Endpoint fluctuations for directed polymers

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We discuss the work [Bol89], primarily following the presentation of [Com17, Chapter 3].

1 Directed polymer model

Let $\pi = (\pi(t))_{t\in\mathbb{Z}_{\geq 0}}$ be the simple symmetric random walk on $\mathbb{Z}^d$ started at 0:

$$\pi(t) = \sum_{s=1}^{t} X_s, \qquad X_s \overset{\text{i.i.d.}}{\sim} \mathsf{Uniform}(\{\pm e_1, \dots, \pm e_d\})$$

We denote by P_{RW} the law of π. This is a probability measure on the set Π of all possible trajectories for π. By the central limit theorem, if $\pi \sim P_{\text{RW}}$, then $\pi(n)$ converges in the diffusive scaling limit to a vector of i.i.d. Gaussians:

$$\lim_{n\to\infty} P_{\text{RW}}\left(\frac{\pi(n)}{\sqrt{n}} \in A\right) = \text{Prob}\big(\mathsf{N}(0, d^{-1}I) \in A\big), \qquad A \subset \mathbb{R}^d.$$

Let $\omega = \big(\omega(t,x)\big)_{(t,x)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}^d}$ be a family of i.i.d. Rademacher random variables: $\mathbb{P}(\omega(t,x) = 1) = \mathbb{P}(\omega(t,x) = -1) = 1/2$. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. The family ω is called the **environment**. For $\beta \in [0,1]$ and $n \in \mathbb{Z}_{\geq 0}$, we define the **Hamiltonian** (or **energy**) of a trajectory $\pi \in \Pi$ by

$$H_n^\omega(\pi) := \prod_{t=1}^{n} (1 + \beta\omega(t, \pi(t))).$$

The **polymer measure** is the (random) probability measure μ_n^ω on Π with Radon–Nikodym derivative

$$\frac{d\mu_n^\omega}{dP_{\text{RW}}}(\pi) := \frac{H_n^\omega(\pi)}{\langle H_n^\omega \rangle},$$

where the angle brackets denote expectation with respect to P_{RW}:

$$\langle H_n^\omega \rangle := \int_\Pi H_n^\omega(\pi)\, dP_{\text{RW}}(\pi).$$

Example 1.1. If $\beta = 0$ then $H_n^\omega(\pi) = 1$ for any ω, n, π, and therefore $\mu_n^\omega = P_{\text{RW}}$.

It is of great physical importance to understand how and whether the presence of disorder (i.e. $\beta > 0$) affects the geometry and statistics of the simple random walk.

2 Gaussian fluctuations in dimension $d \geq 3$

Theorem 2.1 ([Bol89]). *There exists $\beta_0 > 0$ such that if $\beta \in (0, \beta_0)$ and $d \geq 3$, then*

$$\lim_{n\to\infty} \mu_n^\omega \left(\frac{\pi(n)}{\sqrt{n}} \in A \right) = P\big(\mathsf{N}(0, d^{-1}I) \in A\big) \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

Corollary 2.2 ([IS88; Bol89]). *There exists $\beta_0 > 0$ such that if $\beta \in (0, \beta_0)$ and $d \geq 3$, then*

$$\mathbb{E}_{\mu_n^\omega}\left[|\pi(n)|^2\right] = \frac{1}{\langle H_n^\omega \rangle} \left\langle |\pi(n)|^2 H_n^\omega(\pi) \right\rangle \xrightarrow{n\to\infty} 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

We will deduce Theorem 2.1 from the following convergence of moments:

Theorem 2.3 ([Bol89]). *There exists $\beta_0 > 0$ such that if $\beta \in (0, \beta_0)$ and $d \geq 3$, then for all multiindices $\alpha \in \mathbb{Z}^d_{\geq 0}$,*

$$\lim_{n\to\infty} \frac{1}{\langle H_n^\omega \rangle} \left\langle \prod_{i=1}^d \left(\frac{\pi_i(n)}{\sqrt{n}} \right)^{\alpha_i} H_n^\omega(\pi) \right\rangle = E\left[\mathsf{N}(0, d^{-1}I)^\alpha\right] \quad \text{for } \mathbb{P}\text{-a.e. } \omega,$$

where we adopt the usual multiindex notation $x^\alpha := \prod_{i=1}^d x_i^{\alpha_i}$.

3 Proof

3.1 Convergence of the partition function

Let $\mathcal{F}_n := \sigma\left(\{\omega(t,x) : t \leq n,\ x \in \mathbb{Z}^d\}\right)$.

Lemma 3.1. *The process $n \mapsto \langle H_n^\omega \rangle$ is a nonnegative $(\mathcal{F}_n)_{n \in \mathbb{Z}_{\geq 0}}$-martingale with $\mathbb{E}\langle H_n^\omega \rangle = 1$.*

Proof. Using that $\mathbb{E}[1 + \beta\omega(t,x)] = 1$, we get

$$\begin{aligned}
\mathbb{E}\left[\langle H_n^\omega \rangle \mid \mathcal{F}_{n-1}\right] &= \frac{1}{(2d)^n} \sum_{\substack{0=x_0,x_1\ldots,x_n,\\ |x_t - x_{t-1}|=1}} \mathbb{E}\left[\prod_{t=1}^n (1 + \beta\omega(t,x_t)) \mid \mathcal{F}_{n-1}\right] \\
&= \frac{1}{(2d)^n} \sum_{\substack{0=x_0,x_1,\ldots,x_n,\\ |x_t - x_{t-1}|=1}} \prod_{t=1}^{n-1} (1 + \beta\omega(t,x_t)) \\
&= \frac{1}{(2d)^{n-1}} \sum_{\substack{0=x_0,x_1,\ldots,x_{n-1},\\ |x_t - x_{t-1}|=1}} \prod_{t=1}^{n-1} (1 + \beta\omega(t,x_t)) \\
&= \left\langle H_{n-1}^\omega \right\rangle.
\end{aligned}$$

□

Lemma 3.2. *The limit $Z := \lim_{n\to\infty} \langle H_n^\omega \rangle$ exists $\mathbb{P}$-a.s. and satisfies $\mathbb{E}Z = 1$ and $\mathbb{P}(Z > 0) = 1$.*

Proof. The a.s. convergence of $\langle H_n^\omega \rangle$ is a direct application of the martingale convergence theorem, which says that non-negative martingales converge a.s. To upgrade this to convergence in L^1, we analyze the second moment of $\langle H_n^\omega \rangle$. Let π, π' be two independent simple random walks. Then we have

$$\begin{aligned}
\mathbb{E} \langle H_n^\omega \rangle^2 &= \mathbb{E} \langle H_n^\omega(\pi) \rangle \langle H_n^\omega(\pi') \rangle \\
&= \mathbb{E} \langle H_n^\omega(\pi) H_n^\omega(\pi') \rangle \\
&= \left\langle \mathbb{E} \prod_{t=1}^{n} (1 + \beta\omega(t, \pi(t)))(1 + \beta\omega(t, \pi'(t))) \right\rangle \\
&= \left\langle (1+\beta^2)^{L_n(\pi,\pi')} \right\rangle,
\end{aligned}$$

where $L_n(\pi, \pi')$ is the **intersection local time**

$$L_n(\pi, \pi') := \#\{1 \le t \le n : \pi(t) = \pi'(t)\}.$$

We have that $L_n(\pi, \pi') \le L(\pi, \pi') := \#\{t \ge 1 : \pi(t) = \pi'(t)\}$. Observe that $L(\pi, \pi')$ is equal in distribution to the number of times the (non-simple) random walk $\widetilde{\pi} := \pi - \pi'$ returns to 0. Since $d \ge 3$, after each time hitting 0 there is a constant probability $p > 0$ that $\widetilde{\pi}$ never returns to 0 again. [1] This implies an exponential tail $P_{\mathrm{RW}}[L(\pi, \pi') > k] = (1-p)^k$, and in particular for all sufficiently small $\beta > 0$ we have

$$\left\langle (1+\beta^2)^{L(\pi,\pi')} \right\rangle < \infty.$$

In particular, $\sup_n \mathbb{E} \langle H_n^\omega \rangle^2 < \infty$. It follows that $\langle H_n^\omega \rangle \to Z$ in L^2, hence in L^1. [2] So $\mathbb{E}Z = 1$. This implies that $\mathbb{P}(Z > 0) > 0$. It can be shown that $\{Z > 0\}$ is a tail event, so it follows from Kolmogorov's zero-one law that $\mathbb{P}(Z > 0) = 1$. □

3.2 A martingale construction for convergence of moments

Recall that we want to show, for any fixed multi-index $\alpha \in \mathbb{Z}_{\ge 0}^d$, that

$$\lim_{n\to\infty} \frac{1}{\langle H_n^\omega \rangle} \left\langle \left(\frac{\pi(n)}{\sqrt{n}} \right)^\alpha H_n^\omega(\pi) \right\rangle = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left(\frac{u}{\sqrt{d}} \right)^\alpha e^{-|u|^2/2} \,\mathrm{d}u \quad \mathbb{P}\text{-a.s.}$$

[1] That p is constant follows from the strong Markov property, so in particular we have the identity $\mathbb{E}L(\pi, \pi') = \sum_{k\ge 0} P_{\mathrm{RW}}(L(\pi, \pi') > k) = \sum_{k \ge 0} (1-p)^k$. On the other hand, by CLT, $\widetilde{\pi}(n)$ is (approximately) uniformly distributed in a box of volume $\Theta(n^{d/2})$. So we get $\mathbb{E}L(\pi, \pi') = \sum_{n \ge 1} P_{\mathrm{RW}}(\pi(n) = 0) = \Theta\left(\sum_{n\ge 1} n^{-d/2}\right)$, which converges for $d \ge 3$. It follows that $p > 0$.

[2] Let M be a martingale with $\sup_n \mathbb{E}\left[M_n^2\right] < \infty$. One can show that $\mathbb{E}\left[(M_n - M_m)^2\right] = \sum_{k=m}^{n-1} \mathbb{E}\left[(M_{k+1} - M_k)^2\right]$ for any $n \ge m$. Setting $m = 0$ and using that $\sup_n \mathbb{E}\left[M_n^2\right] < \infty$ leads to $\sum_{k=0}^{\infty} \mathbb{E}\left[(M_{k+1} - M_k)^2\right] < \infty$. We conclude that M_n is Cauchy—hence convergent—in L^2.

We have taken care of the factor $1/\langle H_n^\omega \rangle$ So it remains to understand the other factor. To this end, we note the following general fact:

Lemma 3.3. *Let $\varphi : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \to \mathbb{R}$ be a function such that $\varphi(n, \pi(n))$ is a martingale with respect to the filtration generated by π, i.e.*

$$\langle \varphi(n, \pi(n)) \mid \pi(t) : t \leq n-1 \rangle = \varphi(n-1, \pi(n-1)).$$

Then $\langle \varphi(n, \pi(n)) H_n^\omega(\pi) \rangle$ is an $(\mathcal{F}_n)_{n \in \mathbb{Z}_{\geq 0}}$-martingale.

Proof.

$$\begin{aligned}
\mathbb{E}\left[\langle \varphi(n, \pi(n))\, H_n^\omega(\pi) \rangle \mid \mathcal{F}_{n-1}\right] &= \langle \mathbb{E}\left[\varphi(n, \pi(n))\, H_n^\omega(\pi) \mid \mathcal{F}_{n-1}\right] \rangle \\
&= \left\langle \varphi(n, \pi(n))\, H_{n-1}^\omega(\pi)\, \mathbb{E}\left[1 + \beta\omega(n, \pi(n))\right] \right\rangle \\
&= \left\langle \varphi(n, \pi(n))\, H_{n-1}^\omega(\pi) \right\rangle \\
&= \left\langle \langle \varphi(n, \pi(n)) \mid \pi(t) : t \leq n-1 \rangle \; H_{n-1}^\omega(\pi) \right\rangle \\
&= \left\langle \varphi(n, \pi(n))\, H_{n-1}^\omega(\pi) \right\rangle.
\end{aligned}$$

□

Unfortunately, plugging in $(n, \pi(n))$ to the function $\varphi : (t, x) \mapsto \left(x/\sqrt{t}\right)^\alpha$ does **not** produce a martingale, and we need to add a correction. We derive this correction as follows.

Recall that the α-th moment can be accessed by differentiating the moment-generating function:

$$\langle H_n^\omega(\pi)\, \pi(n)^\alpha \rangle = \partial_\lambda^\alpha \left\langle H_n^\omega(\pi)\, e^{\lambda \cdot \pi(n)} \right\rangle \Big|_{\lambda=0} = \left\langle H_n^\omega(\pi)\, \partial_\lambda^\alpha e^{\lambda \cdot x} \Big|_{(\lambda, x) = (0, \pi(n))} \right\rangle,$$

where we write $\partial_\lambda^\alpha := \prod_{i=1}^d \frac{\partial^{\alpha_i}}{\partial \lambda_i^{\alpha_i}}$. This leads us to define $\varphi : \mathbb{Z} \times \mathbb{Z}^d \to \mathbb{R}$ by

$$\varphi(t, x) := \partial_\lambda^\alpha\, e^{\lambda \cdot x - t\varrho(\lambda)} \Big|_{\lambda=0},$$

where ϱ is the log-moment-generating function of the increments $X \sim \mathsf{Uniform}(\{\pm e_1, \ldots, \pm e_d\})$:

$$\varrho(\lambda) := \log \left\langle e^{\lambda \cdot X} \right\rangle = \log \left(\sum_{i=1}^d \frac{e^{\lambda_i} + e^{-\lambda_i}}{2d} \right), \qquad \lambda \in \mathbb{R}^d.$$

The role of the factor $e^{-t\varrho(\lambda)}$ is explained by the following

Lemma 3.4. *$\varphi(n, \pi(n))$ is a martingale with respect to the filtration generated by π.*

Proof. Since $\left\langle e^{\lambda \cdot X_n - \varrho(\lambda)} \right\rangle = 1$, we get that

$$\left\langle e^{\lambda \cdot \pi(n) - n\varrho(\lambda)} \mid \pi(t) : t \leq n-1 \right\rangle = e^{\lambda \cdot \pi(n-1) - (n-1)\varrho(\lambda)}.$$

Using this, we obtain

$$\langle \varphi(n,\pi(n)) \mid \pi(t) : t \le n-1 \rangle = \partial_\lambda^\alpha \left\langle e^{\lambda\cdot\pi(n) - n\varrho(\lambda)} \mid \pi(t) : t \le n-1 \right\rangle \Big|_{\lambda=0}$$
$$= \varphi(n-1, \pi(n-1)).$$

□

Lemma 3.3 now implies the following

Corollary 3.5. *$\langle \varphi(n,\pi(n)) H_n^\omega(\pi)\rangle$ is an $(\mathcal{F}_n)_{n\in\mathbb{Z}_{\ge 0}}$-martingale.*

This is promising, but still not enough. So we analyze $\varphi(t,x)$ more carefully. Notice that we have the decomposition

$$\varphi(t,x) = x^\alpha + \varphi_0(t,x) := x^\alpha + \sum_{\substack{j\ge 1,\ \gamma\in\mathbb{Z}^d_{\ge 0},\\ |\gamma|+2j\le|\alpha|}} A(j,\gamma) t^j x^\gamma,$$

for some coefficients $A(j,\gamma)$. Here, for a multiindex γ we write $|\gamma| := \gamma_1 + \cdots + \gamma_d$. Also, notice the moment-generating function of X approximates that of a Gaussian $G \sim \mathsf{N}(0, d^{-1}I)$:

$$\left\langle e^{\lambda\cdot X}\right\rangle = 1 + \frac{|\lambda|^2}{2d} + o(|\lambda|^2) \sim e^{|\lambda|^2/2d} = \left\langle e^{\lambda\cdot G}\right\rangle \quad \text{as } \lambda \to 0,$$

where we abuse notation by writing $\langle\cdot\rangle$ for the expectation in G. It follows that the function

$$\psi(t,x) := \partial_\lambda^\alpha e^{\lambda\cdot x - t\log\left\langle e^{\lambda\cdot G}\right\rangle}\Big|_{\lambda=0} = \partial_\lambda^\alpha e^{\lambda\cdot x - t\frac{|\lambda|^2}{2d}}\Big|_{\lambda=0}$$

admits a similar decomposition:

$$\psi(t,x) = x^\alpha + \psi_0(t,x) := x^\alpha + \sum_{\substack{j\ge 1,\ \gamma\in\mathbb{Z}^d_{\ge 0},\\ |\gamma|+2j=|\alpha|}} A(j,\gamma) t^j x^\gamma.$$

Notice in particular that φ and ψ have the same coefficients $A(j,\gamma)$ for $|\gamma| + 2j = |\alpha|$. Now we write

$$\left(\frac{\pi(n)}{\sqrt{n}}\right)^\alpha = n^{-|\alpha|/2}\varphi(n,\pi(n)) - \psi_0(1, \pi(n)/\sqrt{n}) + n^{-|\alpha|/2}\left[\psi_0(n,\pi(n)) - \varphi_0(n,\pi(n))\right]$$

and take expectations:

$$\frac{1}{\langle H_n^\omega\rangle}\left\langle \left(\frac{\pi(n)}{\sqrt{n}}\right)^\alpha H_n^\omega(\pi)\right\rangle = \frac{n^{-|\alpha|/2}}{\langle H_n^\omega\rangle}\langle \varphi(n,\pi(n)) H_n^\omega(\pi)\rangle - \frac{1}{\langle H_n^\omega\rangle}\left\langle \psi_0(1,\pi(n)/\sqrt{n}) H_n^\omega\right\rangle + \frac{n^{-|\alpha|/2}}{\langle H_n^\omega\rangle}\langle [\psi_0(n,\pi(n)) - \varphi_0(n,\pi(n))]\, H_n^\omega(\pi)\rangle.$$

We analyze the above terms via induction on $|\alpha|$.

When $\alpha = 0$ then $\varphi = \psi = 1$, so the first term is 1 and the second and third terms are 0. So the above expression is equal to the 0-th moment of a Gaussian.

Suppose we have proven the convergence for all multiindices $|\alpha| \leq k$, and let $|\alpha| = k+1$. One can argue along the lines of Lemma 3.2 to show that $n^{-|\alpha|/2} \langle \varphi(n, \pi(n)) H_n^\omega(\pi) \rangle \to 0$ a.s. (for details, see [Bol89, Lemma 4]). Since we have shown already that $\langle H_n^\omega \rangle \to Z > 0$, we conclude that the first term converges to 0 a.s.

Using the preceding argument and the induction hypothesis, we can further conclude that the third term converges to 0. Indeed, the matching of the coefficients $A(j, \gamma)$ implies that the difference $\psi_0(t, x) - \varphi_0(t, x)$ has degree at most $|\alpha| - 3$.

Finally, we can repeat all of the preceding analysis with the increments X_t replaced by i.i.d. Gaussians G_t to show by induction that the second term above converges to $\langle G^\alpha \rangle$. More precisely, we can use the fact that

$$\langle \psi(1, G) \rangle = \partial_\lambda^\alpha \left\langle e^{\lambda \cdot G - \log \mathbb{E} e^{\lambda \cdot G}} \right\rangle \bigg|_{\lambda = 0} = \partial_\lambda^\alpha 1 \bigg|_{\lambda=0} = 0, \qquad \forall \alpha \neq 0$$

to deduce the identity $\langle \psi_0(1, G) \rangle = \langle G^\alpha \rangle$, and then separately apply the induction hypothesis to establish the convergence

$$\frac{1}{\langle H_n^\omega \rangle} \left\langle \psi_0(1, \pi(n)/\sqrt{n}) H_n^\omega(\pi) \right\rangle \xrightarrow{\text{a.s.}} \langle \psi_0(1, G) \rangle .$$

References

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