In and around the origin of quantum groups.

Vaughan F.R. Jones *

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Abstract

Quantum groups were invented largely to provide solutions of the Yang-Baxter equation and hence solvable models in 2-dimensional statistical mechanics and one-dimensional quantum mechanics. They have been hugely successful. But not all Yang-Baxter solutions fit into the framework of quantum groups. We shall explain how other mathematical structures, especially subfactors, provide a language and examples for solvable models. The prevalence of the Connes tensor product of Hilbert spaces over von Neumann algebras leads us to speculate concerning its potential role in describing entangled or interacting quantum systems.

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1 The representations of $SU(2)$

Since $SU(2)$ is compact, any continuous representation on Hilbert space is unitarizable and the direct sum of a family of irrecuible representations, all of which are finite dimensional. The irreducible unitary representations (henceforth called "irreps") are easy to classify. There is exactly one of each dimension n which is often written $n = 2j + 1$ where j is the "spin" of the representation. Let V_j be the vector space of the spin j irrep. Explicitly, V_j can be constructed from the identity representation on $\mathbb{C}^2$ as the symmetric algebra of $\mathbb{C}^2$. That is to say that $SU(2)$ acts on homogeneous polynomials of two variables x and y of degree $2j + 1$ by extending the action $x \mapsto ax + by$, $y \mapsto cx + dy$ for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SU(2)$.

1.1 Clebsch-Gordon rules

The tensor product decomposition for the irreps of $SU(2)$ is known as the Clebsch-Gordon rule and is simply the following:

$$V_j \otimes V_k = \oplus_{i=|j-k|}^{j+k} V_i$$

where the equation is as $SU(2)$-modules and i goes in steps of 1. This decomposition is easy to prove. Observe that the circle sugroup of diagonal matrices $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ acts in V_j by diagonal matrices with respect to the basis of monomials with eigenvalues $\{z^{2j}, z^{2j-2}, ..., z^{-2j}\}$ (where $z = e^{i\theta}$. These eigenvalues are the "weights" of the representation. It is clear then that $V_j \otimes V_k$ has highest weight z^{2j+2k} with multiplicity one so there is exactly one copy of V_{j+k}. Orthogonal to it we see the weight z^{2j-2} with multiplicity one. Continuing in this way we are done.

When $k = 2$ the Clebsch-Gordon rules say that $V_j \otimes V_{1/2} = V_{j+1/2} \oplus V_{j-1/2}$. Since any irrep is contained in a tensor power of $V_{1/2}$ one may show that this rule alone suffices to determine all the Clebsch-Gordon rules. We may represent this rule graphically as follows:

Here the vertices of the graph, known as A_∞, represent the irreps of $SU(2)$ and an edge between two vertices means that the irrep of one is contained in the tensor product of the other with $V_{1/2}$. This procedure for associating graphs with the irreps of an object, with one privileged one, is obviously quite general and we will use it without further explanation below. Note that if there were multiplicity in the decomposition, one would use multiple edges in the graph.

1.2 Decomposition of the tensor powers of irreps.

If π is a representation of the group G on the vector space V, one looks first for proper subspaces of V which are invariant under π_g for all $g \in G$. If V is a Hilbert space and π is unitary it is natural to ask that the subspace be closed, hence also a Hilbert space. Moreover closed subspaces of Hilbert space are the same as projection operators-continuous linear maps $p : \mathcal{H} \to \mathcal{H}$ with

$$p = p^* \text{ and } p^2 = p.$$

To say that the subspace is invariant is the same as saying that the corresponding projection commutes with π_g for all $g \in G$. Thus the various ways in which a unitary representation decomposes are described entirely by projections that commute with the group. But the set of <u>all</u> continuous operators which commute with the group has the structure of an algebra to which many more techniques can be brought to bear than on the set of its projections. Indeed we have just given one of the equivalent definitions of a **von Neumann algebra**, namely the algebra of operators commuting with a unitary group representation.

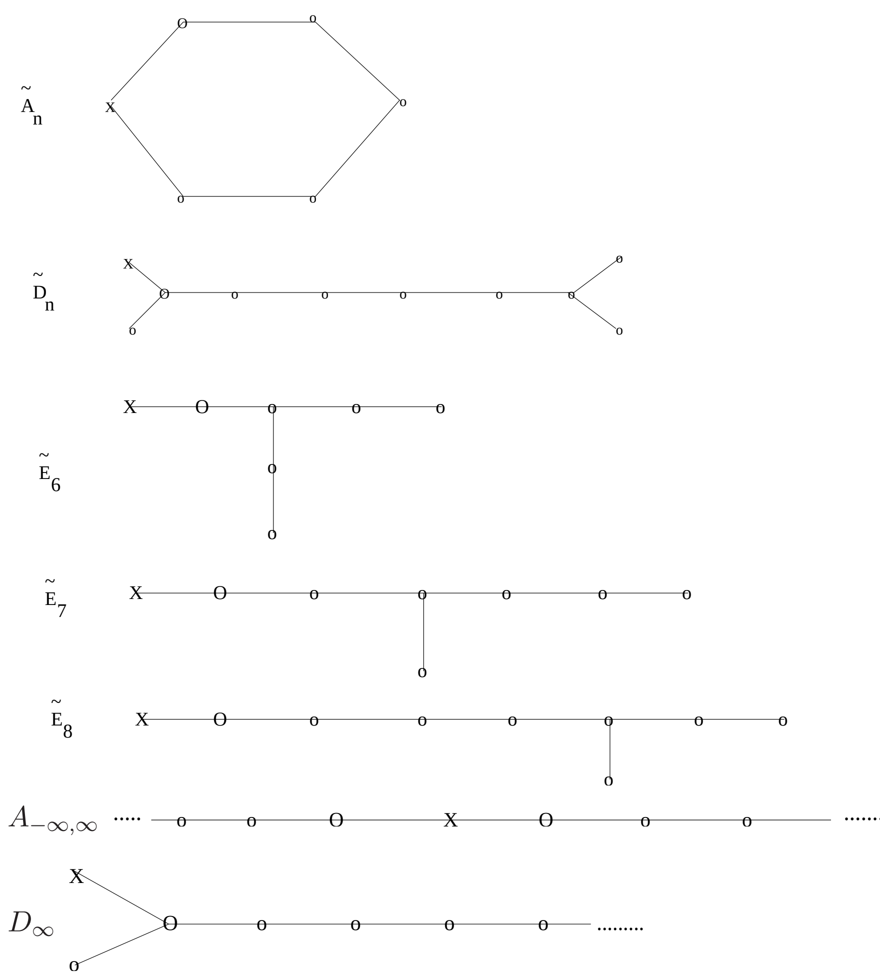
If π is any representation of any group G on the vector space V, there is *always* a canonical algebra of linear transformations of $\otimes^k V$ commuting with $\otimes^k \pi$. That is the algebra generated by the permuation group S_k acting by permuting the various tensor product components (i.e. if σ is a permutation then $\sigma(v_1 \otimes v_2 \otimes ... \otimes v_k) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes .... \otimes v_{\sigma(k)}$ – or is it σ^{-1}?...). Since the permutation group is generated by its transpositions, this algebra is generated by $S_{12}, S_{23}, ...S_{(k-2)(k-1)}$ where $S : V \otimes V \to V \otimes V$ is the map $S(v \otimes w) = w \otimes v$, and for the rest of this paper we make the convention that if $R : V \otimes V \to V \otimes V$ is any linear map then for $1 \leq i \leq k-1$ the linear map $R_{i(i+1)} : \otimes^k V \to \otimes^k V$ is defined by

$$R_{i(i+1)}(v_1 \otimes v_2 \otimes ...v_i \otimes v_{i+1} ... \otimes v_k) = v_1 \otimes v_2 \otimes ...R(v_i \otimes v_{i+1}) ... \otimes v_k$$

Thus one may decompose the tensor powers of π according to the irreps of the symmetric group by projecting on to the subspace of vectors (the so-called "isotypical component") of vectors that transform according to that representation of S_k. Thus the symmetric powers of π are given by the trivial irrep and the antisymmetric powers by the parity irrep. It is a well known result, sometimes called "Schur-Weyl duality", that if $V = \mathbb{C}^n$ and $G = SU(n)$ then the algbebra generated by S_k is in fact the algebra of all operators commuting with G.

2 The McKay correspondence

This is a relation between closed subgroups of $SU(2)$ and the extended Coxeter-Dynkin diagrams $\tilde{A}$, $\tilde{D}$, $\tilde{E}$ $\tilde{A}_{-\infty,+\infty}$ and $\tilde{D}_\infty$ drawn below (and of course A_∞ drawn above).



Let us start with the case of a finite subgroup G. By passing to the quotient $SO(3)$ we see that G is the double cover of either a cyclic group, a dihedral group or the symmetry group of one of the Platonic solids-the tetrahedron, cube/octahedron and the icosahedron/dodecahedron. We now form a graph for G as we did for $SU(2)$

in 1.1. The vertices of the graph are the set of irreps of G and there are k edges between two irreps if the tensor product of one by the two dimensional identity representation of G contains k copies of the other. (In fact no multiplicity higher than one occurs here.) The McKay correspondence asserts that the graph obtained is necessarily an extended Coxeter-Dynkin diagram according to the following scheme.

$$\begin{array}{ccl} \tilde{A}_n & \leftrightarrow & \text{Cyclic Group} \\ \tilde{D}_n & \leftrightarrow & \text{Dihedral Group} \\ \tilde{E}_6 & \leftrightarrow & \text{Tetrahedral Group} \\ \tilde{E}_7 & \leftrightarrow & \text{Cube/Octahedron Group} \\ \tilde{E}_8 & \leftrightarrow & \text{Icosahedral/Dodecahedral Group} \end{array}$$

There are three infinite closed subgroups of $SU(2)$. They are $SU(2)$ itself, the circle group $\mathbb{T}$ and the infinite dihedral group $\mathbb{T}\rtimes\mathbb{Z}/2\mathbb{Z}$. They correspond to the diagrams A_∞, $A_{-\infty,\infty}$ and D_∞ respectively. Here A_∞ is the graph of the Clebsch-Gordon rules for $SU(2)$ and $A_{-\infty,\infty}$ and D_∞ are as above.

For the lovers of the empty set we must mention the trivial group consisiting of the identity element. It has one irreducible representation which, on tensoring with the identity representation gives 2 copies of itself. So the graph of the McKay correspondence could be taken as the graph with one vertex and two edges connecting that vertex to itself...

The cyclic group $\mathbb{Z}/n\mathbb{Z}$ case requires a certain amount of care as the representation is not irreducible so corresponds actually to both vertices on the graph adjacent to the trivial representation. The cyclic groups exist as honest subgroups of $SU(2)$ and as such they give rise to $\tilde{A}_n$'s. As subgroups of $SO(3)$ they are double covered in passing to $SU(2)$ and what happens depends on the parity of n. We leave the somewhat confusing details as an exercise.

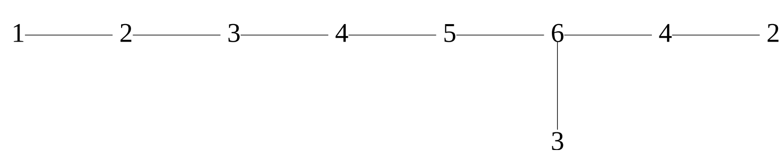
The guiding light here is that the graph must somehow be made up from extended ADE diagrams as there is a third ingredient of the McKay correspondence which is to $p \times q$ matrices with non-negative integer entries whose norm is equal to 2. (The norm of a matrix Λ is the largest stretching factor for unit vectors, or alternatively the square root of the largest eigenvalue of $\Lambda^T\Lambda$.) In this correspondence one takes a bipartite graph with n vertices, with disjoint vertex subsets X and Y not connected to themselves, but $n = \#(X) + \#(Y)$, and constructs the matrix with columns labelled by X and rows labelled by Y. Under certain irreducibility assumptions, if the resulting matrix has norm 2, the graph has to be an extended ADE graph.The importance of norm 2 is explained as follows. From A form the square matrix

$$\Omega = \begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix}$$

(which is actually the adjacency matrix of the graph in the usual sense). The norm of Ω is the same as that of Λ and the Perron Frobenius theorem on matrices with non-negative entries implies that the norm of Ω is the eigenvalue of the unique eigenvector with positive entries. It suffices to exhibit such a vector (whose representation theoretic nature we will describe) for the ADE diagrams to show they have norm 2.

In the other direction one may see an a priori connection with root systems for Lie algebras by forming $2 - \Omega$. Given that the norm of Ω is equal to 2 and Ω is symmetric, $2 - \Omega$ is positive semidefinite so has a symmetric (real) square root Δ. The relation $\Delta^2 = \Omega$ says precisely that the rows of Δ are vectors which are all of length $\sqrt{2}$ and are either orthogonal or at an angle of $120°$ to each other. Since 2 is actually an eigenvalue of Ω, the rows of $2 - \Omega$ only span a subspace of dimension $n - 1$. Up to this detail we are dealing with a root system. In fact if any vertex of the graph is removed the resulting set of vectors will indeed be a root system all of whose roots have the same length. Thus we expect to see the ADE Coxeter-Dynkin diagrams. The details are left as an exercise.

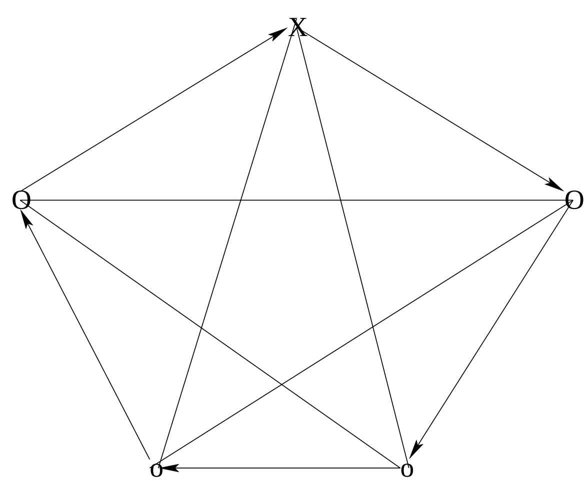
We would like to mention an amusing check on all this stuff. From the point of view of $SU(2)$, the reason the matrix has to have norm 2 is that tensoring a representation by the identity representation multiplies the dimension of the representation by 2 so that *the vector whose entries are the dimensions of the irreps of the closed subgroup G of $SU(2)$* is an eigenvector for Ω of eigenvalue 2. Conversely, if we take the Perron-Frobenius eigenvector for Ω and normalise it so that the component corresponding to the trivial representation is 1, the other entries must all be integers, indeed they must be the dimensions of the irreps of G ! We illustrate with the Perron-Frobenius eigenvector for $\tilde{E}_8$ below:



Note that the sum of the squares of the dimensions is 120. The order of the group of rotational symmetries of the dodecahedron is obviously 60. The factor of 2 is due to the double covering when passing from $SU(2)$ to $SO(3)$.

A curious question arises out of our McKay correspondence. Why did only the extended Coxeter-Dynkin diagrams arise? Are there naturally arising structures whose representations are the vertices of an ordinary ADE diagram and for which the tensor product rule can be interpreted as above? If such structures exist is there a context in which they appear just as naturally as the McKay correspondence? The answer to these questions is provided by subfactors as we shall see.

A more obvious way to extend the McKay correspondence is to do the same thing for $SU(3)$ and beyond. One will not of course obtain the ADE diagrams but rather graphs of norm 3, 4 and so on. Moreover the graphs will have to be directed. The reason for undirected graphs for $SU(2)$ and its subgroups is that the identity representation is self-conjugate. If we had considered $U(2)$ instead we would have had to use directed graphs and would have found graphs with loops and directed edges. As a very simple example for $U(3)$ here is the directed graph (of norm 3 of course) resulting from a copy of $\mathbb{Z}/5\mathbb{Z}$ in $U(3)$:



3 Commuting transfer matrices, the Yang-Baxter equation

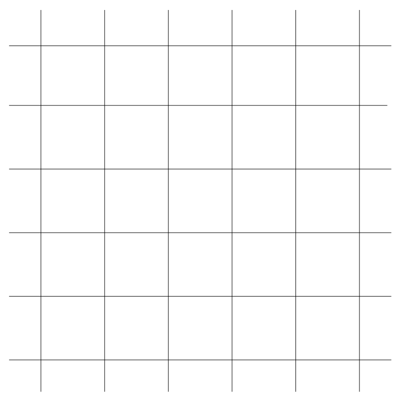
3.1 Generalities

In statistical mechanics systems are sometimes modelled by specifying a set of states $\{\sigma\}$ arising from a collection of locally interacting sites placed on some lattice. An energy is assigned to each state according to the model. If just a finite subset X of the lattice,with N sites , is considered, the number of states may be finite and the "partition function" for X is

$$Z_X = \sum_{\sigma \in \{\text{states of } X\}} e^{\frac{-E(\sigma)}{kT}}$$

Some attention will have to be given to the boundary of X to properly define Z_X. In general we will consider an increasing family of subsets X whose union is the whole system.

For instance the simplest of all such models is the Ising model where the lattice is $\mathbb{Z}^n$ and X is a product of intervals, depicted below for instance when $n = 2$ and X is a 6×6 square:



A state of the system is specified by assigning one of two "spin" states $\uparrow$ and $\downarrow$ to each site (=lattice point). The edges between the lattice points correspond to (nearest neighbour) interactions and the energy of a state σ is the sum:

$$\sum_{\text{edges between lattice points}} E(\sigma_x, \sigma_y)$$

where in the sum x and y are the lattice points at each end of the edge, and $E(i, j)$ (with i and j being $\uparrow$ or $\downarrow$) is the local energy arising from the interaction along the edge.

The boundary conditions can be handled in many ways-one can wrap approximating rectangles on a torus creating periodic boundary conditions. Or one can simply neglect the interactions of the boundary sites with neighbours outside X, (free boundary conditions), or one may fix all the spins along the boundary according to some specified pattern (fixed boundary conditions). Since most of the contribution to the partition function will not involve the boundary, the asymptotic growth rate of the partition function should depend only on the whole system. This rate is called the free energy per site:

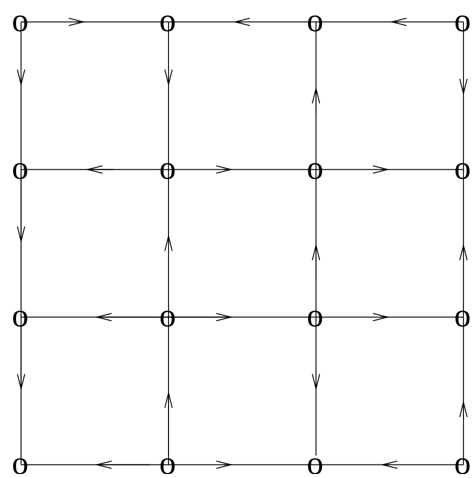
$$F = \lim_{N \to \infty} \frac{1}{N} \log Z_X$$

This free energy may depend on several parameters. Certainly the temperature is one of them, but there may be different horizontal and vertical interactions, an external field and so on.

We will say a model is "solved" if F is expressed as an explicit function of its parameters. Given the complexity of the function that may be involved in such a solution, one may question the usefulness of a solution as opposed to the defining limit. But there are many cases in which the explicit formula is simple enough to read off meaningful results. There are also many other limits one might like to calculate before saying the model is "solved".

The most completely (non-trivial) solved model is the Ising model in 2 dimensions. But we shall be more interested in another kind of model called a "vertex model" on a lattice, where the state of the system is defined by assigning values (in some indexing set) to the edges of the lattice. The "ice-type" model is a vertex model in which the indexing set has two elements (corresponding to the presence or absence of some kind of bond between neighbouring molecules) which can be conveniently represented by arrows on the edges. Thus a state of an approximating square in a 2-d ice-type model might be as below:

Fig. 3.1.1.

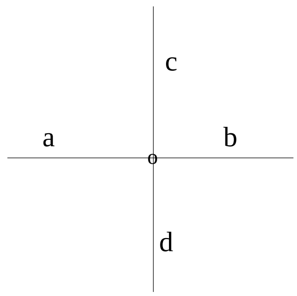


The energy of a state of a vertex model is the sum of energy contributions from each vertex. If the state is given each vertex is surrounded by edges with indices on them so that the energy is specified by assigning an energy to each configuration of indices. In the ice-type model there are 16 such configurations corresponding to the arrow configurations around a vertex.

The partition function is calculated using exponentiated energies. The exponential of the energy is called the Boltzmann weight so that we have Boltzmann weights

$$R(a, b|c, d)$$

assigned to each local index configuration as below:



The partition function for the rectangular subregion X is then

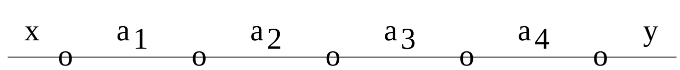
$$Z_X = \sum_{states} \prod_{vertices} R(a, b|c, d)$$

Where conventions must be adopted for how the indices surrounding a vertex (in a given state) are to be used as indices in $R(a, b|c, d)$, and the boundary conditions must be specified.

Remark 3.1.2. *In a large part of the literature what we call "R" below is called* $\check{R}$ *and* $R = S\check{R}$ *with* S *as in section 1.2. We use our notation slightly reluctantly but it seems that the more fundamental formalism is the one where our R-matrix is present but* S *is not. And we do have the justification that* R *is the letter Baxter himself uses in [2]. In quantum group theory it is no doubt the other* R *that is more natural.*

3.2 Transfer Matrices

Transfer matrices are a powerful method for translating the problem of finding the partition function into a problem of linear algebra. The basic idea is that the summation over indices in the partition function becomes the summation over indices in matrix multiplication. For instance if one had a one dimensional vertex model with Boltzmann weights $R(a, b)$ the partition function for a lattice with n sites as below (illustrated with $n = 5$):



is readily seen to be the (x, y) entry of the matrix R^n. The boundary conditions were fixed to be x at the left and y at the right. If the boundary conditions were periodic the partition function would be $Trace(R^n)$.

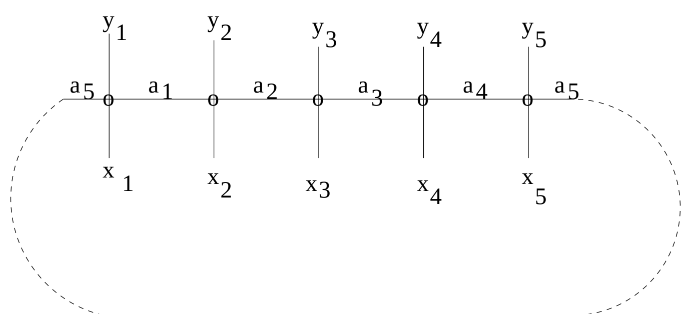
One is interested in the asymptotic behaviour as the subsystem X tends to the whole infinite lattice and one can use linear algebra techniques to understand the asymptotic behaviour of R^n (the behaviour is in general governed by the largest eigenvalue. We leave the solution of the one dimensional vertex model as an easy exercise.

To apply the transfer matrix method to a two dimensional lattice we simply think of each row of the lattice as being an atom in a one dimensional lattice and construct its transfer matrix. The trouble is of course that the the size of the transfer matrix will grow (exponentially) with the size of the system. And the boundary conditions will have to be handled in a more complicated way. Let us first impose periodic horizontal boundary conditions. Then the transfer matrix for a 2-d lattice built up from horizontal rows will be

$$T^{y_1,y_2,\dots y_n}_{x_1,x_2,\dots,x_n} = \sum_{a_1,a_2,\dots,a_n} R(a_n,a_1|x_1,y_1)R(a_1,a_2|x_2,y_2)\dots.R(a_{n-1},a_n|x_n,y_n).$$

as explained diagrmattically below:

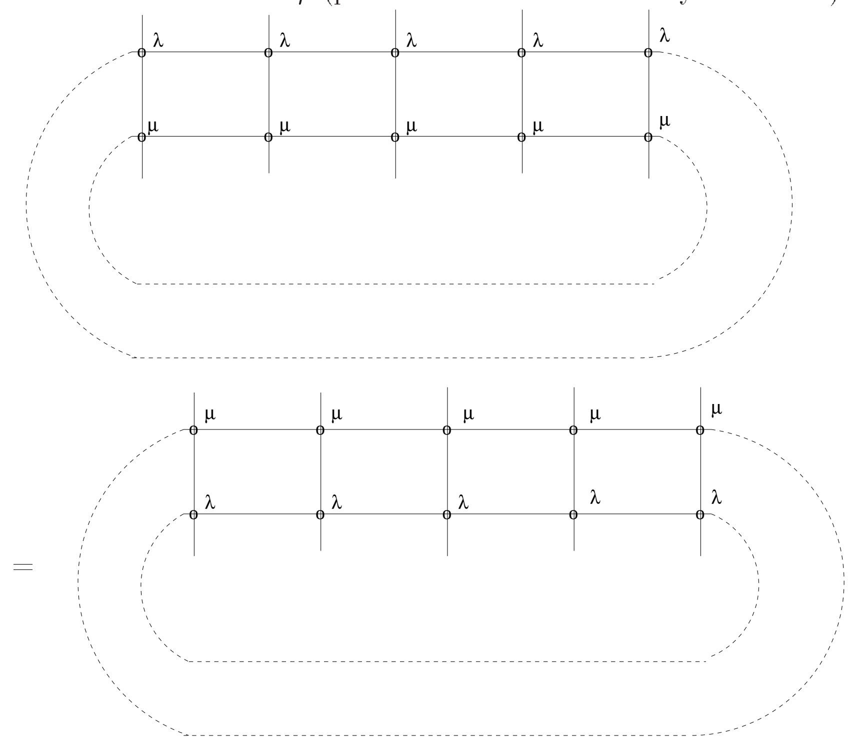
Fig. 3.2.1.



Because of the growing size of T, the problem of calculating its largest eigenvalue becomes formidable and hopeless in general. One of Baxter's great ideas was to look for models in which the transfer matrices <u>commute</u> <u>with</u> <u>each</u> <u>other</u> for different values of their parameters. Then they will have to have a common eigenvector and one may try to deduce enough about how the eigenvalue depends on the parameter to determine it completely. This part of the Baxter program - actual determination of the eigenvalues - has not been completely formalised, but a great machine has evolved for producing examples of models with commuting transfer matrices. That machine is *QUANTUM GROUPS*.

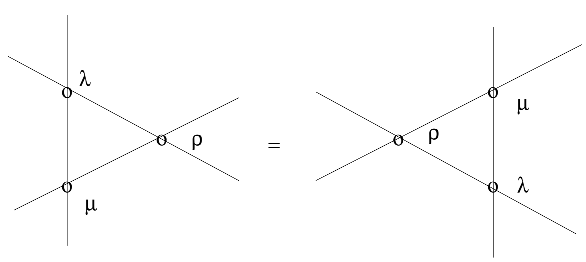
3.3 The Yang-Baxter equation.

The diagram below illustrates what it means for the transfer matrix with value λ (often called the *spectral parameter*) to commute with the transfer matrix with value μ (periodic horizontal boundary conditions):



Here we left out all indices, the convention being that indices are implicit on the boundary edges and summed over for each internal edge. And the value of the spectral parameter to be used for the R matrix is indicated near to the corresponding vertex on the diagram.

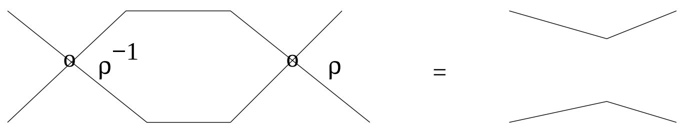
If written out in full, the equations represented by the diagram form a huge system of highly non-linear equations for the Boltzmann weights. The Yang-Baxter equation (YBE) is a set of equations involving ony 3 vertices which implies that the transfer matrices commute. With the same notational conventions as above the YBE asserts the existence of a third value ρ of the spectral parameter (depending of course on λ and μ) for which we have the following equation:



If we use $R(\lambda)$ to denote the matrix of Boltzmann weights with parameter λ then the YBE is, in the notation of 1.2:

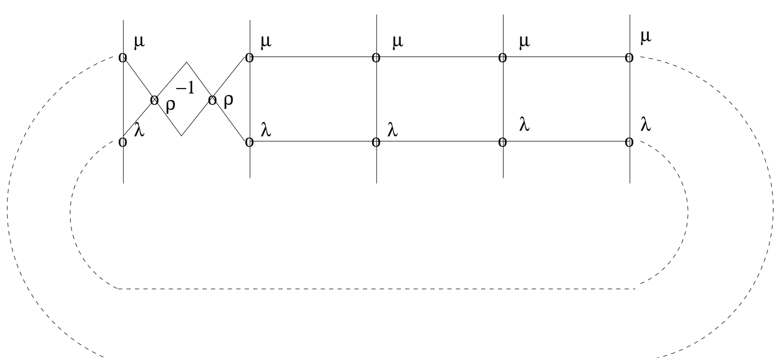
3.3.1. $R_{12}(\lambda)R_{23}(\rho)R_{12}(\mu) = R_{23}(\mu)R_{12}(\rho)R_{23}(\lambda)$.

The argument that the YBE implies commuting transfer matrices is an elegant one which is entirely diagrammatic with our summation convention. We need to make the assumption that the matrix of Boltzmann weights for the third value ρ is invertible. This is precisely the condition that there is another R-matrix which we will denote by the parameter "ρ^{-1}" for which:

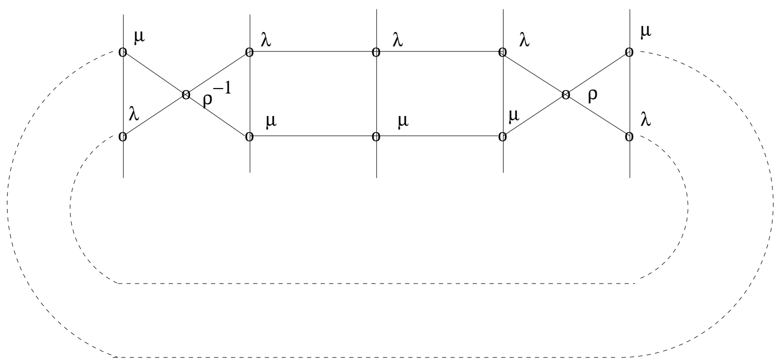


(which of course implies the same thing with ρ and ρ^{-1} interchanged). Note that it is rather important to associate the correct indices of $R(a,b|c,d)$ to the correct edges of the the diagram. How to do this will be obvious from the following argument so we leave it to the reader.

Now take the picture representing one side of the equation for commuting transfer matrices and insert the picture above for ρ "ρ^{-1}" to obtain:



This does not change the partition function. Successive applications of YBE move ρ clockwise around the picture, swapping a λ and a μ each time. After a few steps one obtains:



Eventually the ρ comes right round the circle and meets its inverse with which it cancels and one obtains the other side of the commuting transfer matrix equation!!

3.4 First solution of the YBE

Apart from the most difficult question of how to proceed once we have commuting transfer matrices (which we do not pursue here), the YBE raises many questions. One might ask for instance if the ρ is uniquely defined by λ and μ and if so what are the properties of the operation $(\lambda, \mu) \mapsto \rho$. But most of all there is the question of existence- the YBE is a system of coupled cubic equations with far more equations than unknowns. Without any discussion at this point of how it was found, we present the following $R-$matrix and claim it is a solution of YBE:

3.4.1. $R_q(x) =$

$$\frac{1}{xq - x^{-1}q^{-1}} \begin{pmatrix} xq^{-1} - x^{-1}q & 0 & 0 & 0 \\ 0 & x^{-1}(q^{-1} - q) & x - x^{-1} & 0 \\ 0 & x - x^{-1} & x(q^{-1} - q) & 0 \\ 0 & 0 & 0 & xq^{-1} - x^{-1}q \end{pmatrix}$$

The assiduous reader may check directly that (suppressing q-dependence)

$$R_{12}(x)R_{23}(xy)R_{12}(y) = R_{23}(y)R_{12}(xy)R_{23}(x)$$

but we will see easier ways to check this later on. Note that the factor $\dfrac{1}{x^{-1}q^{-1} - xq}$ is arbitrary but it yields the following pleasant properties:

(i) $R_1(x) = S$ (recall from 1.2 that S is the flip $v \otimes w \mapsto w \otimes v$).

(ii) $R_q(1) = -1$.

(iii) $R_q(x)^{-1} = R_q(x^{-1})$

(iv) If we define R to be $\lim_{x\to 0} R_q(x)$ then

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^2-1 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

satisfies the braid equation

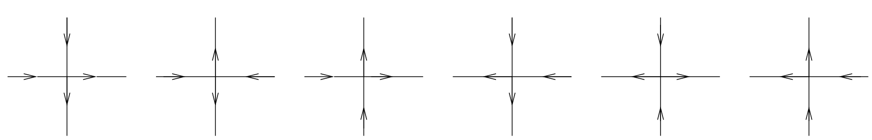
$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

Note that writing $x = e^{\lambda}$ and $q = e^{\theta}$ converts all the entries of the matrix into hyperbolic sines. This R-matrix is called a trigonometric solution of the YBE.

Since the entries of the matrix are supposed to be Boltzmann weights, to be of interest to statistical mechanics there must be values of x and q for which the entries of the matirx are all non-negative. The global multiplying factor is neither here nor there so we see that it suffices to choose x and q positive with $x > x^{-1}$ and $q^{-1} > q$.

3.5 The Ice-type model and the Potts model.

Recall that a Boltzmann weight of 0 corresponds to infinite energy- i.e. a forbidden configuration. If we look at the positions of the zeros of our R-matrix 3.4.1 and think of the rows and columns as indexing arrow configurations as in 3.1.1 we see that the configurations allowed by the R-matrix are the following:



Excluding all but these particular configurations has something to do with the particular physical system under configuration. Lieb solved the ice-type model in [29], not by the method of commuting transfer matrices but by the so-called Bethe Ansatz. For a detailed account of this see [30]. This was the first major class of statistical mechanical models solved after Onsager's solution of the Ising model. It was in a detailed study of Lieb's solution that Baxter observed that the transfer matrices commute and that this property extends to a model with eight non-zero Boltzmann weights. Baxter went on to solve this model by a new method, but the theory of quantum groups has been most successful in extending the six-vertex case, and applications to topology have not used the full eight-vertex model. so

Lieb's work remains at the heart of the whole business. Indeed Lieb's solution of the ice problem motivated Izergin, Coker and Korepin ([17]) to solve the six-vertex model with twisted boundary conditions, which Kuperberg used to give a proof ([27]) of the alternating sign matix conjecture.

The Potts model is not a vertex model. It is like the Ising model in that individual "spins" are located on the vertices of the lattice and a state of the system is specified by assigning a "spin" value from 1 to Q to each of the vertices. The interactions occur along the edges of the lattice so that the total energy of a state is

$$\sum_{\text{edges of the lattice}} E(\sigma, \sigma')$$

where we have suppressed the approximating rectangle and σ and σ' denote the spin values at the ends of the particular edge being summed over. Thus the partition function is

$$\sum_{states} \prod_{edges} w(\sigma, \sigma')$$

where the Boltzmann weights are the exponentiated energies as usual. Thus from a purely mathematical point of view the only data for the lattice model is the $Q \times Q$ matrix $w(\sigma, \sigma')$ of Boltzmann weights. If the edges of the lattice are not directed this must be a symmetric matrix though the geometry of the lattice may allow, say, different Boltzmann weights for vertical or horizontal interactions.

The Potts model is defined by the property that the spin states have no structure other than being different so that the Boltzmann weight $w(\sigma, \sigma')$ depends only on whether $\sigma = \sigma'$ or not. If $V = \mathbb{C}^Q$ with usual basis v_σ, then the transfer matrix which creates a new row with n spins of the lattice will be a linear map from $\otimes^n V$ to itself. To organise the transfer matrix we introduce the maps $p : V \to V$ with all matrix entries equal to $\frac{1}{\sqrt{Q}}$, and the map $d : V \otimes V \to V \otimes V$ with $d(v_\sigma \otimes v_{\sigma'}) = \delta_{\sigma,\sigma'} v_\sigma \otimes v_\sigma$. We then put $E_{2i-1} = 1 \otimes 1 \otimes ... \otimes p \otimes 1 ... \otimes 1$ with the p in the ith. tensor position, and $E_{2i} = \sqrt{Q} d_{i(i+1)}$ using the notation of section 1.2. Then it is an easy exercise to show that, for the Potts model, the transfer matrix with free horizontal boundary conditions is a multiple of

3.5.1. $$\prod_{i=1}^{n-1} (aE_{2i} + 1) \prod_{i=1}^{n} (bE_{2i-1} + 1)$$

where a and b are determined by the horizontal and vertical Boltzmann weights respectively (note for instance that necessarily, up to a constant, $w = Ap + 1$ where w is the matrix given by the Boltzmann weights).

You may be wondering about the bizarre normalisations we have used in defining p and d and the strange indexing of the E_i's. The reason is to get the nicest possible algebra going. It should be checked that the E_i satisfy the following relations:

3.5.2. $$E_i^2 = \sqrt{Q} E_i$$

3.5.3. $$E_i E_{i\pm1} E_i = E_i$$

3.5.4. $$E_i E_j = E_j E_i \qquad \textit{if} \quad |i-j| \geq 2$$

These relations are known as the Temperley-Lieb relations and are somewhat magical. It is fun to check that $P = E_1 E_3 E_5 ... E_{2n-1}$ has the property that

$$PxP = \phi(x)P$$

where x is any word on the E_i's and $\phi(x)$ is a real valued function of x. This means that if X is any element in the algebra generated by the E_i's then $PXP = \phi(x)P$ for some linear functional ϕ on that algebra. Moreover in terms of our statistical mechanical model this functional ϕ gives precisely the partition function for free vertical boundary conditions! Thus in principle the partition function for a rectangular lattice with free boundary conditions is entirely determined by the Temperley-Lieb relations.

So what? To answer this we return to our R-matrix 3.4.1 for the ice-type model. Put

3.5.5. $$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then two things are true. First, if we define E_i on $\otimes^n \mathbb{C}^2$ as $E_{i(i+1)}$ with the notation of 1.2, then the Temperley- Lieb relations hold, and second, $R_q(x)$ is a linear combination of E and the identity. It is thus not surprising that, with the appropriate boundary conditions, the partition function for the ice-type model is the same as that of the Potts model with a (physically bizarre) change of variables. In fact this is only true if the horizontal and vertical interactions of the Potts model satisfy the relation $a = b$, known as "criticality" for various reasons. In [37], Temperley and Lieb showed the equivalence of the ice-type model and the critical Potts model on a square lattice using the relations 3.5.23.5.3 and 3.5.4. This equivalence was subsequently understood on a general planar graph (for the Potts model) and its "medial graph" (for the ice-type model). For a beautiful treatment see chapter 12 of [2].

3.6 How to remember the formula.

My personal way of reconstructing the formula 3.4.1 from simpler ones involves the Hecke algebra of type A_n. This is the algebra with generators $g_1, g_2, ..., g_{n-1}$ and relations

(h1) $g_i^2 = (q-1)g_i + qid$

(h2) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$

(h3) $g_i g_j = g_j g_i \quad \text{if} \quad |i-j| \geq 2.$

Here I am faithfully reproducing a constant disagreement in the literature over the meaning of q. In our Hecke algebra relations we are using q as in [6], which is natural in its context as the number of elements in a finite field. The q in 3.4.1 is the square root of this q.

The relations $h2$ and $h3$ are the braid relations which we have seen as the limit of the YBE as the spectral parameter tends to infinity. In this Hecke algebra case we can reconstruct the YBE from the braid relations as follows:

Step 1: Renormalize the g_i's as G_i's so that relation $h1$ becomes

$$G_i + G_i^{-1} = k\ id$$

Step 2: Define

$$R_i(x) = xG_i + x^{-1}G_i^{-1}$$

.

Then it is an exercise to prove that, in the presence of the braid relations the YBE is equivalent to

3.6.1. $$G_1 G_2^{-1} G_1 + G_1^{-1} G_2 G_1^{-1} = G_2 G_1^{-1} G_2 + G_2^{-1} G_1 G_2^{-1}$$

It is immediate to show 3.6.1 from $G_i + G_i^{-1} = k\ id$. I do not know of any other solutions to 3.6.1.

Of course this begs the question of how to get appropriate solutions of the Hecke algebra relations. One way is to obtain g_i's from the Temperley-Lieb E_i's by $g_i = qE_i - 1$ (which q is this??) and this is indeed how I put together 3.4.1. But there are other solutions as we shall see.

4 Local Hamiltonians

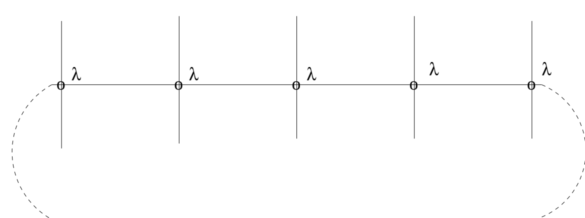
A *quantum spin chain* is a one dimensional array of spins. If the Hilbert space describing an individual spin is $\mathbb{C}^2$, then a quantum spin chain with n spins will be described by $\otimes^n \mathbb{C}^2$. If T is a Hermitian 2×2 matrix defining an observable for a single spin, then that observable for the kth. spin in the chain is (with T in the kth. slot):

$$T_k = 1 \otimes 1 \otimes \ldots \otimes T \otimes 1 \otimes \ldots \otimes 1.$$

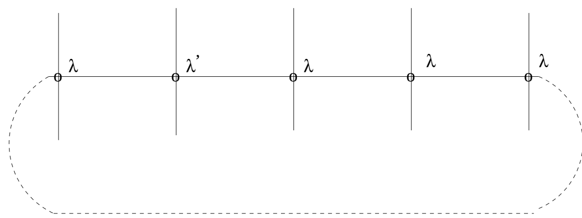
If H is the Hamiltonian for the spin chain, the observables A evolve according to $A_t = e^{iHt} A e^{-iHt}$. By writing down the correlations between observables of spin k at (discretized) time t one sees a strong similarity with expected values of spins in the Ising model whose x coordinate is given by k and y coordinate by the time t, provided one takes as Hamiltonian the logarithm of the transfer matrix (times $\sqrt{-1}$). This is generalised into a powerful equivalence between 2-dimensional statistical mechanics and 1-dimensional quantum mechanics provided time can be anaylytically continued to imaginary time.

This suggests that perhaps the transfer matrices of statistical mechanical models can be used to create Hamiltonians for quantum spin chains. In order to satisfy locality conditions, a Hamiltonian should be expressible as a sum of terms each one only involving spins close to each other on the lattice. The simplest would be a nearest neighbour interaction and if it is translation invariant it must be of the form $\sum_i H_{i(i+1)}$ in the notation of 1.2 where H is some self-adjoint operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$. An ingenious way to do this is to take the logarithmic derivative of the transfer matrix with respect to the spectral parameter at some appropriate value of the spectral parameter. By the conditions after 3.4.1 it is clear that the right value is $x = 1$.

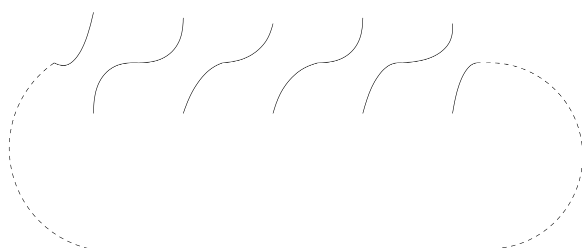
Since the transfer matrix is multilinear in the R matrices used at the vertices we see that the derivate with respect to λ of



is the sum over all ways of putting in one λ' of



where we have symbolically used λ' to stand for the derivative of the R-matrix with respect to λ. If we use 3.4.1, the sign of $R_q(1)$ is irrelevant so we see that the transfer matrix 3.2.1 is represented diagrammatically by:

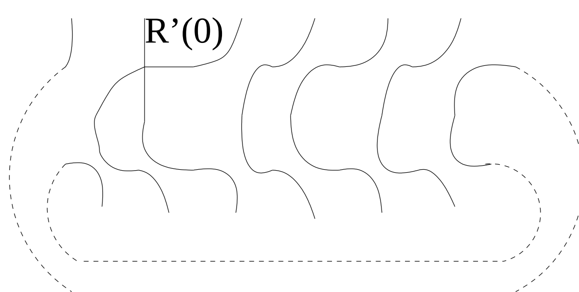


which is of course a rotation if it is represented in a cylinder.

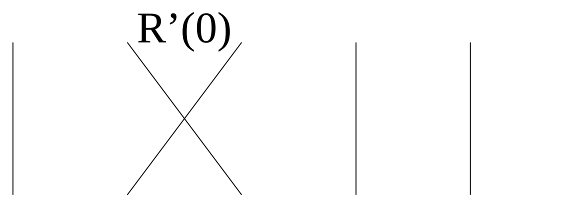
So if we write $T(x)$ for the transfer matrix using 3.4.1, a typical term in the logartithmic derivative

$$T(1)^{-1}\frac{dT}{dx}|_{x=1}$$

can be seen by multiplying the diagrams as below with $R'(0)$ being the derivative of 3.4.1 with respect to x at $x = 1$:



On cleaning up the picture we see that the logarithmic derivative is the sum of all matrices whose diagrams are as below, the crossings occuring between the ith. and $(i+1)$th. strings from the left, with periodic boundary conditions so that the last term would involve a crossing between the first and last strings:



This sum of matrices is clearly a Hamiltonian with nearest neighbour interactions provided it is positive self-adjoint. This positive condition on the R-matrix may be quite different from the positivity of the Boltzmann weights so R-matrices which do not work in the statistical mechanics world may work here. The moral is that if you find a solution to the Yang-Baxter equation that does not admit positive Boltzmann weights, don't necessarily condemn it to the trash as it may give you a solvable quantum spin chain.

I have made the arguments for commuting transfer matrix and local Hamiltonian completely diagrammatic so they will work in any situation where the diagrams make sense as multilinear maps on their inputs. For instance the arguments work just as well for the Potts model and other models known as "IRF models". Several people have axiomatised the diagram calculus-see [28],[3]. I have developed a specific formalism, called "planar algebras" whose special features were driven by subfactors. Certainly the arguments for commuting transfer matrices and local Hamiltonians work in planar algebras.

We should not forget what we have achieved with the local Hamiltonian-since the transfer matrices all commute among themselves, they commute also with the local Hamiltonian so we are armed with a large family of operators commuting with the time evolution which should be extremely useful in diagonalising it. If, for physical reasons, one wanted *local* expressions for these constants of the motion one could take higher logarithmic derivatives of the transfer matrices with respect to the spectral parameter.

In the particular case of our R-matrix 3.4.1 of we do the computation we find the local Hamiltonian

$$\sum_{i=1}^{n} H_{i(i+1)}$$

where the indices are taken mod n and, up to addition of some constant matrix which only changes the whole Hamiltonian by addition of a constant,

$$H_{i(i+1)} = \begin{pmatrix} q^{-1}+q & 0 & 0 & 0 \\ 0 & -(q^{-1}-q) & 2 & 0 \\ 0 & 2 & q^{-1}-q & 0 \\ 0 & 0 & 0 & q^{-1}+q \end{pmatrix}$$

which may mean more to physicists written in terms of the Pauli spin matrices:

$$H_{i(i+1)} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \frac{1}{2}\left((q+q^{-1})(id+\sigma_z\otimes\sigma_z) + (q-q^{-1})(\sigma_z\otimes id - id\otimes\sigma_z) \right)$$

The presence of the term multiplied by $q-q^{-1}$ is the only difference between this and what is known as the XXZ Hamiltonian. But in fact as Barry McCoy pointed out to me these terms cancel when one performs the sum over i. So the XXZ Hamiltonian admits a large family of explicit matrices that commute with it.

4.1 Spin Models

As we have mentioned, quantum groups gave a machine for creating large families of solutions of the Yang-Baxter equation and hence statistical mechanical models with commuting transfer matrices, and quantum spin chains with many commuting Hamiltonians. But the R-matrix coming from a quantum group is that of a vertex model. I have long wondered if there is any such machine that would produce what I call "spin models", that is to say generaisations of the Potts model with an arbitrary (symmetric) matrix of Boltzmann weights. There does not seem to be such a machine. Searches for such models have been very combinatorial and though they have led to some insights in combinatorics (see [4]) there have been few new statistical mechanical models. The one spectacular new spin model was discovered by Jaeger in [19]. He only gives the knot theoretic solution of the Yang-Baxter equation but it can be easily "Baxterised" as in [22] to give a solvable model.

The idea that led to the Jaeger model was to look for models whose matrix $w(\sigma, \sigma')$ of Boltzmann weights was the next simplest after the Potts model. In the Potts model this matrix only has two different entries, one on the diagonal and one off. The first generalisation of this would be to matrices with three distinct entries-one on the diagonal and two others, say x and y. One may then construct a graph whose vertices are the indices of the matrix entries (i.e. the spin states per site) with an edge connecting σ and σ' if $w(\sigma, \sigma') = x$. This puts one in the world of association schemes and their Bose graphs. By applying this kind of idea Jaeger found two remarkable things:

(a) That there is a solvable model as above for which the underlying graph is the Higman-Sims graph on 100 vertices! (The automorphism group of this graph is the Higman-Sims group, one of the first sporadic finite simple groups.)

(b)Together with a couple of simple examples, the Higman Sims graph is the only known graph that can work.

At this stage no-one has gone on to solve the Jaeger model. There is a Temperley-Lieb like duality with a quantum group R-matrix so that the bulk free energy is not too interesting, but the correlation functions must support representations of the Higman-Sims group.

5 Subfactors.

5.1 Factors

For a complete change of pace we treat a topic in analysis. Von Neumann algebras are self-adjoint algebras of bounded linear operators on Hilbert space which contain the identity operator and are closed under the topology of pointwise convergence on the Hilbert space. Factors are von Neumann algebras with trivial centre. We do not want to go into the details more than that but we can suggest the notes from a course given by the author, accessible from his home page, for anyone who wants to know more. Thus a subfactor is a pair $N \subseteq M$ of factors. To avoid technical difficulties we will only talk about the case of "type II_1" factors which are those which are infinite dimensional but have a trace tr which is a linear functional satisfying

$$tr(ab) = tr(ba)$$

and can be normalised so that $tr(1) = 1$, in which case

$$tr(x^*x) > 0 \text{ for } x \neq 0.$$

5.2 Index

There is a notion of index for a subfactor, written $[M : N]$. If $[M : N] < 4$ it was shown in [21] that it must be one of the numbers $4\cos^2 \pi/n$ for $n = 3, 4, 5, \ldots.$ The key ingredient in the proof of this result was the construction of a tower of factors from the original pair and certain operators satisfying the Temperley-Lieb relations! To be precise one can construct an orthogonal projection e_N from M to N

defined by the formula

$$tr(xe_N(y)) = tr(xy)$$

and then show that if $[M : N] < \infty$, the algebra $< M, e_N >$ of linear operators on M generated by M (acting by left multiplication) and e_N is again a II_1 factor and $[< M, e_N >: M] = [M : N]$. This constructs the beginning of the tower:

$$N \subseteq M \subseteq M_1 =< M, e_N > .$$

To continue just repeat the construction to obtain $M_{i+1} =< M_i, e_{M_{i-1}} >$.

If we write $e_i = e_{M_i}$ for short and renormalise by $E_i = \sqrt{[M : N]}\, e_i$ then relations 3.5.2,3.5.3 and 3.5.4 all hold. Moreover there is *-structure for which $E_i^* = E_i$ and the trace has to satisfy

$$tr(x^*x) > 0 \text{ for } x \neq 0.$$

A careful examination of this property of the trace on the algebra generated by the E_i's proves the result about the "quantized" index values-see [15].

To see a more compelling similarity with the previous sections, note that the tower M_i can also be constructed as

$$M_i = M \otimes_N M \otimes_N M \otimes_N ... \otimes_N M$$

with $i+1$ copies of M in the tensor product. (For anyone who has not seen the tensor product over non-commutative algebras, $M \otimes_N M$ is the quotient of the vector space tensor product $M \otimes M$ by the subspace spanned by $\{xn \otimes y - x \otimes ny | x, y \in M \text{ and } n \in N\}$.) The E_i's can then be constructed with the same asymmetry between odd and even i as in the Potts model. Thus

$$E_1(x \otimes y \otimes z...) = \sqrt{[M : N]} e_N(x) \otimes y \otimes z...$$

$$E_2(x \otimes y \otimes z...) = \sum_i xy\lambda_i \otimes \lambda_i^* \otimes z...$$

and so on, where λ_i is any "orthonormal basis" for M over N, i.e. $\sum_i \lambda_i e_N \lambda_i^* = 1$ in $< M, e_N >$. The existence of the λ_i is an easy consequence of the original work of Murray and von Neumann-see [33]. It is easy to show that the E_i's acting on $M_i = M \otimes_N M \otimes_N M \otimes_N ... \otimes_N M$ satisfy the TL relations 3.5.2,3.5.3 and 3.5.4.

In their first paper on subfactors, Pimsner and Popa discovered precisely the representation of the TL relations given by 3.5.5! Not long afterwards D. Evans noticed the connection with statistical mechanical models.

5.3 A speculation on fusion/entanglement/interaction.

I never fail to be struck by the relationship between the tensor product $M \otimes_N M$ and two interacting spins on the spin chain. Connes had earlier introduced the notion of a "correspondence" between von Neumann algebras M_1 and M_2 which is a Hilbert space $M_1 - M_2$ bimodule and had defined a surprisingly subtle notion of tensor product of bimodules-see [9] and [35]. Perhaps the kinematics for a system of two interacting quantum systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with some common observables given by a von Neumann algebra M is the Connes tensor product $\mathcal{H}_1 \otimes_M \mathcal{H}_2$?

If two quantum systems described by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ were so entangled that an an observable x on one were equivalent to an observable y on the other then if $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$ are vectors defining states, there should be no difference between $x\xi \otimes \eta$ and $\xi \otimes y\eta$. If moreover the identification of the x's with the y's were implemented by an antiisomorphism $\phi : M \to \phi(M)$ from some von Neumann algbebra M on $\mathcal{H}_2$ to a von Neumann algebra on $\mathcal{H}_1$ we would have a right action of M on $\mathcal{H}_1$ and there would be no difference between $\xi x \otimes \eta$ and $\xi \otimes x\eta$. We would be forced to take the Connes tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The behaviour of the Connes tensor product is very rich and contains the theory of subfactors. Thus one could account for physicists' assertions that the Hilbert space which describes several Chern Simons "particles" is not the tensor product but a more complicated structure. Thus this notion of fusion might be relevant for the systems proposed by Freedman et al. in [14] in connection with quantum computing. The approach of Wassermann in [38] to the fusion of loop group representations fits exactly into our framework and produces the right fusion algebra.

Note that this notion of fused systems is much stronger than the usual notion of entanglement where individual states of a combined system may be more or less entangled.

The Hilbert spaces of the Andrews-Baxter-Forester and other IRF models for n sites on a lattice furnish another example. The natural basis for these Hilbert spaces is a basis of paths and their dimensions are not simply powers of a given integer. This could be explained if one supposes that proximity on the lattice causes a large algebra of observables for one particle to be identified with observables for its neighbour.

The Connes tensor product is easy to describe in finite dimensinos. If the algebra M is the $n \times n$ matrices and it acts (unitally) on the right on a finite dimensional vector space V, V may be identified tiwh the $p \times n$ matrices for some p, the right action being matrix multiplication.

Similarly a left M-module structure on W means that W is isomorphic to the $n \times q$ matrices for some q. The tensor product $V \otimes_M W$ is then the $p \times q$ matrices. Direct sums behave in the obvious way so this is a complete description of the finite dimensional situation.

A physical setup realising these kinematics would have some surprising properties.

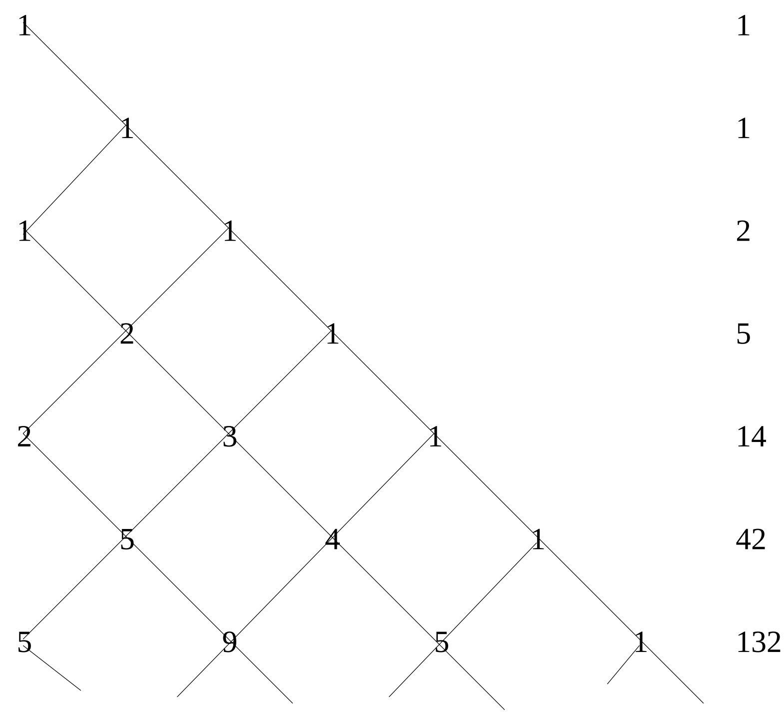
First of all the map $\xi \otimes \eta \mapsto \eta \otimes \xi$ does not pass to the Connes tensor product so fermionic or bosonic statistics would not make sense. It is known however that in many examples of systems of bimodules there is a unitary braiding which could be interpreted as playing the role of an exchange of particles.

Secondly, the only observables for $\mathcal{H}_1$ that pass to the Connes tensor product are those which commute with the fusing algebra M. In particular if the fusing algebra is non-commutative the only fusing observables that remain observable on the combined system are those in the centre of M.

Thirdly, if two sysems are so entangled that every observable of one were equivalent to an observable for the other then the Hilbert space for the fused system would collapse to a one-dimensional one. This is reminiscent of the Pauli exclusion principle.

5.4 Principal graphs.

In the subfactor context it is natural to ask about the algebra generated by the E_i's as above which actually has a von Neumann algebra structure. The algebra generated by the first n E_i's is finite dimensional and naturally included in the next one. A very visual way to describe inclusions of such finite dimensional algebras is by a "Bratteli diagram" ([7] which records the ranks of the minimal projections of the smaller algebra in the simple components of the bigger one. The tower of algebras of the E_i's as above has the Bratteli diagram below (for index ≥ 4).



The numbers on the diagram are the sizes of the matrix algebras which are the simple components, and the numbers to the right are the dimensions of the whole Temperley-Lieb algebra- the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

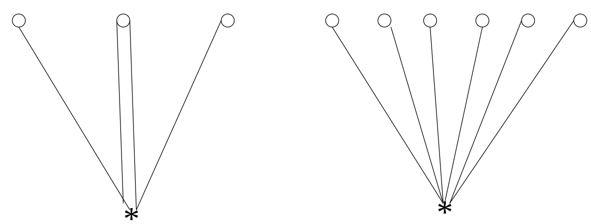
A provocative connection occurs here between this and section 1.2. For if we write $F_i = 1 + S_{i(i+1)}$ then these F_i satisfy the relations 3.5.2,3.5.3 and 3.5.4 for $Q = 4$. So that the algebra generated by the F_i should have (and indeed has for index ≥ 4) , the same Bratteli diagram as 5.4! Thus the decomposition of the tensor powers of the 2-dimensional representation of $SU(2)$ are also described by 5.4. We see that in fact 5.4 is redundant in the sense that its essential information is the graph A_∞ of 1.1. The numbers on the Bratteli diagram are just the number of loops on the graph A_∞ starting and ending at the leftmost vertex. These loops form a basis of the algebra that gives the Bratteli diagram.

But there is a lot more finite dimensional algebra inside the tower M_i. In fact the centralisers $N' \cap M_i = \{x \in M_i | xn = nx \quad \forall n \in N\}$ are all finite dimensional and obviously each one is included in the next. So they also have a Bratteli diagram which can be shown to have the same structure as that for the $E_i's$ - there is a graph Γ with a privileged vertex *, such that the algebra $N' \cap M_i$ is given by loops on Γ based at *. The graph Γ is called the principal graph. There is a duality between $N \subseteq M$ and $M \subseteq M_1$ and the principal graph of

$M \subseteq M_1$ is called the dual principal graph.

There are subfactors for which the principal and dual principal graphs are both A_∞. More interestingly perhaps the subfactors in [21] which give the index values $4\cos^2 \pi/n$ have principal graphs A_{n-1} (with one of the end points being *). We will return to this when we discuss finite "quantum" subgroups of $SU(n)$.

For another example choose an outer action of a finite group G on a II_1 factor M and let $N = M^G$, the subfactor of fixed points under the action of G. Then the principal and dual principal graphs Γ and $\check{\Gamma}$ of $M^G \subseteq M$ are as follows: Γ has a vertex * and a vertex for each irrep of G, with as many edges between * and the irrep as the dimension of the irrep, and no other edges. $\check{\Gamma}$ has a vertex * and one vertex for each element of G with an edge between * and each element of G, and no other edges. Thus for the symmetric group S_3 the graphs are as below:



6 Braid group representations.

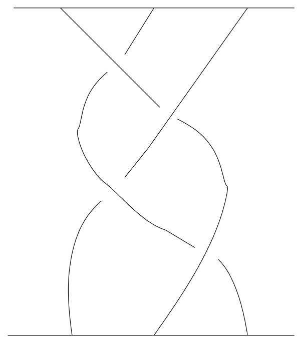
The braid group B_n is the finitely presented group with presentation

$$< \sigma_1, \sigma_2, ..., \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad \mathrm{i} = 1, 2, .., \mathrm{n}-2,$$

$$\text{and} \quad \sigma_\mathrm{i} \sigma_\mathrm{j} = \sigma_\mathrm{j} \sigma_\mathrm{i} \quad \text{for} \quad |\mathrm{i} - \mathrm{j}| \geq 2 >$$

If one puts $\lambda = \mu = \rho$ in the Yang Baxter equation 3.3.1 it is clear that one obtains a braid group representation by sending σ_i to $R_{i(i+1)}(\lambda)$ on $\otimes^n V$ where $R(\lambda)$ acts on $V \otimes V$, provided $R(\lambda)$ is invertible. This was first done in [23] for the matrix of 3.4.1. The resulting braid group representation is of considerable interest. It may well be faithful for generic q. This method of obtaining braid group representations was applied universally after the development of quantum groups. It was shown by Krammer and Bigelow ([26],[5]) that certain of the ensuing finite dimensional representations of the braid group are indeed faithful.

The main reason for the interest in the braid group is that it has geometric interpretations. First, according to its name, it is the group of all braids on n strings. A braid is a way of tying n points on a bar at the top to n points on a bar at the bottom, by strings whose tangent vector always has a non-zero vertical component. Thus the figure below represents a braid on 3 strings.



Alternatively, braids can be thought of as motions of n points in the plane. The position at any time t of the n points being determined by the intersection of a horizontal plane with y co-ordinate equal to t with the strings of the braid. In this way $B_n = \pi_1((\mathbb{C}^n \setminus \Delta)/S_n)$ where Δ is the set of n-tuples $(z_1, z_2, ..., z_n)$ with $z_i = z_j$ for some pair $i \neq k$ and the symmetric group S_n acts in the obvious way.

Knots and links can be formed from braids by tying the tops of the strings to the bottom. The figure-eight knot is obtained by doing this to the braid drawn above. It was realised in [23] that the trace coming from the subfactor origin of the braid group representation furnished an invariant of knots and links.

From the physics point of view the most interesting property of these representations is their unitarity. This is a difficult topic in general for the representations are not unitary for real positive values of q, even though there is a natural Hilbert space structure on the vector spaces on which they act. Fortunately this Hilbert space structure persists enough to supply, for fixed n, a small interval of q values (containing 1) on the circle for which the representation is unitary on the Hilbert space $\otimes^n V$ though the Hilbert space structure will fail if n is increased indefinitely and q is left fixed. For q a root of unity of the form $e^{\frac{2\pi i}{p}}$ there are other statistical mechanical models with Hilbert spaces for which the braid group representations are unitary for all n and fixed q. These are also the values of q for which subfactors of finite depth occur - see Wenzl - [39] and Xu -[42]. Subfactors can be constructed for positive real values of q as well. They are of infinite depth and have been analyzed by Sawin in [36].

7 Detective work

Hopefully the reader has been struck by common threads, notational and otherwise, connecting all the previous sections. They suggest a grand structure in which the formulae we have come up with are a small but significant part. We are not convinced that the last word has been said on this grand structure but for the case of the vertex models the notion of quantum group does the trick with great elegance and power.

Beginning with the decomposition of the tensor powers of the representations of $SU(2)$ we have presented operators that "deform" the permutation operators $S_{i(i+1)}$ of 1.2, the most general of which was the R-matrix 3.4.1. We have only hinted that the theory goes beyond $SU(2)$ but in fact Cherednik gave R-matrices that deform the representation of the symmetric group on the tensor powers of $\mathbb{C}^k$ for $k \geq 2$ and Wenzl independently discovered the same objects in the subfactor/braid group context in [40]. Thus it was natural to hope for an object that would "deform" $SU(k)$ in a way that Schur-Weyl duality would be preserved and the commutant of this object would be generated by the R-matrices. This was done by Jimbo and Drinfeld who were greatly inspired also by a vision of this procedure as a "quantisation" of the theory of integrable systems in Hamiltonian mechanics. See [11],[18] and [13].

There are many accounts of this work and we do not want to dwell on it as we are really interested here in cases where the quantum group formalism does not apply easily but which arise naturally in the subfactor world. Suffice it to say that the final result is the construction of R-matrix solutions (depending in various ways on the spectral parameter) for all (finite dimensional)simple Lie algebras and all representations thereof. In the statistical mechanics formalism the horizontal and vertical directions on the lattice may correspond to different representations of the Lie algebra. And there are extensions to affine Lie algebras.

Appropriate R-matrices can be evaluated at special values of the parameters so that they give braid group representations and all such representations are known to give link invariants where a representation of the Lie algebra can be assigned independently to each component of the link. The invariants are always polynomials in the quantum deformation parameter q. See [34]. The link invariants are powerful but many elementary questions remain unanswered. Perhaps the most galling of these is the question of whether the simplest of the invariants, corresponding to the Lie algebra $sl(2)$ and its 2-dimensional representation, detects knottedness, i.e. is there a non-trivial knot for which this polynomial is the same as for the unknot? The answer is known for links (see [12]) and all knots up to 17 crossings have been checked.

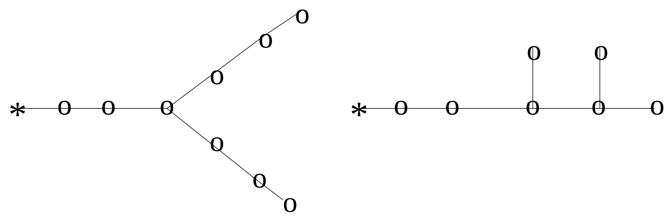
Kohno showed in [25] that the braid group representations coming from quantum groups could be obtained using only the data of the "classical" Yang-Baxter equation (as formulated by Drinfeld) which could be used to define a flat connection on configuration space $(\mathbb{C}^n \setminus \Delta)/S_n$.

In a remarkable demonstration that quantum groups were an idea whose time had come, a notion that was essentially the same was discovered simultaneously by Woronowicz(in [41]) working within operator algebras! He called the object a "compact matrix pseudogroup" and was motivated by duality considerations in the representation theory of groups and Hopf algebras. His ideas have proved extremely fruitful in the analytic side of the subject and have stimulated many discoveries in operator algebras. We have not presented the Woronowicz approach here as the connection with subfactors, while certainly present, is not as compelling.

8 The Haagerup and Haagerup-Asaeda subfactors.

In the light of the huge success of quantum groups and connections with conformal field theory and σ-models which we have not mentioned here, one is entitled to ask if all interesting theories are somehow obtainable from Lie groups and "geometry". This is an important question, particularly for subfactors whose utility would be brought into question if one could obtain all examples from structures outside subfactor theory. In this section and the next we offer two examples of how subfactors can produce examples independently of any geometric input, the first pioneered by Haagerup and the second by Ocneanu.

Haagerup asked the question: what is the (irreducible) subfactor of smallest index greater than 4 that occurs for a II_1 factor? In fact Popa has shown that any index value greater than 4 occurs but those examples have A_∞ as principal graph. So Haagerup's real question was to find the smallest index subfactor with principal graph different from A_∞. This he solved in spectacular fashion. He first showed in [16] that the smallest index value is $\frac{5+\sqrt{13}}{2}$ and identified the principal graph and dual principal graph as being one of the two below (which are dual to each other):



Then in [1] Haagerup and Asaeda showed that there is indeed a subfactor of the hyperfinite II_1 factor which has these graphs as dual principal graphs. They further showed that the next possible index value is $\frac{5+\sqrt{17}}{2}$, constructed the subfactor and calculated the principal graphs.

The techniques for eliminating other graphs and constructing the examples were highly calculatory, relying on Ocneanu's theory of connections. It is a major challenge in the theory to come up with an interpretation of these subfactors subfactors as members of a family related to some other mathematical objects. Izumi in [20] made some good progress in this direction. In [24] we have taken a diagrammatic approach and calculated certain parameters that show that the subfactors of indices $\frac{5+\sqrt{13}}{2}$ and $\frac{5+\sqrt{17}}{2}$ are of a very different kind.

9 Finite "quantum" subgroups of Lie groups.

Subfactors of index less than 4 have principal graphs equal to one of the A,D or E Coxeter-Dynkin diagrams. (Ocneanu showed that in fact only the diagrams D_{2n} and E_6 and E_8 can actually occur- and he constructed subfactors for each of these diagrams.) We saw in the first section how the extended Coxeter-Dynkin diagrams are connected with finite subgroups of $SU(2)$. There is also a subfactor connected with a finite subgroup of $SU(2)$ as follows: Construct a II_1 factor R as $\otimes^\infty M_2(\mathbb{C})$ (completed using the normalised trace to get a von Neumann algebra). The subfactor R_0 is the subalgebra of all elements of R of the form $id \otimes x$ where id is the 2×2 identity matrix. The group $SU(2)$ acts on R by the infinite tensor product of its action on $M_2(\mathbb{C})$ by conjugation. This action preserves the subfactor R_0. So for any subgroup G of $SU(2)$ one may form the subfactor $N \subseteq M$ of fixed points for G: $N = R_0^G$ and $M = R^G$. The principal graph for $N \subseteq M$ is then the extended Coxeter-Dynkin graph for the subgroup! The index is $[M : N] = 4$.

This suggests that one should be able to interpret the vertices of the principal graph as representations of something and the edges as induction/restriction. This is indeed possible and is inevitable in Connes' picture where bimodules over a II_1 factor replace representations of a group. The vertices of the principal graph are a certain family of $N-N$ bimodules and $N-M$ bimodules and the edges count induction/restriction multiplicities. This point of view was first pointed out by Ocneanu.

Putting together the index < 4 and index 4 cases above we see that it is natural to think of the index < 4 subfactors as being some kind of quantum version of subgroups of $SU(2)$ whose "representation theory" is a truncation of the representation theory of the corresponding genuine subgroup of $SU(2)$.

Cappelli, Itzykson and Zuber in [8] also ran into the $A-D-E$ Coxeter-Dynkin diagrams in their attempt to understand modular invariants in conformal field theory. In extending that work DiFrancesco and Zuber in [10] looked for truncations of the representation theory graphs of subgroups of $SU(3)$ that could give modular invariants. Zuber presented a list of such graphs satisfying certain criteria and conjectured that it was complete. Ocneanu used the subfactor point of view to exhibit the complete list for $SU(3)$ and beyond, with slight changes in the requirements for a graph to be on the list. Unfortunately the situation is a little too nice for $SU(2)$ because all its representations are equivalent to their conjugates. To do justice to the subfactor point of view would require a detour beyond the principal graphs so we simply refer to Ocneanu's paper [32] for those interested in this topic.

10 The direct relevance of subfactors to physics.

The connection between subfactors and physics that we have outlined above has been somewhat indirect, passing from certain elements in centraliser towers to quantum spin chains and/or statistical mechanical models. However probably von Neumann's main motivation for studying his algebras was because of their relevance to the quantum mechanical formalism. So one might hope for a more direct connection between subfactors and quantum physics. This does exist and is associated with many names. It goes back to the pioneering work of Haag and Kastler who sought to develop a non-perturbative framework for quantum field theory by taking as basic ingredients the algebras of observables localised in various regions of space-time.

I feel unable to give a satisfactory account of this theory and it is to be hoped that a book on it will appear in the near future. I will just say the following-a subfactor naturally appears by looking at the von Neumann algebras associated to certain regions of space-time. If two regions of space time are such that no light ray can connect them (they are not causally connected), then their von Neumann algebras of local observables should commute. These von Neumann algebras are known to be factors (of type III) so one may obtain a subfactor by taking N to be the algebra of observables localised in a region and M to be the commutant of all observables localised in the causal complement of that region.

Using this and other motivations, Wassermann and I looked for subfactors in the theory of loop group representations. A unitary (projective) representation of the group $LSU(2)$ of smooth functions from the circle to $SU(2)$ can be thought of as a one-dimensional quantum field theory whose currents are given by functions from the circle into the Lie algebra $su(2)$ and whose algebra of observables localised in an interval I of the circle is the von Neumann algebra generated by the normal subgroup $L_I SU(2)$ of loops supported in that interval. In this way we were led to the subfactor (where I^c is the interval complementary to I on the circle)

$$L_I SU(2)'' \subseteq L_{I^c} SU(2)'.$$

Wassermann subsequently showed in [38] that the set of indices of subfactors so obtained does indeed contain the set $\{4\cos^2 \pi/n\}$ and extended this work to $SU(n)$, the diffeomorphism group of the circle, and beyond.

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