An orthogonal approach to the subfactor of a planar algebra

Vaughan Jones*, Dimitri Shlyakhtenko and Kevin Walker

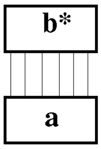
July 11, 2008

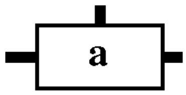
1 Introduction

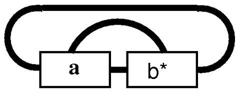
Starting from a subfactor planar algebra, a construction was given in [GJS07] of a tower of II_1 factors whose standard invariant is precisely the given planar algebra. The construction was entirely in terms of planar diagrams and gave a diagrammatic reproof of a result of Popa in [Pop95]. The inspiration for the paper was from the theory of large random matrices where expected values of words on random matrices give rise to a trace (see [Voi83]) on the algebra of noncommutative polynomials. Since that trace is definable entirely in terms of planar pictures it was easy to generalise it to an arbitrary planar algebra, giving the planar algebra a concatenation multiplication to match that of noncommutative polynomials. Unfortunately, though the algbera structure is very straightforward, the inner product is not always easy to work with as words of different length are not orthogonal. In this paper we use a simple diagrammatic orthogonalisation discovered by the third author to reprove the II_1 factor results of [GJS07] in a direct and simple way without the use of full Fock space or graph C^*-algebras. One may capitalise on the advantages of orthogonalisation since the multiplication does not actually become much more complicated when transported to the orthogonal basis. We present the results by beginning with the orthogonal picture and giving a complete proof of the tower result. Then we show that this orthogonal structure is actually isomorphic to that of [GJS07].

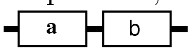
*Supported in part by NSF Grant DMS0401734, Auckland University and the NZIMA

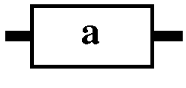
2 Setup.

Let $\mathfrak{P} = (P_n)_{n=0,1,2,\ldots}$ be a subfactor planar algebra. Let $Gr_k(\mathfrak{P})$ be the graded vector space $\oplus_{n\geq 0} P_{n+k}$ equipped with the prehilbert space inner product $<,>$ making it an orthogonal direct sum and for which, within $P_{n,k}$, $< a, b >= \delta^{-k}$ 

. Write $P_{n,k}$ for P_{n+k} when it is considered as the n-graded part of $Gr_k(\mathfrak{P})$. We will attempt to keep the pictures as uncluttered as possible by using several conventions and being as implicit as possible. Shadings for instance will always be implicit and we will eliminate the outside boundary disc whenever convenient. An element $a \in P_{n,k}$ will be represented whenever possible in a picture as: 

where the thick lines to the left and right of the box represent k lines and the think line at the top represents $2n$ lines. If the multiple lines have to be divided into groups the number off lines in each group will be indicated to the minimal extent necessary. The distinguished first interval in a box will always be the top left of the box. Thus the inner product above of $< a, b >$ for elements of $P_{n,k}$ will be 

. In the original works on planar algebras (e.g. [Jon99]), each P_k ia an associative *-algebra whose product, with these conventions, would be to view P_k as $P_{0,k}$ and $ab =$ 

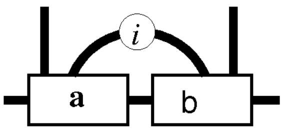
. There are unital inclusions of P_k in P_{k+1} by identifying a with 

. The identity element of P_k is thus represented by a single thick horizontal line. It is also the identity element of $Gr_k(\mathfrak{P})$. The trace (often called the Markov trace) tr onP_k is normalised so as to be compatible with the inclusions by $tr(a) = \delta^{-k} < a, 1 >$. We extend this trace to $Gr_k(\mathfrak{P})$ by the same formula so that the trace of an element is the Markov trace of its zero-graded piece. Each P_n is a finite dimensional C*-algebra whose norms are also compatible with the inclusions.

3 *-Algebra structure on $Gr_k(\mathfrak{P})$

Definition 3.1. *If $a \in P_{m,k}$ and $b \in P_{n,k}$ are elements of $Gr_k(\mathfrak{P})$ we define their product to be*

$$a \star b = \sum_{i=0}^{min(2m,2n)}$$

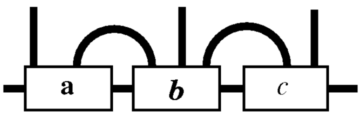


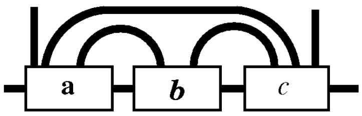
(where the i means there are i parallel strings. The numbers of other parallel strings are then implicitly defined by our conventions.)

*The *-structure on $P_{n,k}$ is just the involution coming from the subfactor planar algebra.*

Proposition 3.2. *$(Gr_k(\mathfrak{P}), \star, ^*)$ is an associative *-algebra.*

Proof. The property $(a \star b)^* = b^* \star a^*$ is immediate from the properties of a planar *-algebra. For associativity note that both $a \star (b \star c)$ and $(a \star b) \star c$ are given by the sum over all epi (see section 5) diagrams where no strand has both of its endpoints of a, or both of its endpoints on b, or both of its endpoints on c. Here are two typical examples:



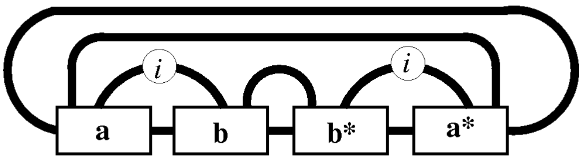


□

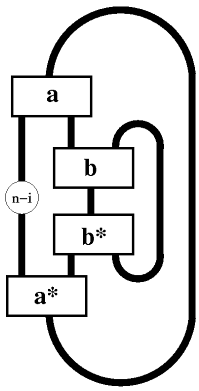
The inner product $< a, b >$ is clearly equal to $tr(ab^*)$ and is positive definite by definition. We would like to perform the GNS construction but since there is no C*-algebra available we need to show by hand that left (and hence right) multiplication by elements of $Gr_k(\mathfrak{P})$ is bounded.

Theorem 3.3. *Let $a \in Gr_k(\mathfrak{P})$. Then the map $L_a : Gr_k(\mathfrak{P}) \to Gr_k(\mathfrak{P})$, defined by $L_a(\xi) = a \star \xi$ is bounded for the prehilbert space structure.*

Proof. We may suppose $a \in P_{n,k}$ for some n. Then L_a is a sum of $2n+1$ maps L_a^i from an orthogonal direct sum of finite dimensional Hilbert spaces to another, respecting the orhtogonal decomposition,L_a^i being the map defined by the ith. term in the sum defining $\star$. Thus it suffices to show that the norm of the map $L_a^i : P_{m,k} \to P_{m+n-i,k}$ is bounded independently of m, the number of i values being $\leq 2n+1$. Clearly we may suppose that $m >> n+k$ which simplifies the number of pictures to be considered. So if $b \in P_{m,k}$ we must estimate $< ab, ab >$ which is the following tangle:

δ^{-k} 

We may suppose $i \leq n$ since the norm of an operator is equal to that of its adjoint and the roles of i and $2n-i$ are reversed in going between L_a^i and $(L_a^i)*$. Then we may isotope the picture, putting a and b in boxes with the same number ($k+n$ and $k+m$ respectively) of boundary points on the top and bottom, also possibly rotating them, to obtain the following equivalent tangle:

δ^{-k} 

Note that the multiplicities of all the strings are determined by the "$n-i$" and our conventions.

Neglecting powers of δ which do not involve m we see $< \tilde{a}\tilde{b}, \tilde{a}\tilde{b} >$ where $\tilde{a}$ is a with $m-i$ strings to the right and $\tilde{b}$ is b with $n-i$ strings to the left. The strings to the right do not change the norm of a (as an element of the finite dimensional C*-algebra P_{2k+2n}) by the uniqueness of the C*-norm. The (L^2-) norm of $\tilde{b}$ differs from that of b by an m-independent power of δ. Hence we are done.

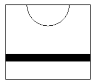
□

$Gr_k(\mathfrak{P})$ has thus been shown to be what is sometimes called a "Hilbert Algebra" or "unitary algebra".

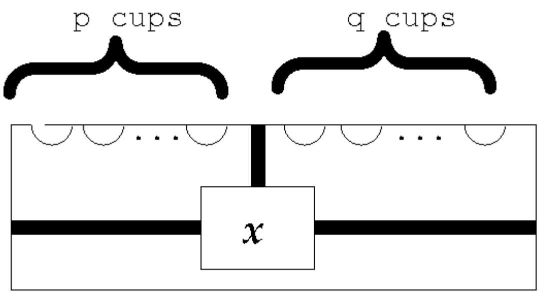
4 The von Neumann algebras M_k.

Definition 4.1. *Let M_k be the finite von Neumann algebra on the Hilbert space completion of $Gr(\mathfrak{P}_k)$ generated by left multiplication by the L_a.*

Since right multiplication is also bounded, the identity in $P_{0,k}$ is a cyclic and separating trace vector for M_k defining the faithful trace tr as usual, and the right multiplications generate the commutant of M_k. We shall first show that each M_k is a factor.

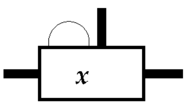
Definition 4.2. *The element $\cup_k \in P_{2,k}$ will be the following:*  *(where we have included the boundary to avoid disembodiment). The subalgebra of $Gr(\mathfrak{P}_k)$ generated by $\cup_k$ will be denoted $\mathfrak{A}_k$ and its weak closure in M_k will be called A_k.*

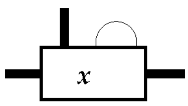
Definition 4.3. *For $x \in P_{n,k}$, $n \geq 0$ and $p, q \geq 0$ let $x_{p,q} \in P_{n+p+q,k}$ be $\frac{1}{(\sqrt{\delta})^{p+q}}$ times the following:*



Definition 4.4. *Let $W_0 = \{0\} \subset P_{0,k}$ and for each $k \geq 1$ let W_n be the span in $P_{n,k}$ of $\{x_{1,0}\}$ and $\{x_{0,1}\}$ for $x \in P_{n-1,k}$. For $n \geq 0$ let $V_n = W_n^{\perp}$.*

Lemma 4.5. *For $x \in P_{n,k}$, $n \geq 1$,*

$x \in V_n \iff$ 

$= 0 =$ 

.

Proof. Taking the inner product of these two elements with an arbitrary element in $P_{n-1,k}$ we see the inner product of element s in $W_{n,k}$ with x. □

Corollary 4.6. *Let $v \in V_m$, $m \geq 0$ and $v' \in V_n$, $n \geq 1$, then*

$$< v_{p,q}, v'_{p',q'} >= \begin{cases} < v, v' > & \text{if } p = p', q = q' \\ 0 & \text{otherwise} \end{cases}$$

Proof. If either $p \neq p'$ or $q \neq q'$, the left or rightmost pair of boundary points of v or v' will be capped off to give zero. □

Corollary 4.7. *If $v \in V_n$ for $n > 0$ is a unit vector, the $v_{p,q}$ are an orthonormal basis for the $\mathfrak{A} - \mathfrak{A}$ bimodule $\mathfrak{A}v\mathfrak{A}$.*

Proof. By the previous lemma it suffices to show that the span of the $v_{p,q}$ is invariant under left and right multiplication (using $\star$) by $\cup_k$. In fact we have the formula:

$$\cup_k \star v_{p,q} = \begin{cases} \sqrt{\delta} v_{1,q} + v_{0,q} & \text{if } p = 0 \\ \sqrt{\delta} v_{p+1,q} + v_{p,q} + \sqrt{\delta} v_{p-1,q} & \text{otherwise} \end{cases}$$

And there is an obvious corresponding formula for right multiplication by $\cup_k$. □

Lemma 4.8. *The linear span of all the $v_{p,q}$ for $v \in V_n$ for all n is $Gr(\mathfrak{P}_k)$.*

Proof. By a simple induction on n these vectors span $P_{n,k}$ for all k. □

Let us summarise all we have learnt using the unilateral shift S (with $S^*S = 1$) on $\ell^2(\mathbb{N})$.

Theorem 4.9. *Suppose $\delta > 1$. As an $A - A$ bimodule,*

$$L^2(M_k) = P_{0,k} \otimes \ell^2(\mathbb{N}) \oplus \{\mathfrak{H} \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})\}$$

with $\cup_k$ acting on the left and right on $P_{0,k} \otimes \ell^2(\mathbb{N})$ by $id \otimes (\sqrt{\delta}(S+S^*)+SS^*)$, on the left on $\mathfrak{H} \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ by $id \otimes (\sqrt{\delta}(S+S^*)+1) \otimes id$ and on the right on $\mathfrak{H} \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ by $id \otimes id \otimes (\sqrt{\delta}(S+S^*)+1)$. ($\mathfrak{H}$ is an auxiliary infinite dimensional Hilbert space.)

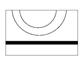
Proof. Obviously $P_{0,k}$ commutes with A so the first term in the direct sum is the result of a simple calculation. Choosing an orthonormal basis for each V_n gives the rest by corollary 4.7.

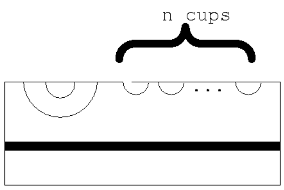
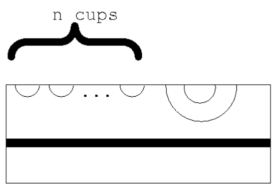
□

Corollary 4.10. $A' \cap M_k = AP_{0,k}$

Proof. It suffices to show that no non-zero $\xi \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ satisfies $(S + S^*)\xi = \xi(S+S^*)$. But such a ξ would be a Hilbert-Schmidt operator on $\ell^2(\mathbb{N})$ commuting with $S+S^*$ and $S+S^*$ would leave invariant a finite dimensional subspace and hence have an eigenvalue. But $S + S^*$ is Voiculescu's semi-circular element and is well-known not to have an eigenvalue (this follows immediately from a direct proof). □

Corollary 4.11. *Suppose $\delta > 1$. For each k, $M_0' \cap M_k = P_{0,k}$ (as an algebra).*

Proof. The element $\alpha =$  is in $M_0 \subset M_k$ so it is enough to show that the only elements in the Hilbert space closure of $P_{0,k}A$ that commute with it are elements of $P_{0,k}$. We define

$\lambda_n = \frac{1}{(\sqrt{\delta})^n}$  and $\rho_n = \frac{1}{(\sqrt{\delta})^n}$ 

We have

$$[\alpha, 1_{0,n}] = (\lambda_n - \rho_n) + \frac{1}{\sqrt{\delta}}(\lambda_{n-1} - \rho_{n-1})$$

So if an element c in the closure of $P_{0,k}A$ is written as an ℓ^2 sum $\sum_{n=0}^{\infty} c_n \star 1_{0,n}$ with $c_n \in P_{0,k}$ we find

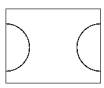
$$[\alpha, c] = \sum_{n=1}^{\infty}(c_n + \frac{1}{\sqrt{\delta}}c_{n+1}) \star (\lambda_n - \rho_n).$$

The terms in the sum are orthogonal for different n so for c to commute with α we would have $c_{n+1} = -\delta c_n$ for $n > 1$ which forces $c_n = 0$ for $n \geq 1$ since $c_n \in \ell^2$. So $c \in P_{0,k}$. □

Corollary 4.12. *If $\delta > 1$, M_k is a type II_1 factor.*

Proof. If x were in the centre of M_k it would have to be in $P_{0,k}$. But a trivial diagrammatic argument shows that the only elements in $P_{0,k}$ that commute with  are scalar multiples of the identity. □

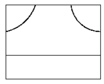
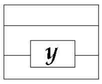
We now want to identify the II_1 factors M_k with the tower coming from the subfactor $M_0 \subset M_1$ obtained by iterating the basic construction of [Jon83]. For simplicity we will do it for the case M_0, M_1, M_2, the general case being the same argument with heavier notation.

Definition 4.13. *The element $e \in M_2$ will be $\frac{1}{\delta}$*  .

Proposition 4.14. *This e is a projection and for $x \in M_1, exe = E_{M_0}(x)e$ and $E_{M_1}(e) = \delta^{-2}id$ where E_{M_i} is the trace preserving conditional expectation onto M_i.*

Proof. Easy computation with diagrams. □

Lemma 4.15. *The von Neumann algebra $\{M_1, e\}''$ is a II_1 factor.*

Proof. If x is in the centre of $\{M_1, e\}''$ then it commutes with M_0 so by 4.11 we know that $x \in P_{0,2}$. But x also has to commute with  which forces x to be of the from 

for $y \in P_1$. But for this to commute with e forces it to be a scalar multiple of the identity. □

Corollary 4.16. *For $z \in \{M_1, e\}'', ze = \delta^2 E_{M_1}(ze)e$.*

Proof. By algebra, M and MeM span a *-subalgebra of $\{M_1, e\}''$ which is thus wealkly dense. The assertion is trivial for $z \in M$ and a simple calculation for $z \in M_1 e M_1$. And E_{M_1} is continuous. □

Corollary 4.17. *The map $x \mapsto \delta xe$ from M_1 to $\{M_1, e\}''e$ is a surjective isometry intertwining E_{M_0} on $L^2(M_1)$ and left multiplication by e.*

Proof. Surjectivity follows from the previous lemma. The intertwining property is a calculation. □

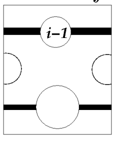
Corollary 4.18. *$\{M_1, e\}''$ is the basic construction for $M_0 \subset M_1$ and $[M_1 : M_0] = \delta^2$.*

Proof. The basic construction is the von Neumann algebra on $L^2(M_1)$ generated by E_{M_0} and M_1. By lemma 4.15, $\{M_1, e\}''$ as a subalgebra of M_2 is the same as it is acting on $\{M_1, e\}''e$ by left multiplication. And this is the basic construction by the previous corollary. The index is then just a matter of evaluating the trace of e, by uniqueness of the trace on a factor. □

Corollary 4.19. *$\{M_1, e\}'' = M_2$.*

Proof. The same argument as above applied to $M_1 \subset M_2$ shows that $[M_2 : M_1] = \delta^2$. But then $[M_2 : \{M_1, e\}''] = 1$. □

Summing up the above arguments applied to the whole tower we have the following:

Theorem 4.20. *Let $\mathfrak{M}_n$ be the II_1 factor obtained by the basic construction from $\mathfrak{M}_{n-2} \subset \mathfrak{M}_{n-1}$ with $\mathfrak{M}_0 = M_0$ and $\mathfrak{M}_1 = M_1$ and e_n be the projection of the basic construction generating $\mathfrak{M}_{n+1}$ from $\mathfrak{M}$. Then there is a (unique) isomorphism of towers from $\mathfrak{M}_n$ to M_n which is the identity on M_1 and sends e_i to $\frac{1}{\delta}$* 

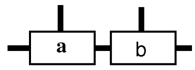
.

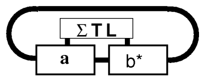
Theorem 4.21. *Given a subfactor planar algebra $\mathfrak{P} = (P_n)$ with $\delta > 1$ the subfactor M_0 constructed above has planar algebra invariant equal to $\mathfrak{P}$.*

Proof. It is well known ([Jon99]) that the planar algebra structure is determined by knowledge of the $e_i's$, the multiplication and the embeddings $P_n \subset P_{n+1}$ corresponding to the inclusions $M_1' \cap M_{n+1} \subset M_0' \cap M_{n+1}$ and $M_0' \cap M_n \subset M_0' \cap M_{n+1}$. But the conditional expectations onto these are just given by the appropriate diagrams. □

5 Change of basis

In this section we show that the pre-Hilbert space $Gr_k(\mathfrak{P})$ defined above is isometric and isomorphic as a *-algebra to the pre-Hilbert space also called $Gr_k(\mathfrak{P})$ defined in [GJS07]. To distinguish between them we will call the latter pre-Hilbert space $Hr_k(\mathfrak{P})$.

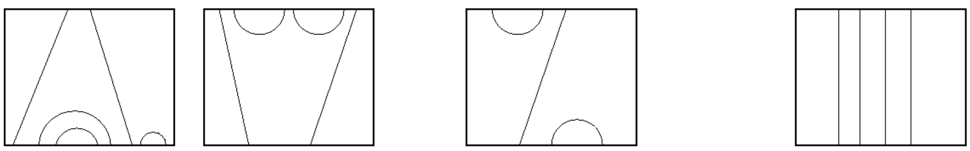
Recall that $Hr_k(\mathfrak{P})$ is defined on the same underlying vector space $\oplus_{n\geq 0} P_{n+k}$, but with a simpler multiplication and more complicated inner product. The multiplication is simple juxtaposition, $a \bullet b =$ 

, while the inner product $\langle\langle a, b\rangle\rangle$ of $a \in P_{m,k}$ and $b \in P_{n,k}$ is 

where ΣTL denotes the sum of all loopless Temperley-Lieb diagrams with $2(m+n)$ strands on the boundary. Note that while the multiplication respects the grading, the inner product does not.

We will define an upper-triangular change of basis in $\oplus_{n\geq 0} P_{n+k}$ which induces an isomorphism between $Gr_k(\mathfrak{P})$ and $Hr_k(\mathfrak{P})$.

Recall that an epi TL diagram is one in which each point on the top/outgoing side of the rectangle is connected to the bottom/incoming side of the rectangle. A monic diagram is defined similarly, but with the roles of the sides reversed:



epi   monic   neither epi nor monic   both epi and monic

Note that each TL diagram factors uniquely as epi followed by monic.

We will think of a TL diagram with $2i$ strands on the bottom of the rectangle and $2j$ strands on the top of the rectangle as a linear map from $P_{i,k}$ to $P_{j,k}$.

Define $X : \oplus_{n\geq 0} P_{n+k} \to \oplus_{n\geq 0} P_{n+k}$ to be the sum of all epi TL diagrams. Thus the j, i block of X is the (finite) sum of all epi TL diagrams from $2i$ strands to $2j$ strands, which is the identity if $i = j$ and zero if $i < j$.

Define a non-nested epi TL diagram to be one where each "turn-back" or "cap" on the bottom of the rectangle encloses no other turn-backs. Define $Y : \oplus_{n\geq 0} P_{n+k} \to \oplus_{n\geq 0} P_{n+k}$ to be the sum of all non-nested epi TL diagrams, with the coefficient in the i, j block equal to $(-1)^{i-j}$.

Remark 5.0.1. *In the special case of a vertex model planar algebra ([Jon99]) the graded vector space is the (even degree) non-commutative polynomials. Voiculescu in [Voi83] defined a map from these polynomials to full Fock space, the vacuum component of which is the trace on what we have called $Hr_k(\mathfrak{P})$. in this case the map X gives Voiculescu's map in its entirety and Y is its inverse. We presume that these formulae are known perhaps in some slightly different form but have been unable to find them explicitly in the literature.*

Lemma 5.1. $XY = 1 = YX$.

Proof. $X_{jm}Y_{mi}$ is equal to the sum of all products of a non-nested TL diagram from i to m (with $i - m$ turn-backs) followed by a general epi TL diagram from m to j, with sign $(-1)^{i-m}$. The number of times a given diagram D appears in this sum is is equal to the number of subsets of size $i - m$ taken from the innermost turn-backs of D. It follows that the total coefficient of D in $\sum_m X_{jm}Y_{mi}$ is $\sum_p (-1)^p \binom{t}{p} = 0$ (assuming $p > 0$), where t is the total number of innermost turn-backs of D. Thus the off-diagonal blocks of XY are zero, and it is easy to see that the diagonal blocks of XY are all the identity.

The proof that $YX = 1$ is similar, with outermost turn-backs playing the role previously played by innermost turn-backs. □

Lemma 5.2. $X(a \bullet b) = X(a) \star X(b)$.

Proof. Let $a \in P_{m,k}$ and $b \in P_{n,k}$. Each epi diagram from $2(m + n)$ to $2j$ appearing in the definition of $X(a \bullet b)$ factors uniquely as $T \cdot (L|R)$, where L is an epi diagram from $2m$ to $2m'$, R is and epi diagram from $2n$ to $2n'$, $L|R$ denotes L and R placed side by side, and T is an epi diagram from $2(m'+n')$ to $2j$ where each turn-back has one end in the m' side and the other end in the n' side. L corresponds to a diagram used in the definition of $X(a)$, R corresponds to a diagram used in the definition of $X(b)$, and T corresponds to a diagram used in the definition of $\star$ in $X(a) \star X(b)$. □

Lemma 5.3. $\langle\langle a, b\rangle\rangle = \langle X(a), X(b)\rangle$.

Proof. Let $a \in P_{m,k}$ and $b \in P_{n,k}$. Let D be a TL diagram in ΣTL used in the definition of $\langle\langle a, b\rangle\rangle$. We can think of D is a TL diagram from $2m$ strands to $2n$ strands, and from this point of view it has a unique factorization $E \cdot M$, where E is an epi diagram starting at $2m$, and M is a monic diagram ending at $2n$. E is an epi diagram figuring in the definition of $X(a)$, and M^* is an epi diagram figuring in the definition of $X(b)$. The way in which E and M^* are glued together corresponds to the definition of $\langle \cdot, \cdot \rangle$. □

References

[GJS07] A. Guionnet, V. F. R. Jones, D. Shlyakhtenko. Random matrices, free probability, planar algebras and subfactors. arXiv:0712.2904

[Jon83] V.F.R. Jones. Index for subfactors. *Invent. Math*, 72:1–25, 1983.

[Jon99] V.F.R. Jones. Planar algebras. Preprint, Berkeley, 1999.

[Jon00] V. F. R. Jones. The planar algebra of a bipartite graph. In *Knots in Hellas '98 (Delphi)*, volume 24 of *Ser. Knots Everything*, pages 94–117. World Sci. Publ., River Edge, NJ, 2000.

[Pop95] S. Popa. An axiomatization of the lattice of higher relative commutants of a subfactor. *Invent. Math.*, 120(3):427–445, 1995.

[Voi83] Symmetries of some reduced free product C^*-algebras. Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), 556–588, Lecture Notes in Math., 1132, Springer, Berlin, 1985.