# Quadratic Tangles in Planar Algebras

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19th October 2003

## 1 Introduction

A planar algebra $\mathcal{P}$ consists of vector spaces $P_{n,\pm}$ together with multilinear operations between them indexed by planar tangles-large discs with internal ("input") discs all connected up by non-intersecting curves called strings. Thus a planar algebra may be thought of as made up from generators $R_i \in P_{n,\pm}$ to which linear combinations of planar tangles may be appplied to obtain all elements of $\mathcal{P}$ .

It was shown in [] that a certain kind of planar algebra, called a subfactor planar algebra, is equivalent to the standard invariant of an extremal finite index subfactor. We define this notion carefully in section 2.3.1, after which the term "planar algebra" will mean subfactor planar algebra.

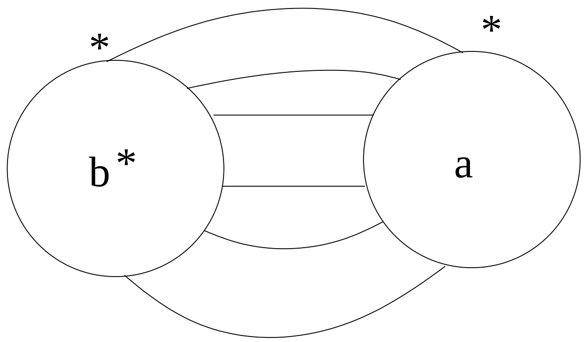
The simplest tangles are ones without input discs and they supply a quotient of the Temperley-Lieb (TL) algebra in any planar algebra. Tangles of the next level of complexity are the annular tangles first introduced in []. They form a category and planar algebras can be decomposed as a modules over the corresponding algebroid. This was begun in []. Note that annular tangles also appear in free probability theory-see [].

In the present paper we begin the much more difficult taks of studying tangles with two input discs which we call "quadratic tangles". They are not closed under any natural operation and one problem is to uncover their mathematical structure. To begin this we would need to list the quadratic tangles which is easy enough. However there is no guarantee that the elements of $\mathcal{P}$ obtained from different tangles will be linearly independent. Indeed even for the TL tangles this may not be so and it is precisely this linear dependence that gives rise to the discrete spectrum of the index for subfactors and

*Supported in part by NSF Grant DMS93–22675, the Marsden fund UOA520, and the Swiss National Science Foundation.

the existence of $SU(2)$ TQFT's ([],[]). Linear dependence of labelled tangles is most easily approached if there is a positive definite sesquilinear form given by the tangle depicted below (on $P_3^+$).

**Fig. 1.0.1.**




The dimension of the vector space spanned by a set of labelled tangles is then given by the rank of the matrix of inner products. In this way we were able to use the powerful results of [] to obtain the dimensions of the relevant Grahamlehrer
modules over the annular category in [],[] and [].

There are two main ideas in this paper:

(i) All inner products between quadratic labelled tangles are determined by closed tetrahedral tangles.

(ii) The number of tetrahedral configurations required is much less than one might think.

It is not hard to see why (i) is true by contemplating figure 1.0.1 when $a$ and $b$ are labelled quadratic tangles. To obtain (ii) we may suppose that the labels are lowest weight vectors of irreducible annular TL modules and that any connections between vertices of the tetrahedron with too many strings can be reduced by knowledge of how the labels multiply. We treat this in the section 3. For simplicity we limit ourselves to just one label and this is as far as the applications will go in this paper.

Given the tetrahedral structure constants it is possible, in principle, to calculate all inner products between quadratic tangles. We have not yet carried this beyond the simplest cases. The first step is to orthogonalise with respect to annular tangles. For this it would be most convenient to have an orthonormal basis of annular tangles that behaves well under the rotation. This is an interesting open problem which we solve only in an easy case. The dual basis to the diagrammatic basis of labelled annular tangles would also suffice but this seems quite hard to determine.

In section 6 we apply our ideas under certain hypotheses on the Poincaré series of $\mathcal{P}$ . The *critical depth* of $\mathcal{P}$ is the smallest $n$ for which $dimP_{n,\pm}$

is greater than $\frac{1}{n+1}\binom{2n}{n}$. If $dimP_{n,\pm} = \frac{1}{n+1}\binom{2n}{n} + 1$ we may choose an essentially unique $R \in P_{n,+}$ and look at quadratic tangles labelled by $R$. Such tangles in $P_{n+1,\pm}$ give lower bounds on the dimension of $P_{n+1,\pm}$ or alternatively knowledge of $dimP_{n+1,\pm}$ gives constraints on the tetrahedral structure constants.

In section 7 we treat concrete examples, using the results of 6 to obtain many obstructions for graphs to be principal graphs of subfactors, and some new information about the Haagerup and Haagerup-Asaeda subfactors of [], including a skein-theoretic presentation of the Haagerup planar algebra in the sense of [].

Throughout this paper we will use diagrams with a small number of strings to define tangles where the number of strings is arbitrary. This is a huge savings in notation and we shall strive to use enough strings so that the general situation is clear.

# 2 Planar Algebras

The definition of a planar algebra has evolved a bit since the original one in [] so we give a detailed definition which is, we hope, the ultimate one.

## 2.1 Planar Tangles.

**Definition 2.1.1.** *Planar k-tangles.*

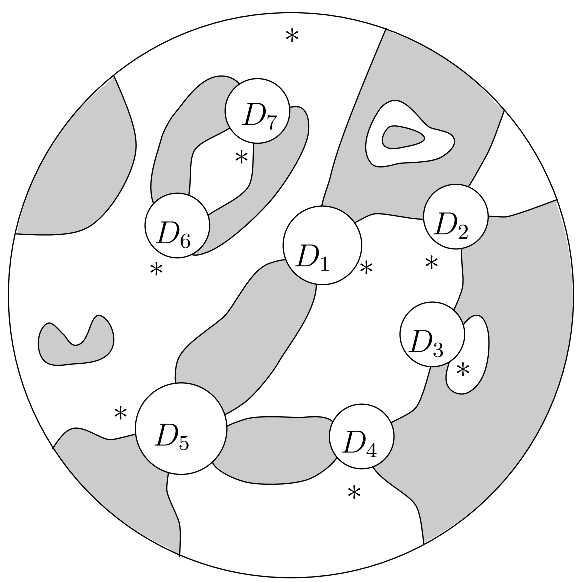
*A* planar $k$-tangle *will consist of the closed unit disc $D$ $(= D_0)$ in $\mathbb{C}$ together with a finite (possibly empty) set of disjoint smoothly embedded discs $D_1, D_2, \ldots, D_n$ in the interior of $D$. Each disc $D_i$, $i \geq 0$, will have an even number $2k_i \geq 0$ of marked points on its boundary (with $k = k_0$). Inside $D$ there is also a finite set of disjoint smoothly embedded curves called* strings *which are either closed curves or whose boundaries are marked points of the $D_i$'s. Each marked point is the boundary point of some string, which meets the boundary of the corresponding disc transversally. The strings all lie in the complement of the interiors $\overset{\circ}{D}_i$ of the $D_i$, $i \geq 0$. The connected components of the complement of the strings in $\overset{\circ}{D} \setminus \bigcup_{i=1}^{n} D_i$ are called* regions. *Those parts of the boundaries of the discs between adjacent marked points (and the whole boundary if there are no marked points) will be called* intervals. *The regions of the tangle will be shaded black and white so that two regions with a common* *boundary are shaded differently. (Such a shading is always possible.) The shading will be considered to extend to the intervals which are part of the boundary of a region. Finally, to each disc in a tangle there is a distinguished interval on its boundary(which may be shaded black or white).*

**Definition 2.1.2.** *The set of* internal *discs of a tangle $T$ will be denoted $\mathcal{D}_T$.*

**Remark 2.1.3.** Observe that diffeomorphisms of $D$ (which are not necessarily the identity on the boundary of $D$) act on tangles in the obvious way-if $\phi$ is such a diffeomorphism, which may be orientation reversing, and $I$ is the distinguished boundary interval of a disc $D_i$ in the tangle $T$, then $\phi(I)$ is the distinguished boundary interval of $\phi(D_i)$ in $\phi(T)$.

**Definition 2.1.4.** *The set of all planar $k$-tangles for $k > 0$ will be called $\mathcal{T}_k$. If the chosen interval of $D$ for $T \in \mathcal{T}$ is shaded white, $T$ will be called* positive *and if it is black, $T$ will be called* negative. *Thus $\mathcal{T}_k$ is the disjoint union of sets of positive and negative tangles: $\mathcal{T}_k = \mathcal{T}_{k,+} \sqcup \mathcal{T}_{k,-}$.*

We will often have to draw pictures of tangles. To indicate the distinguished interval on the boundary of a disc we will place a $*$, near to that disc, in the region whose boundary contains the distinguished interval. An example of a 4-tangle illustrating all the above ingredients is given below.

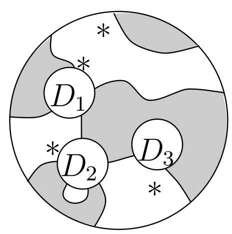


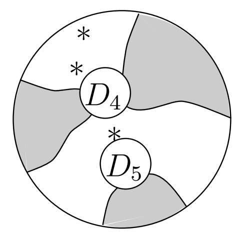

## 2.2 Operadic Structure

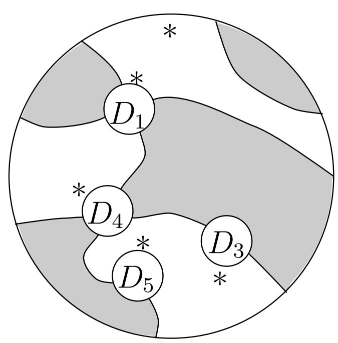
Planar tangles admit a partially defined "gluing" or "composition" operation which we now define.

Given a planar $k$ and $k'$-tangles $T$ and $S$ respectively, and a disk $D_i$ of $T$ with $k_i = k'$, and such that the distinguished boundary intervals of $D_i$ in $T$ and $D$ in $S$ have the same shading, we define the $k$-tangle $T \circ_i S$ by isotoping $S$ so that its boundary, together with the marked points, coincides with that of $D_i$, and the distinguished boundary interval for $D_i$ (in $T$) coincides with the distinguished boundary interval for $D$ in $S$. The strings may then be joined at the boundary of $D_i$ and smoothed. The boundary of $D_i$ is then removed to obtain the $k$-tangle $T \circ_i S$. This composed tangle depends on the isotopy used to move $S$ to fit inside $D_i$ and on the smoothing near the boundary of $D_i$, but its isotopy class depends only on the isotopy classes of $S$ and $T$. Note that if $k_i = 0$ the shading in $T \circ_i S$ is consistent across the boundary of $D_i$.

We exhibit the comoposition of tangles in the picture below.

$T =$ 


$S =$ 


$T \circ_2 S$ 


If we say that each disc in a tangle is "coloured" by the number of its marked boundary points and the shading of the distinguished boundary interval, then it seems natural to consider the object we have defined-the set of isotopy classes of planar tangles- to be a "coloured operad". Such a notion has properties in common with the notion of a category and that of an operad.

## 2.3 Planar Algebras

Before giving the formal definition of a planar algebra we recall the notion of the cartesian product of vector spaces over an index set $\mathcal{I}$, $\times_{i\in\mathcal{I}} V_i$. This is the set of functions $f$ from $\mathcal{I}$ to the union of the $V_i$ with $f(i) \in V_i$. Vector space operations are pointwise. Multilinearity is defined in the obvious way, and one convert multinearity into linearity in the usual way to obtain $\otimes_{i\in\mathcal{I}} V_i$, the tensor product indexed by $\mathcal{I}$. A cartesian product over the empty set will mean the scalars, which at this stage are an arbitrary field or even a commutative ring which we will call $K$.

**Definition 2.3.1.** *Planar algebra.*
*A (unital) planar algebra $\mathcal{P}$ will be a family of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces indexed by the set $\{\mathbb{N} \cup \{0\}\}$, where $P_{k,\pm}$ will denote the $\pm$ graded space indexed by $k$. To each planar tangle $T$ there will be a multilinear map*

$$Z_T : \times_{i\in\mathcal{D}_T} P_i \to P_{D_0}$$

*where $P_i$ is the vector space indexed by half the number of marked boundary points of $i$ and graded by $+$ if the distinguished interval of $i$ is shaded white and $-$ if it is shaded black.*

*The maps $Z_T$ are subject to the following two requirements:*

*(i) (Isotopy invariance) If $\varphi$ is an orientation preserving diffeomorphism of the unit disc then*

$$Z_T = Z_{\varphi(T)}$$

*where the sets of internal discs of $T$ and $\varphi(T)$ are identified using $\varphi$.*

*(ii)(Naturality)*

$$Z_{T\circ_i S} = Z_T \circ_i Z_S$$

*Where $i$ is an internal disc in $T$, and to define the right hand side of the equation, first observe that $\mathcal{D}_{T\circ_i S}$ is naturally identified with $(\mathcal{D}_T - \{i\}) \cup \mathcal{D}_S$ (using the isotopy and smoothing data required to define the tangle composition). Thus given a function $f$ on $\mathcal{D}_{T\circ_i S}$ to the appropriate vector spaces, we may define a function $\tilde{f}$ on $\mathcal{D}_T$ by*

$$\tilde{f}(D) = \begin{cases} f(D) & \text{if } D \neq i \\ Z_S(f|_{\mathcal{D}_S}) & \text{if } D = i \end{cases}$$

*Finally the formula $Z_T \circ_i Z_S(f) = Z_T(\tilde{f})$ defines the right hand side.*

**Remark 2.3.2.** *It is sometimes convenient to work with just one of $P_{n,+}$ and $P_{n,-}$. We agree on the convention that $P_n$ will mean $P_{n,+}$.*

For the convenience of the reader we give an example.

**Example 2.3.3.** *The Temperley Lieb algebra* $TL$.
Let $\delta$ be an arbitrary element of $K$. We will define a planar algebra $TL(\delta)$ or just $TL$ for short. We must first define the vector spaces $TL_{k,\pm}$. We let $TL_{k,+}$ be the vector space whose basis is the set of all isotopy classes of connected $k$-tangles with no internal discs, and for which the distinguished interval on the boundary is shaded white. (Here "connected" simply means as a subset of $\mathbb{C}$, i.e. there are no strings that are closed loops.) It is well known that the dimension of $TL_{k,+}$ is the Catalan number $\frac{1}{k+1}\binom{2k}{k}$. Similary define $TL_{k,-}$, requiring the disitinguished interval to be shaded black.

The definition of the maps $Z_T$ is transparent: a multilinear map is defined on basis elements so given a $k$-tangle $T$ it suffices to define a linear combination of tangles, given basis elements associated to each internal disc of $T$. But the basis elements are themselves tangles so they can be glued into their corresponding discs as in the composition of tangles. The resulting tangle will in general not be connected as some closed loops will appear in the gluing. Just remove the closed loops, each time multiplying the tangle by $\delta$, until the tangle is connected. This multiple $\tilde{T}$ of a basis element is the result of applying $Z_T$.

If the tangle $T$ has no internal tangles we must specify a linear map from $K$ to $TL$. Create $\tilde{T}$ from $T$ as above and set $Z_T(x) = x\tilde{T}$.

**Remark 2.3.4.** *In the example above there is a constant* $\delta \in K$ *with the property that the partition function of a tangle containing a closed contractible string is* $\delta$ *times that of the same tangle with the string removed. (In fact the string does not even have to be contractible but one may find examples in [] where contractibility is essential.) We shall call such a planar algebra a planar algebra with parameter* $\delta$.

## 2.4 Labelled Tangles

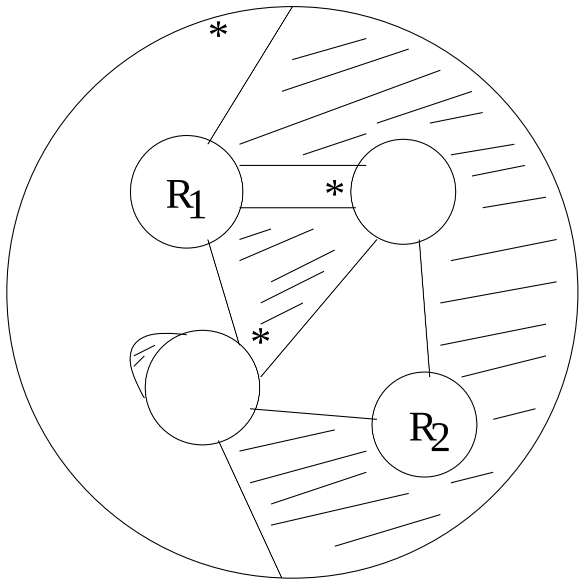
If $\mathcal{P}$ is a planar algebra and $T$ a tangle, and we are given elements $x_i \in P_i$ for $i$'s in some subset $S$ of the internal discs $\mathcal{D}_T$ of $T$ (with $P_i$ as in 2.3.1), we form the "labelled tangle" $T_{x_i}$ by writing each $x_i$ in its $i$ and then forming the linear map

$$Z_{T_{x_i}} : \bigotimes_{D' \in \mathcal{D}_T \setminus S} P_{D'} \to P_{D_0}$$

in the obvious way.

Here is an example of a labelled tangle which defines a map from $P_{2,+} \otimes P_{3,-}$ to $P_{1,+}$.

**Fig. 2.4.1.**




We will say that $T$ is fully labelled if $S = \mathcal{D}_T$ in the above.

The special case where $dim P_{0,\pm} = 1$ is common. In this case we will typically leave out the external boundary disc from the diagram-see for example 1.0.1. We will also leave out the shading when it is defined by other knowledge. For instance in 1.0.1, since we know that $a$ and $b$ are in $P_{3,+}$ the shading is defined by the distinguished intervals indicated with $*$'s.

Spherical invariance is most easily expressed in terms of labelled tangles. We say that a planar algebra $\mathcal{P}$ is spherical if $dim P_{0,\pm} = 1$ and the partition function of any fully labelled 0-tangle is invariant under spherical isotopy. Note that spherical isotopies can pass from tangles with the outside region shaded to ones where it is unshaded.

We will sometimmes be a little sloppy by confusing a fully labelled tangle with its partition function.

## 2.5 Star Structure

Another natural operation on planar tangles will be crucial in $C^*$-algebra considerations.

**Definition 2.5.1.** *If $T$ is a planar $k$-tangle we define the* conjugate *tangle $\bar{T}$ by applying complex conjugation to $T$ itself. The distinguished intervals on the discs of $\bar{T}$ are defined to be the images under complex conjugation of the original ones.*

**Proposition 2.5.2.** *If $T$ and $S$ are isotopic tangles then so are $\bar{T}$ and $\bar{S}$.*

Proof. If $\phi$ is an orientation preserving diffeomorphism of $D$ with $\phi(T) = S$ then the conjugate of $\phi$ by complex conjugation is orientation preserving and maps $\bar{T}$ to $\bar{S}$. $\square$

Thus the conjugate operation on tangles passes to isotopy classes. Since the compostion of two orientation reversing diffeomorphisms preserves orientation, any orientation reversing diffeomorphism could be used in place of complex conjugation to give the same operation on istotopy classes of tangles.

**Definition 2.5.3.** *A planar *-algebra will be a planar algebra $\mathcal{P}$ over $\mathbb{C}$ with a conjutate-linear involution $*$ on each $P_{n,\pm}$ such that for any tangle $T$,*

$$Z_{\overline{T}}(x_1^*, x_2^*, ..., x_k^*) = Z_T(x_1, x_2, ..., x_k)^*.$$

The Temperley-Lieb planar algebra is a planar *-algebra if $\delta \in \mathbb{R}$ and the involution on each $P_{n,\pm}$ is the conjugate-linear extension of conjugation of tangles.

## 2.6 Subfactor planar algebras.

**Definition 2.6.1.** *A subfactor planar algebra $\mathcal{P}$ will be a spherical planar *-algebra with $dim P_{n,\pm} < \infty$ for all $n$ and such that the inner product defined by figure 1.0.1 is positive definite for all $n$ and grading $\pm$.*

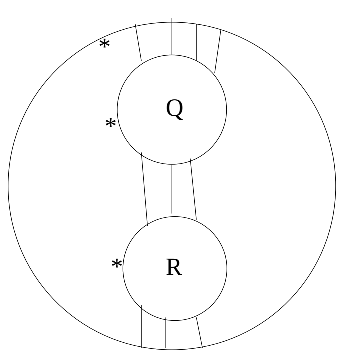
It was shown in [] that a subfactor planar algebra is the same thing as the standard invariant of a finite index extremal subfactor of a type $\mathrm{II}_1$ factor.

Here are some well known facts concerning subfactor planar algebras.

(i) The parameter $\delta$ is $> 0$.

(ii) Algebra structures on $P_{n,\pm}$ are given by the following tangle (with both choices of shading).

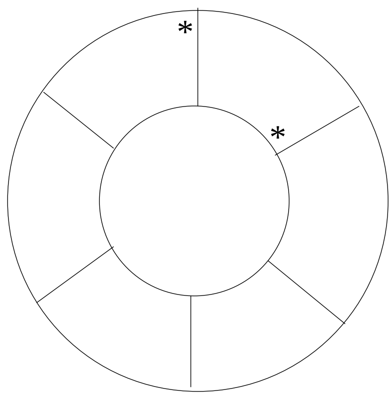
RQ =



(iii) There is a pair of pointed bipartite graphs, called the principal graphs $\Gamma, *$ and $\Gamma', *$ such that $P_{n,+}$ and $P_{n,-}$ have bases indexed by the loops of length $2n$ based at $*$ on $\Gamma$ and $\Gamma'$ respectively. The multiplication of these basis elements is easily defined using the first half of the first loop and the second half of the second, assuming the second half of the first is equal to the first half of the second (otherwise the answer is zero).

(iv) The rotation tangles below (with both choices of shading) give vector space isomporphisms between $P_{n,+}$ and $P_{n,-}$ (for $n > 0$) called the "Fourier transforms".

**Fig. 2.6.2.**

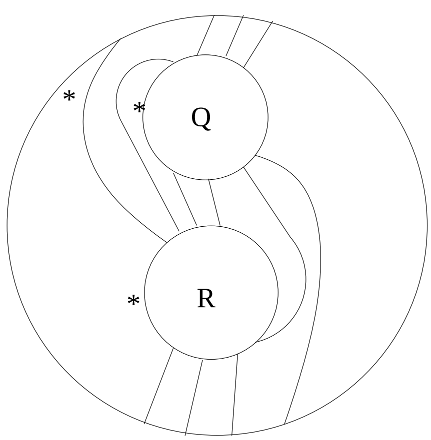



In particular one may pull back the multiplication on $P_{n,-}$ to $P_{n,+}$. If $n$ is odd the two algebra structures are naturally *-anti-isomorphic using the $n$th. power of the Fourier transform tangle $\mathcal{F}$. We record the tangle defining this multiplication for which we use the wedge symbol:

changed the direction of Fourier-picture below probably wrong

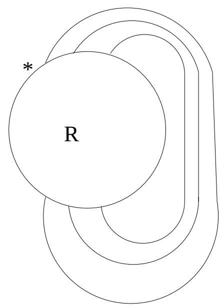
$$R \wedge Q = \mathcal{F}^{-1}(\mathcal{F}(R)\mathcal{F}(Q))$$

$R \wedge Q =$



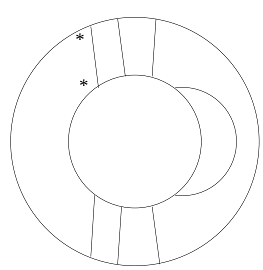
(v) $\Gamma$ is finite iff $\Gamma'$ is in which case $\delta$ is the norm of the adjacency matrix of $\Gamma$ and $\Gamma'$ and the subfactor/planar algebra is said to have finite depth.

(vi) There is a trace $Tr$ on each $P_{n,\pm}$ defined by the following $0-tangle$:

$Tr(R) =$ 


The algebra $P_{n,\pm}$ is semisimple over $\mathbb{C}$ and its simple components are matrix algebras indexed by the vertices of the principal graph at distance $n$ from $*$. The trace $Tr$ is thus given by assigning a "weight" by which the usual matrix trace must be multiplied in each simple component. This multiple is given by the Perron-Frobenius eigenvector (thought of as a function on the vertices) of the adjaceny matrix of the principal graph, normalised so that the value at $*$ is 1. Because of the bipartite structure care may be needed in computing this eigenvector. To be sure, one defines $\Lambda$ to be the (possibly non-square) bipartite adjacency matrix and constructs the eigenvector for the adjacency matrix $\begin{pmatrix} 0 & \Lambda \\ \Lambda^t & 0 \end{pmatrix}$ as $\begin{pmatrix} \Lambda v \\ \delta v \end{pmatrix}$ where $v$ is the Perron Frobenius eigenvector (of eigenvalue $\delta^2$) of $\Lambda^t\Lambda$.

(vii) There are "partial traces" or "conditional expectations" the simplest of which is $\mathcal{E}$, the map from $P_{n,\pm}$ to $P_{n-1,\pm}$ defined by the following tangle:

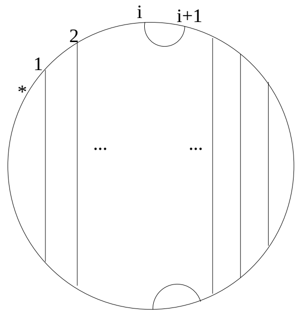
**Fig. 2.6.3.** $\mathcal{E} =$ 


It is obvious that

$$Tr(\mathcal{E}(x)) = Tr(x).$$

(viii) The TL diagrams span a subalgebra of $P_{n,+}$ and $P_{n,-}$ which we will call TL for short. We will call $E_i$, for $1 \leq i \leq n-1$ the element of $P_{n,\pm}$ defined by the tangle below:

**Fig. 2.6.4.**




The algebra generated by the $E_i$ is a 2-sided ideal in TL. Its identity is the JW projection $p_n$ which is uniquely defined up to a scalar and the property

**2.6.5.** $E_i p_n = p_n E_i = 0 \quad \forall i$

One has the formula

**2.6.6.** $Tr(p_n) = [n+1]$

where $\delta = q + q^{-1}$ and $[r]$ is the quantum integer $\dfrac{q^r - q^{-r}}{q - q^{-1}}$.

# 3 The tetrahedral structure constants.

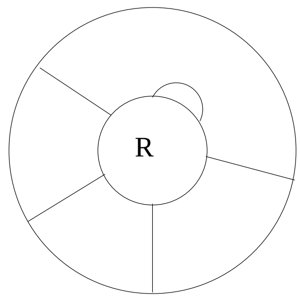
## 3.1 Lowest weight vectors.

We would like to organise the values of labelled 0-tangles with four input discs. For simplicity we will assume that all the labels are the same and that the label is a self-adjoint lowest weight vector for an annular TL-module as in []. To be precise, in this section $R$ will be an element of $P_n$ so that

(i) $R^* = R$.

(ii)

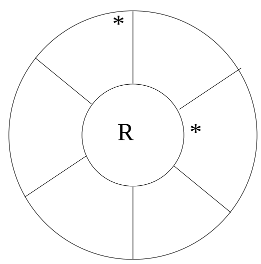
**Fig. 3.1.1.**




$= 0$ for any positions of $*$.

(iii)

**Fig. 3.1.2.**




$= \omega R$

for some $n$th. root of unity $\omega$.

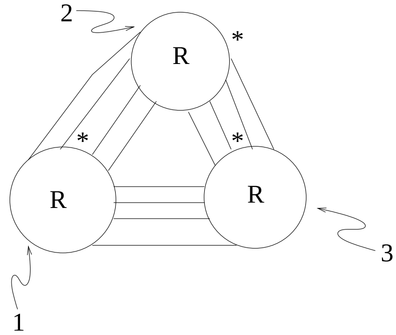
The diagram is a little ambiguous and confusing to apply so let us say verbally what it means: "if you see an input disc $D$ labelled $R$, then the tangle is $\omega$ times the same tangle but with the $*$ of $D$ rotated counterclockwise by 2".

As a warm up to the tetrahedral tangles we will do the triangular case.

## 3.2 Triangular structure constants.

**Definition 3.2.1.** *Let $a_1, a_2$ and $a_3$ be integers between 0 and $n$, all equal mod 2. $\Delta(a_1, a_2, a_3)$ will be the value of the partition function of the labelled tangle of fig 3.2.2 below, where the distinguished interval for the discs marked $1, 2$ and $3$ are the $a_1$th., $a_2$th. and $a_3$th. from the outside in the clockwise direction. (So that $a_1 = 2$,$a_2 = 0$ and $a_3 = 6$ in 3.2.2.)*

**Fig. 3.2.2.** $\Delta(a_1, a_2, a_3) =$



**Proposition 3.2.3.** *Any* $0$*-tangle with* $3$ *input discs all labelled with* $R$ *is zero or isotopic on the* $2$*-sphere to a tangle as in figure 3.2.2.*

*Proof.* If there are not the same number $n$ of strings connecting each pair of the three discs, then for at least one of the discs a pair of its boundary points must be connected to each other. By a spherical istopy those points may be assumed adjacent so the tangle is zero by 3.1.1. Once there are $n$ strings between each pair, spherical isotopies can be used to arrange the strings exactly as in figure 3.2.2. □

**Proposition 3.2.4.** *The triangular structure constants enjoy the following symmetries (with all indices mod* $2n$*):*

*(i) If* $b_1, b_2$ *and* $b_3$ *are integers between* $0$ *and* $n$ *then*

$$\Delta(a_1 + 2b_1, a_2 + 2b_2, a_3 + 2b_3) = \omega^{b_1+b_2+b_3}\Delta(a_1, a_2, a_3)$$

*(ii)*

$$\Delta(a_1, a_3, a_2) = \omega^{-(a_1+a_2+a_3)}\overline{\Delta(a_1, a_2, a_3)}$$

*Proof.* The first property is an immediate consequence of 3.1.2. To see the second, reflect figure 3.2.2 in a straight line through the disc numbered 1 and half way between the discs numbered 2 and 3. Assume for simplicity that the $a_i$'s are all even. To send the *'s of each disc to the outside will then require $a_1 + a_2 + a_3$ clockwise rotations. Then to send them to their original positions will require $a_1 + a_2 + a_3$ more. Each two clockwise rotations count a factor of $\omega^{-1}$. Since $R = R^*$ we are done by 2.5.3. □

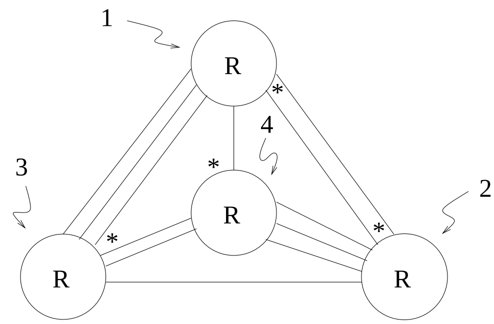
**Proposition 3.2.5.** $\Delta(0, 0, 0) = Tr(R^3)$ *and* $\Delta(1, 1, 1) = Tr(\mathcal{F}(R)^3)$*.*

*Proof.* Inspection of the tangles. □

## 3.3 Tetrahedral structure constants.

Figure 3.3.1 illustrates a fully labelled tangle with four input discs all labelled by the same lowest weight element $R \in P_n$. of a planar algebra. There are $n_1$ strings between the disc numbered 1 and the disc numbered 2, $n_2$ between disc 2 and disc 4 and $n_3$ between disc 3 and disc 1. (So that $n_1 = 2$, $n_2 = 1$ and $n_3 = 3$ in the example.) The distinguished intervals marked with a * will be $a_1, a_2, a_3$ intervals clockwise around from the outside for discs $1, 2$ and 3 and, for disc 4, $a_4$ intervals clockwise from the interval in the region that touches discs 2,3 and 4. The first 3 $a_i$ are necessarily the same mod 2 but $a_4$ will be the same or different mod 2 if $n_2$ is even or odd respectively. In figure 3.3.1 $a_1 = 1, a_2 = 5, a_3 = 3$ and $a_4 = 2$.

**Fig. 3.3.1.**

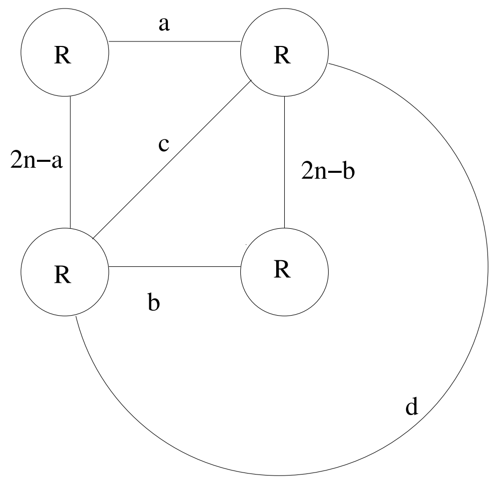



**Proposition 3.3.2.** *If $T$ is a tangle as in fig. 3.3.1 then there are $n_1$ strings connecting discs 3 and 4, $n_2$ strings connecting discs 1 and 4 and $n_3$ strings connecting discs 2 and 4, So that $n_1 + n_2 + n_3 = 2n$.*

*Proof.* This follows immediately from the fact that each disc has $2n$ strings attached to it. □

**Lemma 3.3.3.** *Let $T$ be a connected fully labelled $0-$tangle with 4 input discs all labelled by the same element $R$. Then $T$ is either zero or isotopic on the 2-sphere to a tangle as in fig 3.3.1*

*Proof.* We may suppose that there are no strings connecting a disc to itself or the partition function would be zero. First choose one of the discs of $T$ and use spherical isotopies if necessary to place that disc in the position of disc 1 in 3.3.1 so that a ray leaving that disc and going vertically upwards does not intersect any strings of the tangle. The first string around disc 1 after the ray in clockwise order is connected to another disc which can be moved to the place of disc 2. If any strings in counterclockwise order from the ray on disc 1 are attached to disc 2 then use a spherical isotopy to make them the first strings attached to disc 1. Let $n_1$ be the number of consecutive strings attaching disc 1 to disc 2 starting from the first. The disc attached to the last string on disc 1 is attached to a different disc than disc 2 since the diagram is connected. Move that disc to the position of disc 3 in 3.3.1 and let $n_3$ be the number of strings consecutively counterclockwise connecting disc 1 to disc 3. We claim that all the strings around disc 1, between the first $n_1$ and the last $n_3$ (if there are any) are connected to the fourth disc. If so that disc can be moved to disc 3 in 3.3.1 abd we are done. In fact there are patterns to be excluded here which could not be excluded if the four input discs of $T$ had different numbers of boundary points. The main thing to exclude is the configuration shown below and other isotopic versions of it.




where the number of strings connecting discs has been marked on a single string. Adding up the number of strings connected to discs 2 and 4 we obtain a contradiction unless $c = d = 0$ in which case the configuration is as in 3.3.1 with $n_2 = 0$.

We leave the rest of the details to the reader. □

**Definition 3.3.4.** *Let $n_1, n_2$ and $n_3$ be non-negative integers adding up to $2n$ amd let $a_1, a_2, a_3, a_4$ be integers between 0 and $2n-1$ with $a_1, a_2, a_3$ and $a_4 + n_1$ all equal mod 2. Then*

$$\Gamma\begin{pmatrix} n_1 & n_2 & n_3 & \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

*is the partiton function of the labelled tangle of 3.3.1 with these parameters.*

**Proposition 3.3.5.** *If $b_1, b_2, b_3$ and $b_4$ are integers between 0 and $n$ then*

$$\Gamma\begin{pmatrix} n_1 & n_2 & n_3 & \\ a_1+b_1 & a_2+b_2 & a_3+b_3 & a_4+b_4 \end{pmatrix} = \omega^{b_1+b_2+b_3+b_4}\Gamma\begin{pmatrix} n_1 & n_2 & n_3 & \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}.$$

*Proof.* Use 3.1.2. □

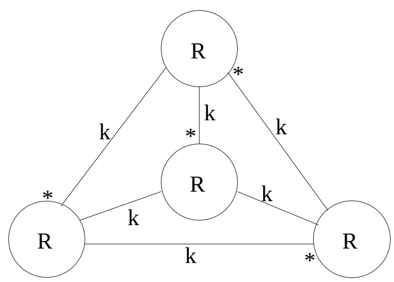
The whole symmetric group $S_4$ acts on the $\Gamma$ symbols. The rotation symmetries form the subgroup $A_4$ of $S_4$ and reflections in various planes give orientation-reversing maps. By 3.3.2 the numbers $n_i$ are the same on opposite edges of the tetrahedron so are in fact associated to the three diagonals of the tetrahedron which are naturally acted on by the symmetry group. So if $\sigma$ is an element of $S_4$ it makes sense to consider the symbol $\Gamma\begin{pmatrix} n_{\sigma(1)} & n_{\sigma(2)} & n_{\sigma(3)} & \\ a_{\sigma(1)} & a_{\sigma(2)} & a_{\sigma(3)} & a_{\sigma(4)} \end{pmatrix}$ which is of course equal to $\Gamma\begin{pmatrix} n_1 & n_2 & n_3 & \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$ if $\sigma$ is in $A_4$. In the event that all the $n_i$ are the same (and hence all equal to $k = \frac{2n}{3}$) we have:

**Proposition 3.3.6.** *If $\sigma \in A_4$,*

$$\Gamma\begin{pmatrix} k & k & k & \\ a_{\sigma(1)} & a_{\sigma(2)} & a_{\sigma(3)} & a_{\sigma(4)} \end{pmatrix} = \omega^{[\sigma]}\Gamma\begin{pmatrix} k & k & k & \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

*where $[\sigma]$ is the element of $\mathbb{Z}/3\mathbb{Z}$ given by the abelianisation of $A_4$.*

*Proof.* It is obvious that the two tetrahedral symbols are equal up to some power of $\omega$. The first observation is that the power of $\omega$ does not depend on the $a_i$'s. For this one may put the tetrahedron in some standard form and see that the effect of changing one of the $a_i's$ has the same effect on the tetrahedral symbols before and after applying $\sigma$. To calculate the multiplicative factor observe that it must be a group homomorphism and can be obtained easily for the obvious rotation of the following tangle:




□

It is an interesting corollary that if $\omega$ is not a $k/2$th. root of unity then the tetrahedral symbols of the previous proposition are zero.

The following proposition, together with the previous results, allows one to calculate the effect of any orientation reversing symmetry on the tetrahedral symbols.

**Proposition 3.3.7.**

$$\Gamma\begin{pmatrix} n_3 & n_2 & n_1 & \\ 0 & 0 & 0 & 0 \end{pmatrix} = \overline{\Gamma\begin{pmatrix} n_1 & n_2 & n_3 & \\ 0 & 0 & 0 & 0 \end{pmatrix}}.$$

*Proof.* Just reflect 3.3.1 in a vertical line. □

Some tetrahedral symbols are determined by known structure. Here are some values that will be useful.

**Proposition 3.3.8.** *We have*

$$(i) \qquad \Gamma\begin{pmatrix} n & 0 & n & \\ 0 & 0 & 0 & 0 \end{pmatrix} = Tr(R^4)$$

*(ii)* $\Gamma \begin{pmatrix} n & 0 & n & \\ 1 & 1 & 1 & 1 \end{pmatrix} = Tr(\mathcal{F}(R)^4)$

*(iii)* $\Gamma \begin{pmatrix} n-1 & 0 & n+1 & \\ 0 & 0 & 0 & 0 \end{pmatrix} = Tr(\mathcal{E}(R^2)^2)$

*Proof.* Just draw the pictures. □

It is clear that if $R^2$ is a linear combination of $R$ and a TL tangle then any tetrahedral symbol with one of the $n_i$ at least $n$ can be evaluated in terms of triangular symbols. We will evaluate some of these when we have more information on $R^2$. In the meantime we introduce the following notion, which is not essential to most of this paper.

**Definition 3.3.9.** *A tangle $T$ will be called essential if no internal disc is connected to itself and no pair $D_1$ and $D_2$ of internal discs with $n_1$ and $n_2$ distinguished boundary poinst is connected by more than* $\min(n_1/2, n_2/2)$ *strings. A tetrahedral symbol* $\Gamma \begin{pmatrix} n_1 & n_2 & n_3 & \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$ *is called essential if $n_1, n_2$ and $n_3$ are all less than $n$.*

The first essential tetrahedral symbol is when $n = 3$ and all the $n_i$ are equal to 2. Curiously, the first planar algebra for which this arises is the one associated with the Coxeter-Dynkin diagram $E_6$ which is sometimes known as the quantum tetrahedron... We will see more essential symbols for the Haagerup subfactor, for which $n = 4$.

# 4 Supertransitive subfactors.

Transitivity of a group action on a set $X$ is measured by the number of orbits on the Cartesian powers $X, X^2, X^3$, etc. The smallest number of orbits is attained for the full symmetric group $S_X$ and we say the group action is $k$-transitive if it has the same number of orbits on $X^k$ as does $S_X$. Transitivity can be further quantified by the number of orbits that an $S_X$ orbit breaks into on $X^k$ when the action fails to be $k-transitive$. There is a planar algebra $\mathcal{P}$ associated to a group action on $X$ for which $dim P_k$ is the number of orbits on $X^k$. The planar algebra contains a copy of the planar algebra for $S_X$ and the action is $k$-transitive if $P_k$ is no bigger than the symmetric group planar algebra. Thus the symmetric group planar algebra is universal in this situation. We will call it the *partition* planar algebra. It depends on $X$ of course but only through $\#(X)$. Other planar algebras may be universal for other situations. The Fuss-Catalan subalgebra is such an example for subfactors which are not maximal. Another one, sometimes called the string algebra, is universal for the Wassermann subfactors for representations of compact groups. Bhattacharya discovered another one which is universal for bipermutation matrix planar algbebras (see []). But the truly universal planar algebra in this regard is the TL planar algebra (a quotient of)which is contained in any planar algebra. Motivated by this discussion we will define a notion of *supertransitivity* measured by how small the algebra is compared to its TL subalgebra.

## 4.1 The partition planar algebra.

In [] we defined a planar algebra $\mathcal{P}$ , called the "spin model planar algebra" associated to a vector space $V$ of dimension $k$ with a fixed basis numbered $1, 2, ..., k$. For $n > 0$ $P_{n,\pm}$ is $\otimes^n V$ and $P_{0,+} = V$, $P_{0,-} = \mathbb{C}$. The planar operad acts by representing an element of $\otimes^n V$ as a tensor with $n$ indices (with respect to the given basis). The indices are associated with the shaded regions and summed over internal shaded regions in a tangle. There is also a subtle factor in the partition function coming from the curvature along the strings. This factor is only necessary to make a closed string count $\sqrt{k}$ independently of how it is shaded whereas without this factor a closed string would count $k$ if the region inside it is shaded and 1 otherwise. For more details see []. Nothing in the planar algebra structure differentiates between the basis vectors so the symmetric group $S_k$ acts on $P$ by planar *-algebra automorphisms.

**Definition 4.1.1.** *If $G$ is a group acting on the set $\{1, 2, ..., k\}$ we define $P^G$ to be the fixed point subalgebra of the spin model planar algebra under the action of $G$. The partition planar algebra $\mathcal{C} = \{C_{n,\pm} | n = 0, 1, 2, ...\}$ is $P^{S_k}$.*

**Remark 4.1.2.** If $G$ acts transitively, passing to the fixed point algebra makes $dim C_{0,+} = dim C_{0,-} = 1$ and spherical invariance of the partition function is clear, as is positive definiteness of the inner product. So $P^G$ is a subfactor planar algebra. The subfactor it comes from is the "group-subgroup" subfactor- choose an outer action of $G$ on a $\text{II}_1$ factor $M$ and consider the

subfactor $M^G \subseteq M^H$ where $H$ is the stabilizer of a point in $\{1, 2, ..., k\}$ .

**Proposition 4.1.3.** *The action of $G$ is $r$-transitive iff $P_r^G = C_r$.*

*Proof.* By definition the dimension of $P_{r,\pm}^G$ is the number of orbits for the action of $G$ on $\{1, 2, ..., k\}^r$. ☐

## 4.2 Supertransitivity, Excess.

**Definition 4.2.1.** *The planar algebra $P$ will be called* *r-supertransitive* *if* $P_{r,\pm} = TL_{r,\pm}$.

**Example 4.2.2.** *The $D_{2n}$ planar algebra with $\delta = 2\cos\pi/(2n-2)$ is r-supertransitive for $r < 2n-2$ but not for $r = 2n-2$.*

The following question indicates just how little we know about subfactors. Analogy with the Mathieu groups suggests that the answer could be interesting indeed.

**Question 4.2.3.** *For each $r > 0$ is there a planar algebra (subfactor) which is $r$-transitive but not equal to $TL$ (for $\delta > 2$)?*

**Example 4.2.4.** *The partition planar algebra $\mathcal{C}$ is $3-$supertransitive but not 4-supertransitve.*

This is because there are just 5 orbits of $S_X$ on $X^3$ but 15 on $X^4$ (at least for $\#(X) > 3$).

To further quantify the transitivity of a group action one might say that a group action on $X$ has *$k$-excess $p$* if there are $p$ more orbits on $X^k$ than there are for the whole symmetric group. (Thus the action would have $k$-excess 0 iff it is $k$-transitive.

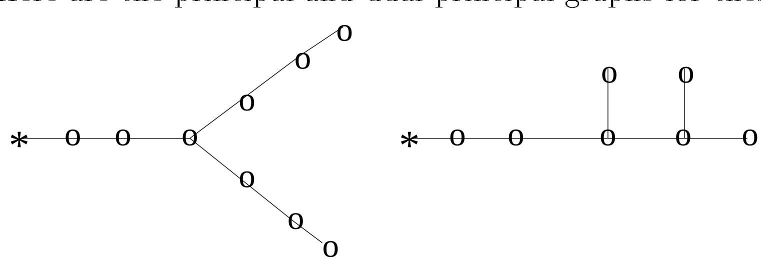
**Definition 4.2.5.** *A planar algbera $\mathcal{P}$ has $k$-*excess *$p$ if the codimension of $TL_{k,\pm}$ in $P_{k,\pm}$ is $p$.*

Thus $k$-supertransitive means that the $k$-excess is zero.

**Example 4.2.6.** *The $D_{2n}$ planar algebra with $\delta = 2\cos\pi/(2n-2)$ has $2n-2$-excess equal to one.*

**Example 4.2.7.** *The Haagerup subfactors of index $\frac{5+\sqrt{13}}{2}$ have 4-excess equal to one.*

Here are the principal and dual principal graphs for these examples.

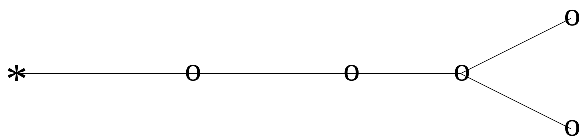


The weights of the trace are all easy to deduce except at $a, b$ and $c$ where they are $\delta^2 - 2$, $\delta$ and $\delta^2$. The other weights can be determined from these by symmetry and the fact that they must be the same as the TL weights up to and including the first branching point on the graph (i.e. as long as supertransitivity holds).

**Example 4.2.8.** *The partition planar algebra has* $4-$*excess equal to one.*

This is equivalent to the statement that the principal graphs for $\mathcal{C}$ is as below for distance $\leq 4$ from $*$.

**Fig. 4.2.9.**




For reasons that will become apparent we would like to calculate both principal graphs for $\mathcal{C}$ (for $k > 4$) to distance 5 from $*$, together with the weights of the trace $Tr$. Our method will be a bit ad hoc but adapted to the needs of this paper. Let $e$ and $f$ be the central projections corresponding to the rightmost points of the principal graph. Our first step will be to calculate $\alpha = Tr(e)$ and $\beta = Tr(f)$. By the principal graph we know that

$$\mathcal{C}_{4,+} \cong M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus e\mathbb{C} \oplus f\mathbb{C}$$

and that the trace of a minimal projection in $M_2(\mathbb{C})$ is 1, that of a minimal projection in $M_3(\mathbb{C})$ is $\delta^2 - 1$ and that

**4.2.10.** $$\alpha + \beta = \delta^4 - 3\delta^2 + 1$$

To obtain another relation on $\alpha$ and $\beta$ one may proceed as follows. By [] Symmgroup and potts model
$\mathcal{C}_{4,+}$ is spanned by TL and the flip transposition $S$ on $V \otimes V$. With attention to normalisation one gets $Tr(S) = \delta^2 (= k)$. But in the 2-dimensional representation $S$ is the identity and in the 3-dimensional one it fixes one basis element and exchanges the other two so it has trace 1. We may assume $eS = e$ and $fS = -f$ for if both reductions had the same sign $S$ would be in TL. Thus

**4.2.11.** $$\alpha - \beta + 2 + (\delta^2 - 1) = \delta^2$$

Combining this with 4.2.10 we obtain

**4.2.12.** $$\alpha = \frac{\delta^4 - 3\delta^2}{2} \qquad \beta = \frac{\delta^4 - 3\delta^2 + 2}{2}$$

We may now repeat the argument for $\mathcal{C}_{4,-}$. In this case the trace is unchanged but $S$ is in fact a projection which is the identity in the 2-dimensional representation and zero in the 3-dimensional one. We can suppose $Sp = p$ and so that the equation corresponding to 4.2.11 becomes simply

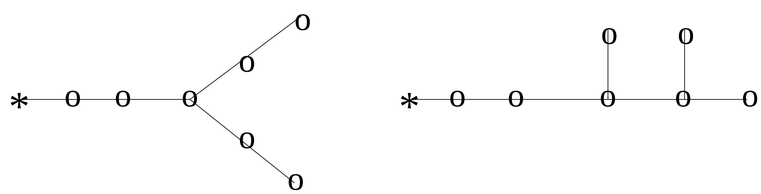
$$2 + \alpha = \delta^2.$$

Combining this with 4.2.10 we obtain

**4.2.13.** $$\alpha = \delta^2 - 2 \qquad \beta = \delta^4 - 4\delta^2 + 3$$

Note that $e, f, \alpha$ and $\beta$ in the second part of the argument are quite different from in the first since they are in different algebras.

Now we can deduce the algbebra structures of $\mathcal{C}_{4,\pm}$. By counting orbits we see that the principal graphs to distance five must have precisely two extra vertices (for $k > 4$). But since the traces form an eigenvector for the adjacency matrix, nothing else can be attached to the vertex with trace $\delta^4 - 2$ in the graph for $_{4,-}$, and something must be attached to both vertices for $_{4,+}$. The conclusion is that the two principal graphs are as below to distance 5 from $*$.



Note the identity between the principal graphs of the Haagerup subfactor and the partition planar algebra for distance up to five from $*$. The partition algebra has been studied by several people with generic values of the parameter $k$. One might be tempted to think that the Haagerup subfactor is some kind of specialisation of the partition algebra but this is not at all the case. For the element $S$ above is invariant under the rotation on $\mathcal{C}_4$ whereas we shall see later that for the Haagerup subfactor the rotation is $-1$ on the orthogonal complement of the TL subspace of the 4-string algebra. So these two planar algberas must be considered very distant cousins indeed. get refs here-Ram?

## 4.3 Excess one.

It is obvious that if a planar algebra has $n$-excess one for some $n$ then that $n$ is unique. We want to show some simple results about excess one subfactors. First a definition.

**Definition 4.3.1.** *If $\mathcal{P}$ is a planar algebra with $n$-excess one, the* <u>*chirality*</u> *of $\mathcal{P}$ is that nth. root of unity for which $\rho = \omega$ on $P_n/TL_n$.*

Now let $\mathcal{P}$ be a planar algebra with $n$-excess one and chirality $\omega$. The orthogonal complement $TL^{\perp}$ of $TL$ in $P_n$ is invariant under both $\rho$ and the adjoint map. So there is a self-adjoint eigenvector (with eigenvalue $\omega$) for $\rho$ spanning $TL^{\perp}$ which is unique up to a real multiple. We would like to make an explicit choice of this element linking it to the principal graph. So as in 4.2.8 choose minimal central projections of $P_n$ with

$$e + f = p_n$$

$p_n$ being the JW projection (2.6.5). Let $\alpha$ and $\beta$ be $Tr(e)$ and $Tr(f)$ respectively so that

$$\alpha + \beta = [n+1]$$

and, to be as precise as possible, suppose $\alpha \leq \beta$ so that

**4.3.2.** $$r = \frac{\beta}{\alpha} \geq 1.$$

Our choice of $R$ orthogonal to $TL$ is

**4.3.3.** $$R = -re + f.$$

Since $ef = 0$, algebra is easy and we obtain

**4.3.4.** $$R^k = (-1)^k r^k e + f$$

so that

**4.3.5.** $$R^2 = (1-r)R + p_n$$

**4.3.6.** $$Tr(R^2) = Tr(r^2 e + f) = r[n+1],$$

**4.3.7.** $$Tr(R^3) = Tr(-r^3 e + f) = r(r-1)[n+1],$$

and

**4.3.8.** $$Tr(R^4) = Tr(r^4 e + f) = r(1 - r + r^2)[n+1].$$

We will also need formulae involving $\mathcal{E}$ of 2.6.3.

By multiplying $\mathcal{E}(R^2)$ and $\mathcal{E}(p_n)$ by the $E_i$'s of 2.6.4 we see that they are both multiples of $p_{n-1}$ and on taking the trace we get

**4.3.9.** $$\mathcal{E}(R^2) = \frac{r[n+1]}{[n]} p_{n-1} \qquad \textit{and} \qquad \mathcal{E}(p_n) = \frac{[n+1]}{[n]} p_{n-1}$$

so that

**4.3.10.** $$Tr(\mathcal{E}(R^2)^2) = \frac{r^2[n+1]^2}{[n]}.$$

To be even-handed we need to do the same for $P_{n,-}$ so we will adopt the convention of using a $\check{}$ symbol to indicate the corresponding objects for $P_{n,-}$. Thus we have $\check{e}$ and $\check{f}$ and

**4.3.11.** $$\check{r} = \frac{\check{\beta}}{\check{\alpha}} \geq 1,$$

and all the above formulae have $\check{}$ versions. What is most interesting though is the relation between $R$ and $\check{R}$. For $\mathcal{F}(R)$ is obviously orthogonal to $TL$ in $P_{n,-}$ so that there is a number $A$ with

**4.3.12.** $$\mathcal{F}(R) = A\check{R}.$$

Since $\mathcal{F}$ is an isometry we have $Tr(R^2) = Tr(\mathcal{F}(R)\mathcal{F}(R)^*)$. But by 2.6.2 we have $\mathcal{F}(R)^* = \omega^{-1}\mathcal{F}(R)$ so that $Tr(R^2) = A^2\omega^{-1}Tr(\check{R}^2)$ and by 4.3.6 we obtain

$$\mathcal{F}(R) = \omega^{\frac{1}{2}} \sqrt{\frac{r}{\check{r}}}\check{R}$$

which yields

**4.3.13.** $$Tr(\mathcal{F}(R)^3) = \omega^{\frac{3}{2}} \sqrt{\frac{r}{\check{r}}} r(\check{r}-1)[n+1].$$

Note that the choice of square root of $\omega$ is important. We will see examples with the same $\omega$ but different square roots!

**Definition 4.3.14.** *Let $\mathcal{P}$ be a planar algebra of $n$-excess one and chirality $\omega$. The* full chirality *of $\mathcal{P}$ will be the $2n$-th. root of unity $\omega^{\frac{1}{2}}$ defined by equation 4.3.13.*

We summarise the results with some triangular and tetrahedral structure constants.

**Proposition 4.3.15.** *Using $R$ as above in 3.2.2 we have*

**4.3.16.** $$\Delta(0,0,0) = r(r-1)[n+1]$$

**4.3.17.** $$\Delta(1,1,1) = \omega^{\frac{3}{2}} \sqrt{\frac{r}{\check{r}}} r(\check{r}-1)[n+1].$$

**4.3.18.** $$\Gamma\begin{pmatrix} n & 0 & n & \\ 0 & 0 & 0 & 0 \end{pmatrix} = r(1-r+r^2)[n+1]$$

**4.3.19.** $$\Gamma\begin{pmatrix} n & 0 & n & \\ 1 & 1 & 1 & 1 \end{pmatrix} = \omega^2 \frac{r^2}{\check{r}}(1-\check{r}+\check{r}^2)[n+1]$$

**4.3.20.** $$\Gamma\begin{pmatrix} n-1 & 0 & n+1 & \\ 0 & 0 & 0 & 0 \end{pmatrix} = r^2 \frac{[n+1]^2}{[n]}.$$

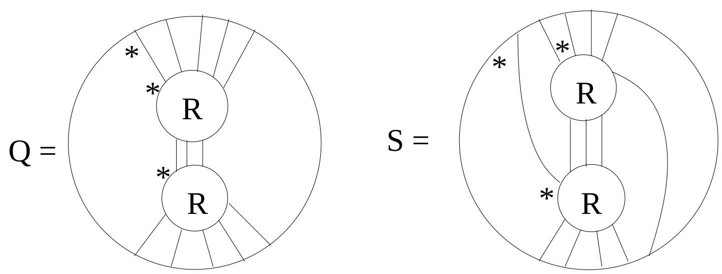
*Proof.* Immediate consequences of the previous calculations. □

# 5 Inner product formulae.

## 5.1 Setup

Let $\mathcal{P}$ be a planar algebra with $n$-excess equal to 1. Let $R \in P_n$ be as in 4.3.3. Our goal is to orthogonalise as much as possible the subspace of $P_{n+1}$ spanned by TL, the annular TL module generated by $R$ and quadratic tangles labelled by $R$. To this end we define the two labelled tangles $Q$ and $S$ in $P_{n+1}$ below.

**Fig. 5.1.1.**




Together with their rotations, $Q$ and $S$ are all the essential quadratic labelled tangles. The inner products between them are mostly essential tetrahedral symbols but some are not. We record these below.(Recall $\langle x, y\rangle = Tr(y^*x)$.)

**Lemma 5.1.2.** *We have*

*(a)* $$\langle Q, Q\rangle = \Gamma\begin{pmatrix} n-1 & 0 & n+1 & \\ 0 & 0 & 0 & 0 \end{pmatrix} = r^2\frac{[n+1]^2}{[n]}.$$

*(b)* $$\langle S, S\rangle = \Gamma\begin{pmatrix} n-1 & 0 & n+1 & \\ 1 & -1 & -1 & 1 \end{pmatrix} = r^2\frac{[n+1]^2}{[n]}.$$

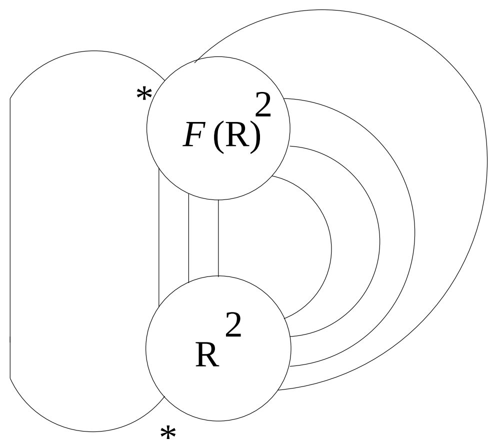
*(c)* $$\langle S, Q\rangle = \Gamma\begin{pmatrix} n-1 & n & 1 & \\ 0 & 0 & 0 & n \end{pmatrix}$$

$$= [n+1]r^2\left(\omega^{1/2}(r^{1/2}-r^{-1/2})(\check{r}^{1/2}-\check{r}^{-1/2}) + (-1)^{n+1}\frac{\omega}{[n]}\right).$$

*Proof.* (a) follows immediately from drawing the picture. For (b) the picture immediately gives the tetrahedral symbol which is visibly

$$Tr\Bigg(\mathcal{E}\Bigg(\mathcal{F}(R)^2\Bigg)\mathcal{E}\Bigg((\mathcal{F}(R)^*)^2\Bigg).$$

But $\mathcal{F}(R) = A\check{R}$ with $A$ as in 4.3.12. Using $A^4 = \omega^2 \frac{r^2}{\check{r}^2}$ we are through. For (c), the picture immediately gives the tetrahedral symbol. Inside that picture we recognize $R^2$ and $\mathcal{F}(R)^2$. After applying those operations we have the following fully labelled tangle:



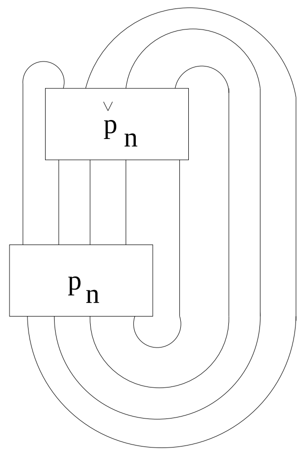

which we recognise as $Tr(\mathcal{F}(R)^2\mathcal{F}(R^2))$. This becomes

$$\langle A(1-\check{r})\mathcal{F}(R) + \check{r}A^2\check{p}_n, \mathcal{F}\Bigg((1-r)R + rp_n\Bigg)\rangle$$

and since $R$ is orthogonal to any TL tangles, this becomes

$$A(1-r)(1-\check{r})\langle R, R\rangle \quad + \quad A^2 r\check{r}\langle \check{p}_n, \mathcal{F}(p_n)\rangle$$

We know $\langle R, R\rangle$ from 4.3.6. The other term is given by the following tangle where we have preferred to use rectangles rather than circles for the inputs ( the convention is that the distinguished boundary interval always contains the the left edge of the rectangle): :




By 2.6.5 there is only one TL basis diagram that gives a non-zero contribution when inserted in the input disc containing $p_n$ and that is the word $E_{n-1}E_{n-2}....E_1$ whose coefficient in $p_n$ is, $\dfrac{(-1)^{n+1}}{[n]}$ by lemma 4.6 of []. When that tangle is inserted we get $Tr(p_n) = [n+1]$. Using $A = \omega^{1/2}\sqrt{\dfrac{r}{\check{r}}}$ we are done.

□

By themselves the formulae of 5.1.2 are not very useful. To get more precise information we need to subtract off the projections of $Q$ and $S$ onto TL and the annular consequences of $R$ itself.

**Definition 5.1.3.** *Let $\mathfrak{A}$ be the linear span of the image of $R$ under all annular tangles defining maps from $P_n$ to $P_{n+1}$ and let $\mathfrak{T}$ be the TL subspace of $P_{n+1}$.*

As observed in [], $\mathfrak{A}$ and $\mathfrak{T}$ are orthogonal.

## 5.2 Orthogonalisation with respect to $\mathfrak{A}$ and $\mathfrak{T}$.

**Proposition 5.2.1.** *The orthogonal projections $P_{\mathfrak{T}}(Q)$ and $P_{\mathfrak{T}}(S)$ of $Q$ and $S$ onto $\mathfrak{T}$ are:*

*(i)* $$P_{\mathfrak{T}}(Q) = r\frac{[n+1]}{[n+2]}p_{n+1}.$$

*(ii)* $$P_{\mathfrak{T}}(S) = r\frac{[n+1]}{[n+2]}\mathcal{F}(\check{p}_{n+1}).$$

*Proof.* The TL subspace is invariant under the annular category so $P_{\mathfrak{T}}$ commutes with left and right multiplication by the $E_i$. Hence $E_iP_{\mathfrak{T}}(Q) =$

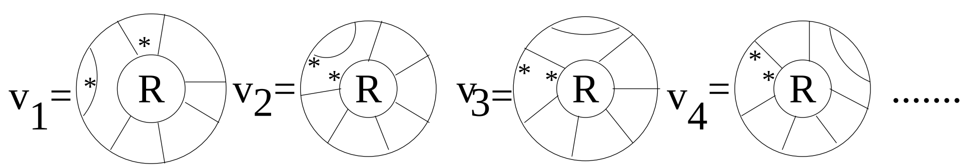
$P_{\mathfrak{T}}(E_i p_{n+1}) = 0$. Hence by 2.6.5 $P_{\mathfrak{T}}(Q)$ is a multiple of $p_{n+1}$, say $\lambda p_{n+1}$. But then $\langle P_{\mathfrak{T}}(Q), 1\rangle = \langle Q, 1\rangle = \lambda\langle p_{n+1}, 1\rangle$ so that $\lambda = \dfrac{Tr(R^2)}{[n+2]}$. Similarly observe that $P_{\mathfrak{T}}(S) = \mu\mathcal{F}(\check{p}_{n+1})$ and

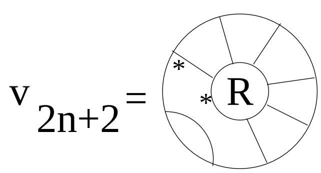
$$\langle \mathcal{F}^{-1}(P_{\mathfrak{T}}(S)), 1\rangle = \omega^{-1}Tr(\mathcal{F}(R)^2) = \omega^{-1}A^2 Tr(\check{R}^2) = r[n+1]$$

. □

To calculate the orthogonal projection onto $\mathfrak{A}$ we will choose the basis below. Linear independence follows from [] or directly from the determinant formula we shall use.

**Definition 5.2.2.** *Let $v_1, v_2, ..., v_{2n+2}$ be the elements of $P_{n+1}$ given by the labelled annular tangles below.*




*Thus all the $v_i$ except $v_1$ are defined by the condition that the ith. and $(i-1)$th. external boundary points (indices mod $2n+2$) are connected by a string and the internal and external distinguished intervals form parts of the boundary of the same region. And $v_1$ is as above.*

**Proposition 5.2.3.** *We have the following inner products:*

**5.2.4.**

$$\langle Q, v_i\rangle = \begin{cases} 0 & if \quad i \neq 1 \quad or \quad n+2 \\ \omega^{-\frac{1}{2}} r^{\frac{3}{2}} (\check{r}^{\frac{1}{2}} - \check{r}^{-\frac{1}{2}})[n+1] & for \quad i = 1 \\ r(r-1)[n+1] & for \quad i = n+2 \end{cases}$$

$$\langle S, v_i\rangle = \begin{cases} 0 & if \quad i \neq 2 \quad or \quad n+3 \\ \omega^{\frac{1}{2}} r^{\frac{3}{2}} (\check{r}^{\frac{1}{2}} - \check{r}^{-\frac{1}{2}})[n+1] & for \quad i = n+3 \\ r(r-1)[n+1] & for \quad i = 2 \end{cases}$$

*Proof.* The zero values are clear as in these cases an internal disc has two boundary points connected to each other.

For $\langle Q, v_1\rangle$ one obtains the triangular symbol $\Delta(1,-1,-1)$ which is $\omega^{-1}Tr(\mathcal{F}(R)^3)$ by 3.2.5. For $\langle Q, v_{n+2}\rangle$ one obtains $\Delta(0,0,0)$ directly. Similarly $\langle S, v_2\rangle = \Delta(0,0,0)$ and $\langle S, v_{n+3}\rangle = \Delta(1,-1,1)$

□

To complete our calculation of $P_{\mathfrak{A}}(Q)$ and $P_{\mathfrak{A}}(S)$ we need the matrix $\langle v_i, v_j\rangle$. Luckily it is the easiest of matrices to diagonalise.

**Lemma 5.2.5.**

$$\langle v_i, v_j\rangle = \begin{cases} 0 & \textit{if} \quad i-j \neq 0 \quad \textit{or} \quad 1 \ (\ \textit{mod}\ 2n+2) \\ r[n+1][2] & \textit{if} \quad i = j \\ r[n+1] & \textit{for} \quad j = i+1 \quad \textit{and} \quad i \neq 2n+2 \\ \overline{\omega} r[n+1] & \textit{if} \quad i = 2n+2,\ j = 1 \end{cases}$$

*Proof.* Orthogonality of most of the $v_i$'s is clear from 3.1.1. The other cases follow immediately from diagrams and $Tr(R^2) = r[n+1]$. □

So the matrix $\langle v_i, v_j\rangle$ is the $(2n+2)\times(2n+2)$ matix below:

$$r[n+1]\begin{pmatrix} [2] & 1 & 0 & . & . & . & \omega \\ 1 & [2] & 1 & 0 & . & . & 0 \\ 0 & 1 & [2] & 1 & 0 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \overline{\omega} & 0 & . & . & . & 1 & [2] \end{pmatrix}$$

**Definition 5.2.6.** *For $\lambda$ a $(2n+2)th.root$ of $\omega$ let*

$$w_\lambda = \sum_{i=1}^{2n+2} \lambda^{i-1}\, v_i.$$

One checks that $\{w_\lambda \quad | \quad \lambda^{2n+2} = \omega\}$ diagonalise the form $\langle,\rangle$ so that

**5.2.7.** $\langle w_\lambda, w_\mu\rangle = (2n+2)r([2]+\lambda+\lambda^{-1})[n+1]\delta_{\lambda,\mu}.$

Thus

**5.2.8.**

$$P_{\mathfrak{A}}(Q) = \frac{1}{(2n+2)r[n+1]}\sum_{\lambda^{2n+2}=\omega}\frac{\langle Q, v_1\rangle + \lambda^{-n-1}\langle Q, v_{n+2}\rangle}{[2]+\lambda+\lambda^{-1}} w_\lambda$$

*and*

$$P_{\mathfrak{A}}(S) = \frac{1}{(2n+2)r[n+1]} \sum_{\lambda^{2n+2}=\omega} \frac{\lambda\langle S, v_2\rangle + \lambda^{-n-2}\langle S, v_{n+3}\rangle}{[2]+\lambda+\lambda^{-1}} w_\lambda$$

To exploit 5.2.8 we need a couple of elementary calculations. (We thank Antony Wassermann for suggesting the logarithmic derivative at this point.)

**Definition 5.2.9.** *For all $m \in \mathbb{N}$ we set*

$$W_m(q,\omega) = q^m + q^{-m} - \omega - \omega^{-1}$$

.

Set $\delta = q + q^{-1}$

**Lemma 5.2.10.**

**5.2.11.** $\displaystyle\sum_{\lambda^m=\omega} \frac{1}{\delta+\lambda+\lambda^{-1}} = (-1)^m \frac{m[m]}{W_m(-q,\omega)}$

**5.2.12.** $\displaystyle\sum_{\lambda^m=\omega} \frac{\lambda}{\delta+\lambda+\lambda^{-1}} = m\Big\{\frac{(-1)^{m+1}[m-1]-\omega}{W_m(-q,\omega)}\Big\}$

*For $m = 2k$*

**5.2.13.** $\displaystyle\sum_{\lambda^m=\omega} \frac{\lambda^k}{\delta+\lambda+\lambda^{-1}} = (-1)^k m \frac{(1+\omega)[k]}{W_m}$

**5.2.14.** $\displaystyle\sum_{\lambda^m=\omega} \frac{\lambda^{k+1}}{\delta+\lambda+\lambda^{-1}} = (-1)^{k+1} m \Big\{\frac{\omega[k+1]+[k-1]}{W_m}\Big\}.$

*Proof.* $W_m(q,\omega)$ and $\prod_{\lambda^m=\eta}(\delta-\lambda-\lambda^{-1})$ are polynomials in $q$ of the same degree and vanishing whenever $q$ is an $m$th root of $\omega$ or $\omega^{-1}$, with double roots when $\omega = \pm 1$. So they are equal. Taking the logarithmic derivative of both sides gives $\displaystyle\sum_{\lambda^m=\omega} \frac{\lambda^k}{\delta-\lambda-\lambda^{-1}} = m\frac{[m]}{W_m}$. Replacing $q$ by $-q$ one gets 5.2.11 since $[m]_{-q} = (-1)^{m+1}[m]_q$.

Now observe that $\displaystyle\frac{2\lambda}{\delta-\lambda-\lambda^{-1}} = -1 + \frac{q}{q-\lambda} - \frac{q}{q-\lambda^{-1}} + \frac{q+q^{-1}}{\delta-\lambda-\lambda^{-1}}$. Summing over $\lambda$, with logarithmic differentiations similar to above, and a little algebra gives 5.2.12.

To obtain 5.2.13 and 5.2.14, choose a square root $\eta$ of $\omega$ and observe that the sum breaks into two parts according to $\lambda^k = \eta$ or $\lambda^k = -\eta$. Each of these parts may be summed using 5.2.11 or 5.2.12 and the answer is obtained by simple algebra including the observation that $W_k(q,\eta)W_k(q,-\eta) = W_{2k}(q,\omega)$. □

## 5.3 Bessel's inequality.

With notation as above let $\alpha = \sqrt{r} - \frac{1}{\sqrt{r}}$ and $\beta = \sqrt{\check{r}} - \frac{1}{\sqrt{\check{r}}}$.

**Lemma 5.3.1.**

$$||P_{\mathfrak{A}}(Q)||^2 = ||P_{\mathfrak{A}}(S)||^2 =$$

$$\frac{r^2[n+1]}{W_{2n+2}(q,\omega)}\left([2n+2](\alpha^2+\beta^2) + (-1)^{n+1}2\alpha\beta(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}})[n+1]\right)$$

*Proof.* By orthogonality of the $w_\lambda$ we have

$$(2n+2)[n+1]r||P_{\mathfrak{A}}(Q)||^2 = \sum_{\lambda^{2n+2}=\omega} \frac{|\langle Q, v_1\rangle + \lambda^{-n-1}\langle Q, v_{n+2}\rangle|^2}{[2]+\lambda+\lambda^{-1}}.$$

By 5.2.4 this becomes

$$\frac{2n+2}{[n+1]r^2}||P_{\mathfrak{A}}(Q)||^2 = \sum_{\lambda^{2n+2}=\omega} \frac{|\omega^{-\frac{1}{2}}\beta + \lambda^{-n-1}\alpha|^2}{[2]+\lambda+\lambda^{-1}}.$$

By 5.2.11 and 5.2.13 this becomes

$$||P_{\mathfrak{A}}(Q)||^2 = \frac{[n+1]r^2}{W_{2n+2}}\Big((\alpha^2+\beta^2)[2n+2] + (-1)^{n+1}\alpha\beta 2\Re(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}})[n+1]\Big).$$

The calculation for $P_{\mathfrak{A}}(S)$ is the same. □

**Theorem 5.3.2.** *Let $P$ be a planar algebra with $n$-excess equal to 1, full chirality $\omega^{\frac{1}{2}}$ and parameters $\alpha$, $\beta$ as above. Then we have the following inequality:*

$$\frac{W_{2n+2}}{[n][n+1]} \geq \alpha^2 + \beta^2 + (-1)^{n+1}(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}})\frac{2\alpha\beta}{q^{n+1}+q^{-n-1}}$$

*Proof.* Bessel's inequality gives $||Q||^2 \geq ||P_{\mathfrak{A}}(Q)||^2 + ||P_{\mathfrak{T}}(Q)||^2$. By 5.2.1, 2.6, 5.1.2 and the previous lemma we get

$$r^2\frac{[n+1]^2}{[n]} \geq$$

$$r^2\frac{[n+1]^2}{[n+2]} + \frac{r^2[n+1]}{W_{2n+2}}\left([2n+2](\alpha^2+\beta^2) + (-1)^{n+1}2\alpha\beta(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}})[n+1]\right).$$

A little algebra gives the answer. □

## 5.4 Cauchy-Schwarz inequality

To get a powerful inequality we need to project $Q$ and $S$ onto the orthogonal complement of $\mathfrak{A}$ and $\mathfrak{T}$ . If $E = P_{\mathfrak{A}} + P_{\mathfrak{T}}$ then since $||Q|| = ||S||$ and $||E(Q)|| = ||E(S)||$, the Cauchy-Schwarz inequality is

$$|\langle Q, S\rangle - \langle E(Q), E(S)\rangle| \leq ||Q||^2 - ||E(Q)||^2.$$

The only term which we have not yet calculated in the above is $\langle E(Q), E(S)\rangle$.

**Lemma 5.4.1.**

$$\langle P_{\mathfrak{A}}(Q), S\rangle = \frac{r^2[n+1]}{W_{2n+2}}\Big((-1)^n([n{+}2]{+}\omega^{-1}[n])(\alpha^2{+}\beta^2){-}2\omega^{-\frac{1}{2}}\alpha\beta(\omega{+}[2n{+}1])\Big)$$

$$\langle P_{\mathfrak{T}}(Q), S\rangle = (-1)^n r^2\frac{[n+1]}{[n+2]}$$

*Proof.* Combining 5.2.12. 5.2.14, 5.2.4 and 5.2.8 we get the first formula. To obtain the second, 5.2.1 gives $r^2\frac{[n+1]^2}{[n+2]^2}\langle p_{n+1}, \mathcal{F}(\check{p}_{n+1})\rangle$ and this inner product was calculated for part (c) of 5.1.2 to be $(-1)^n\frac{[n+2]}{[n+1]}$. □

**Corollary 5.4.2.** *We have*
$\langle Q, S\rangle - \langle E(Q), E(S)\rangle =$

$$(-1)^{n+1}\frac{[n+1]r^2}{W_{2n+2}}\Bigg\{W_{2n+2}\Big(\frac{\omega^{-1}}{[n]} + \frac{1}{[n+2]}\Big) +$$

$$(-1)^{n+1}\omega^{-\frac{1}{2}}\alpha\beta\Big([2n+2]\delta + \omega - \omega^{-1}\Big) + (\alpha^2+\beta^2)(([n+2]+\omega^{-1}[n])\Bigg\}$$

# 6 Poincaré Series restrictions.

## 6.1 Maximal supertransitivity

Continuing the analogy with tranisitivity of group actions, we can further measure the supertransitivity of a planar algebra of $n$-excess one by the size of $P_{n+1}$. It was shown in [] that annular diagrams and the TL subalgebra mean that (for $\delta > 2$) the dimension of $P_{n+1}$ is at least $\frac{1}{n+2}\binom{2n+2}{n+1}$.

**Definition 6.1.1.** *We will say that a planar algebra is* maximally supertransitive of $n$-excess one *(MSn for short) if* $\dim(P_{n+1}) = \frac{1}{n+2}\binom{2n+2}{n+1}$.

The Haagerup planar algebra of [] is MS4 and the Haagerup-Asaeda planar algebra of [] is MS6. The partition planar algebra of section 4.1 is MS4. The Fuss-Catalan algebra of [] is MS2 if one of the two circle parameters is $\sqrt{2}$ (corresponding to an intermediate subfactor of index 2).

**Theorem 6.1.2.** *Let $P$ be an MSn planar algebra with chirality $\omega$ and $r, \check{r}$ as above. Suppose $r \leq \check{r}$ (which can be assured by passing to the dual if necessary). Then $n$ is even,*

$$\check{r} = \frac{[n+2]}{[n]}$$

*and*

$$r + \frac{1}{r} = 2 + \frac{2+\omega+\omega^{-1}}{[n][n+2]}$$

*Proof.* Since $P$ is MSn, the elements $Q$ and $S$ are in the linear span of $\mathfrak{A}_{n+1}$ and $\mathfrak{T}_{n+1}$ which means the Bessel inequality is actually an equality. So by 5.3.2 and 5.4.2 we have

**6.1.3.** $$\frac{W_{2n+2}}{[n][n+1]} = \alpha^2+\beta^2+(-1)^{n+1}(\omega^{\frac{1}{2}}+\omega^{-\frac{1}{2}})\frac{2\alpha\beta}{q^{n+1}+q^{-n-1}} \quad \text{and}$$

**6.1.4.** $$W_{2n+2}\Big(\frac{\omega^{-1}}{[n]}+\frac{1}{[n+2]}\Big) = (-1)^n\omega^{-\frac{1}{2}}\alpha\beta\Big([2n+2]\delta+\omega-\omega^{-1}\Big)$$
$$-(\alpha^2+\beta^2)([n+2]+\omega^{-1}[n])$$

Miraculously but readily, the two linear equations for $\alpha\beta$ and $\alpha^2+\beta^2$ reduce to

$$\alpha\beta = (-1)^n\frac{(\omega^{\frac{1}{2}}+\omega^{-\frac{1}{2}})(q^{n+1}+q^{-n-1})}{[n][n+2]} \quad \text{and}$$

$$\alpha^2+\beta^2 = \frac{(q^{n+1}+q^{-n-1})^2+(\omega^{\frac{1}{2}}+\omega^{-\frac{1}{2}})^2}{[n][n+2]}.$$

Now if $n$ is odd, there is a trace-preserving isomorphism between $P_{n,+}$ and $P_{n,-}$ given by a suitable power of the Fourier transform. Thus $\alpha=\beta$ which with these equations gives $q^{n+1}+q^{-n-1}+(-1)^{n+1}(\omega^{\frac{1}{2}}+\omega^{-\frac{1}{2}})=0$ which is impossible if $\delta>2$. So $n$ is even.

It then follows that

$$(\alpha \pm \beta)^2 = \frac{(q^{n+1} + q^{-n-1} \pm (\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}))^2}{[n][n+2]}$$

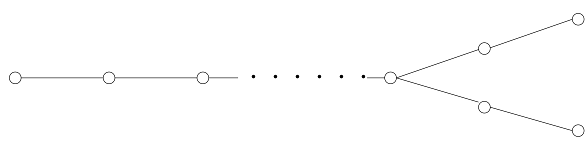
so that if $\alpha \leq \beta$ (which is the same as $r \leq \check{r}$), we get

$$\alpha = \frac{q^{n+1} + q^{-n-1}}{\sqrt{[n][n+2]}} \quad \text{and} \quad \beta = \frac{\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}}{\sqrt{[n][n+2]}}.$$

These immediately imply the desired formulae. □

**Corollary 6.1.5.** *The principal graph (corresponding to $\check{r}$ ) is as below for vertices of distance $\leq n+2$ from *:*

*The other principal graph is as below:*



**Corollary 6.1.6.** *The Haagerup planar algebra has chirality $-1$, but the Haagerup-Asaeda planar algebra has chirality* 1.

*Proof.* As soon as one of the principal graphs has a symmetry exchanging the two vertices at distance $n$ from *, one has that $\check{r} = 1$. This is the case for the Haagerup subfactor. The Asaeda-Haagerup value can be deduced from the Perron Frobenius data. □

# 7 Applications to subfactors.