors compose; the identity is a functor; there’s a category whose objects are categories and whose morphisms are functors.

Example. $\text{Hom}_{\text{categories}}(BG, BH) = \text{Hom}_{\text{groups}}(G, H)$.

But $\text{Hom}(C, D)$ can itself be given the structure of a category. Given two functors $f, g : C \to D$, a natural transformation $f \Rightarrow g$ is a collection of morphisms in $D$, one for each object of $C$

$$\eta_c : f(c) \to g(c)$$

such that for any map $c \to c'$, one has $\eta_c \circ f = f \circ \eta_{c'}$. Natural transformations can be composed.

Example. $\text{Hom}_{\text{functors}}(\text{id}_{BG}, \text{id}_{BG}) = Z(G)$, the center of $G$.

Given any category $C$, there’s another one you get by keeping the objects and turning around the arrows. It’s called the opposite category, and written $C^{\text{op}}$. It’s kind of like taking duals, for vector spaces. $\text{Hom}$ is a pairing:

$$\text{Hom} : C^{\text{op}} \times C \to \text{set}$$

$$(x, y) \mapsto \text{Hom}_C(x, y)$$

If one understands the product of categories have objects as pairs of objects and morphisms as pairs of morphisms, then the $\text{Hom}$ above is a functor. Looking one factor at a time:

$$h^x : C \to \text{set}$$

$$y \mapsto \text{Hom}_C(x, y)$$

$$h_y : C^{\text{op}} \to \text{set}$$

$$x \mapsto \text{Hom}_C(x, y)$$

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2The fact that the hom sets in the category of categories are themselves categories means that the category of categories is a “category enriched in categories”, or a “2-category”. We will hopefully never say this again here.
That is, elements of $C$ give functors on $C^{\text{op}}$, and elements of $C^{\text{op}}$ give functors on $C$. Recalling now the functoriality in the other factor, we have functors

$$C^{\text{op}} \rightarrow \text{Functors}(C, \text{set})$$

$$x \mapsto h^x$$

$$C \rightarrow \text{Functors}(C^{\text{op}}, \text{set})$$

$$y \mapsto h_y$$

Note that what’s being asserted is that homs in $C$ or $C^{\text{op}}$ become natural transformations of functors. (You should check this!) That is, that there’s a map $\text{Hom}_C(y, y') \rightarrow \text{Hom}_{\text{Functors}(C^{\text{op}}, \text{set})}(h_y, h_{y'})$.

We can do it again. From a functor $F : C^{\text{op}} \rightarrow \text{set}$, we form

$$h_F : (C^{\text{op}} \rightarrow \text{set})^{\text{op}} \rightarrow \text{set}$$

$$G \mapsto \text{Hom}_{C^{\text{op}} \rightarrow \text{set}}(G, F)$$

We can restrict to the image of the map $C \rightarrow \text{Functors}(C^{\text{op}}, \text{set})$, to get a new functor

$$h_F|_{h_C} : C^{\text{op}} \rightarrow \text{set}$$

This is a universal procedure from turning functors $C^{\text{op}} \rightarrow \text{set}$ into functors $C^{\text{op}} \rightarrow \text{set}$. Actually, this ridiculous construction doesn’t make anything new:

**Yoneda’s lemma.** For a functor $F : C^{\text{op}} \rightarrow \text{set}$, and any $y \in C$, one has $\text{Hom}(h_y, F) = F(y)$.

In particular, $\text{Hom}(h_y, h_{y'}) = h_{y'}(y) = \text{Hom}_C(y, y')$.

Similarly, for any functor $G : C \rightarrow \text{set}$ and any $x \in C^{\text{op}}$, one has $\text{Hom}(h^x, G) = G(x)$. In particular, $\text{Hom}(h^x, h^{x'}) = h_{x'}(x) = \text{Hom}_C(x', x) = \text{Hom}_{C^{\text{op}}}(x, x')$.

**Proof.** Since $F$ is a functor (on $C^{\text{op}}$), a map $x \rightarrow y$ in $C$ determines maps $F(y) \rightarrow F(x)$. That is, there’s a map

$$h_y(x) \times F(y) = \text{Hom}_C(x, y) \times F(y) \rightarrow F(x)$$

One checks that, for any given $\phi \in F(y)$, the resulting map $h_y(x) \rightarrow F(x)$ determines a natural transformation $h_y \rightarrow F$ (do it!).

Going in the other direction, from a natural transformation $h_y \rightarrow F$, one can evaluate at $y$ to get a map $\text{Hom}(y, y) \rightarrow F(y)$. The image of $1_y$ gives an element of $F(y)$.

Yoneda’s lemma is like the assertion that every vector space embeds in its double dual.

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3Here and elsewhere, $= \rightarrow$ means an isomorphism so canonical I won’t write it.
Its most used consequence is the following. Suppose given any functor $F : C^{op} \to \text{set}$ or $G : C \to \text{set}$. One can ask: is $F \simeq h_y$, is $G \simeq h^x$? The point is that if the answer is yes, then it determines the objects $y, x$ uniquely up to unique isomorphism.

Indeed, given $h_y \simeq F \simeq h_y$, we get $h_{y'} \simeq h_y$. But $\text{Hom}(h_{y'}, h_y) = \text{Hom}(y', y)$ hence we get an isomorphism $y' \to y$. (Exercise: why is it an iso?)

That is, Yoneda’s lemma tells us that by naming a functor, we name uniquely an object, if such an object exists. You’ve probably seen this idea before under the name ‘universal property’. The Yoneda lemma is a way of formalizing what precisely it means to be a universal property.

Remark. The idea of first naming the functor, then looking for the object which ‘represents’ it, is the category-theory version of the analysts’ procedure of first looking for a distribution (linear functional on functions), then looking for the function such that integrating against it gives that distribution. Not all distributions are representable by functions; e.g. the delta “function”. But also: sometimes one just works with distributions; so too one often just gives up on the original category and works with functors on it instead.

The Yoneda lemma lets us ask a given category if it has certain kinds of objects, in such a way that if the answer is yes, then the object is unique. Here are common such questions:

- An initial object has a unique map to any other object. I.e., it’s an object $in$ such that $h_{in}(x)$ is the constant functor with value $\{\emptyset\}$.
- A final object has a unique map from any other object. I.e., it’s an object $fin$ such that $h_{fin}(x)$ is the constant functor with value $\{\emptyset\}$.
- For objects $x, y$, a coproduct $x \coprod y$ is an object which has some maps $x \to x \coprod y \leftrightarrow y$ such that given any maps $x \to z$ and $y \to z$, there’s a unique way to factor them through $x \coprod y \to z$. That is, $h^{x \coprod y} = h^x \times h^y$.
- For objects $x, y$, a product $x \times y$ is an object which has some maps $x \leftarrow x \times y \to y$ such that given any maps $z \to x$ and $z \to y$, there’s a unique way to factor them through $z \to x \times y$. That is, $h_{x \times y} = h_x \times h_y$.

Said differently, Yoneda’s lemma lets us import ideas about sets to arbitrary categories. Here’s another set theoretic construction. Given sets $S \to T \leftarrow U$, one can form $S \times_T U$, the subset of $S \times U$ of pairs which have the same image in $T$. This lets us make:

- For objects $x \leftarrow w \to y$, a pushout $x \coprod_w y$ satisfies $h^{x \coprod_w y} = h^x \times_{h_w} h^y$.
- For objects $x \to w \leftarrow y$, a pullback $x \times_w y$ satisfies $h_{x \times_w y} = h_x \times_{h_w} h_y$.

(Exercise: give more down to earth descriptions of the above, similar to products and coproducts.)

After all that, let’s return to $R$-modules. The abelian group with one element can serve as a module for any ring. It has a unique map to, and a unique map from, any other module (in the language of category theory, it is initial and final). We denote it $0$. 
There’s a product of modules. Its underlying set is $M \times N = \{(m, n) | m \in M; n \in N\}$, and you add and act coordinate-wise. Let’s consider the coproduct. Note that if it exists, the maps
\[ (1 \times 0) : M \to M \times 0 \to M \times N \]
\[ (0 \times 1) : N \to 0 \times N \to M \times N \]
guarantee the existence of $M \coprod N \to M \times N$. In fact one can check that this morphism is an isomorphism, i.e. that $M \times N$ is also the coproduct. We’ll denote this common object by $M \oplus N$.

Here’s an amazing fact. Suppose we just knew the above categorical statements, i.e. that the map $M \coprod N \to M \times N$ is an isomorphism. The identity map $M \to M$ gives a map $M \coprod M \to M$; composed with the inverse of the above morphism we get $M \times M \to M$. Exercise: it’s the addition! So the category $R$-modules, just as a category, already knows that the modules are abelian groups. A similar argument tells you that the Hom spaces are also abelian groups.

Even more is true. Exercise: show that in any category, natural transformations from the identity functor to itself act on all morphism sets. Show that for the category of $R$-modules, said natural transformations are just the ring $R$ itself. (For this, it’s important that $R$ is commutative.) Show that the resulting $R$-module structure on the Hom spaces is given by the usual formula. That is, the category of $R$-modules knows the ring $R$.

We turn to quotients and exact sequences. One usually says what an exact sequence is in terms of objects. We can also think about $0 \to A \to B \to C \to 0$ using the diagram
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C
\end{array}
\]

It’s left-exact if this diagram is a pullback; right exact if this diagram is a pushout, and exact if it’s both. This formulation is useful because we’ll learn shortly that certain categorical considerations (existence of adjoints) guarantee that functors preserve pushouts or pullbacks, hence exactness.

The first isomorphism theorem says that whenever $A \subset B$, there’s a pushout-pullback diagram
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B/A
\end{array}
\]

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4The fact that the categorical product agrees with the cartesian product is not an accident. We’ll see next time that this happens because the forgetful functor to sets has an left adjoint (the construction of free modules).

5Here we only used the fact that $R$-mod has an initial and final object $0$, so this statement is true in any such category. However, many categories don’t have such an object, e.g. the category of sets (initial object $\emptyset$ and final object $\{\emptyset\}$) and the category of rings (initial object $\mathbb{Z}$ and final object $0$).