Some Moment calculations.

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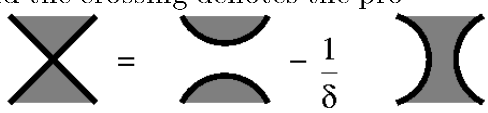
1 Introduction

We calcluate the moment generating function for some Temperley Lieb elements arising in the graded algebra of a planar algebra. Or at least we obtain algebraic equations satisfied by them.

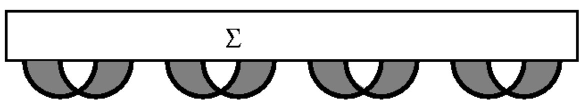
2 Preliminaries

The main goal is to calculate the moments of the following element of M_0:

where we are using planar algebra diagrams with the boundary disc being an implicit horizontal line containing the end points of the strings and the graded algebra structure is given by concatenation-place one element to the right of the other. And the crossing denotes the projection $1-e_1$, in Kauffman diagram language:



Thus our goal is to caluclate the following scalars

$a_n =$ 

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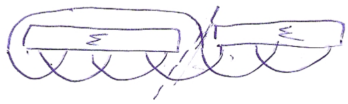
Where there are n crossings (so the actual picture is of a_4), and the box containing $\sum$ is the sum over all Temperley-Lieb connections of the $4n$ boundary points.

The first method of calculating a_n will be painful ①
e will inevitably involve calculation of the following
C_n which we start with as a warmupt:

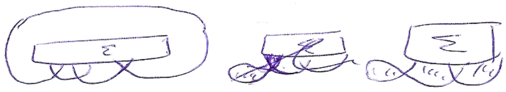
C_n = 

$c_0 = \delta$, $n \geq 1$
$c_1 = \delta^2 - 1$

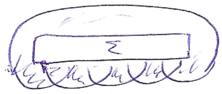
we argue on the possible connections in Σ of the leftmost boundary point. For parity reasons it must be connected to one at an ~~even~~ odd distance from it so in general we get a sum



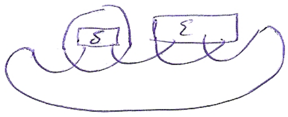
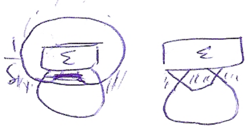
but this is a connected sum on the dotted line so is $\frac{1}{\delta}$

then  which is 0 since  $= 0$.

So we have only the extremal case:

C_n = 

Now argue on the first connection in Σ; conveniently moving the top string to the bottom

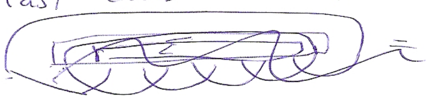
C_n = sum of terms like  $= \frac{1}{\delta}$  C_j

using 

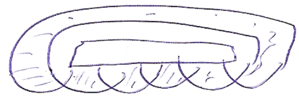
$= \frac{1}{\delta}$  $n-j-1$ C_j

The first and last cases are exceptional:

$j = 0$

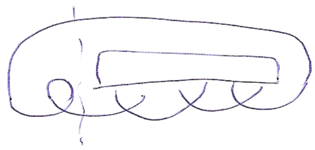


$=$



$=$ projection proper $\frac{1}{\delta} c_{n-1} c_0$ (So in fact ok in sum)

and $j = n-1$



$= \frac{1}{\delta} c_1 c_{n-1}$

hence $$c_n = \frac{1}{\delta}\sum_{j=0}^{n-2} c_{n-j-1} c_j + \frac{1}{\delta} c_{n-1} c_1 \qquad \text{for } n \geq 2$$

$$= \frac{1}{\delta}\sum_{j=0}^{n-1} c_{n-j-1} c_j - \frac{1}{\delta} c_{n-1} c_0 + \frac{1}{\delta}(\delta^2 - 1) c_{n-1}$$

$$= \frac{1}{\delta}\sum_{j=0}^{n-1} c_{n-j-1} c_j + \frac{1}{\delta}(\delta^2 - \delta - 1) c_{n-1}$$

so $$z^n c_n = z\left(\frac{1}{\delta}\sum_{j=0}^{n-1} c_{n-j-1} c_j z^{n-1}\right) + \frac{z}{\delta}\left((\delta^2 - \delta - 1) c_{n-1} z^{n-1}\right)$$

and $$C(z) = \sum_{n=0}^{\infty} c_n z^n = \delta + (\delta^2 - 1)z + \frac{z}{\delta}\left(C^2(z) - \delta^2\right) + \frac{z}{\delta}(\delta^2 - \delta - 1)\left(C(z) - \delta\right)$$

$$C(z) = \cancel{\delta + (\delta^2 - 1)z - z\delta - z(\delta^2 - \delta - 1)} + \frac{z}{\delta} C^2 + \frac{z}{\delta}(\delta^2 - \delta - 1) C(z)$$

or $$\boxed{\frac{z}{\delta} C^2 + \left(\frac{z}{\delta}(\delta^2 - \delta - 1) - 1\right) C + \delta = 0}$$

by independent calculation $c_0 = \delta$, $c_1 = \delta^2 - 1$, $c_2 = \delta^3 + \delta^2 - 2\delta + \frac{1}{\delta} - 1$,

$c_3 = \delta^4 + 3\delta^3 - 2\delta^2 + 2 + 3/\delta - \frac{1}{\delta^2}$ & coefficients of z, z^2, z^3 all check.

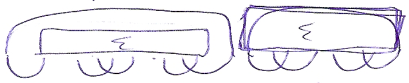
③

Now to the general case, define   so $a_n = b_{1,n-1}$

$b_{m,n} =$ 

m X's n X's

There is a simple relationship between $b_{0,n}$ and a_n:

$b_{0,n} = \sum$  $\sum_{j=0}^{n-1} a_j a_{n-j}$ (for $n \geq 1$)

(other connections 0) j $+ \delta a_n a_0$

$$z^n b_{0,n} = z^n \sum_{j=0}^{n} a_j \cdot a_{n-j} + (\delta - 1) a_n a_0 z^n$$

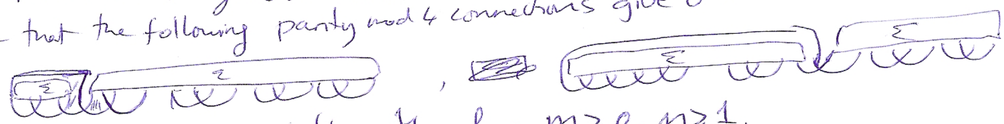
summing from 1 to ∞ gives

$$B_0(z) - \delta = A^2(z) - 1 + (\delta - 1)(A(z) - 1)$$

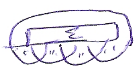
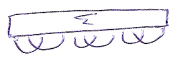
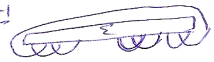
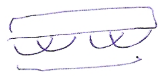
$$\boxed{B_0 = A^2 + (\delta - 1) A}$$

Now on to the general case:

Note that the following parity mod 4 connections give 0:



so we are left with, for $m \geq 0$, $n \geq 1$,

$b_{m,n} =$   $+ \sum_{j=1}^{n-1}$  

$b_{m,0}$ $b_{1,n-1}$ $b_{m+1,n-j}$ j $b_{1,j-1}$

$+$ 

$b_{m+1,n-1}$

for $m \geq 0$, $n \geq 1$

or $b_{m,n} = b_{m,0} b_{1,n-1} + \sum_{j=0}^{n-2} b_{m+1,n-j} b_{1,j} + b_{m+1,n-1}$

$\times z^n$ $\sum$ $b_{m,n} z^n = z(b_{m,0} b_{1,n-1}) z^{n-1} + z^2 \sum_{j=0}^{n-2} b_{m+1,n-j} b_{1,j} + z(b_{m+1,n-1} z^{n-1})$

let $B_m = \sum_{n=0}^{\infty} b_{m,n} z^n$: $B_m - b_{m,0} = z\, b_{m,0} B_1 + z^2 B_{m+1} B_1 + z B_{m+1}$

Now put $B(z,w) = \sum_{m=0}^{\infty} w^m B_m(z)$ ④

by previous,

$$w\left(B_m w^m - b_{m,0} w^m\right) = zw\, b_{m,0} w^m B_1 + z^2 w^{m+1} B_{m+1} B_1 + z w^{m+1} B_{m+1}$$

now sum from $m=0$ to ∞:

$$w\left(B(z,w) - B(0,w)\right) = zw\, B(0,w) B_1 + z^2 B_1 (B - B_0) + z(B - B_0)$$

so $\quad (w - z^2 B_1 - z) B = w B(0,w)(1 + zB_1) - zB_0(1+zB_1)$

with $A = 1+zB_1$ $\quad (w + zA) B = w A B(0,w) - zAB_0$

so if $w = -zA$ $\quad -\cancel{z}A^2 B(0,-zA) = \cancel{z}\cancel{A}B_0$

$$-A\, C(-zA) = A^2 + (\delta-1)A$$

$$-C(-zA) = A + (\delta - 1)$$

so $\quad \frac{zA}{\delta}\left(A + (\delta-1)\right)^2 + \left(\frac{zA}{\delta}(\delta^2 - \delta - 1) - 1\right)\left(A + (\delta-1)\right) + \delta = 0$

coeff. of A^2: $\quad \frac{zA}{\delta}(\delta^2-\delta-1)$ $\quad = \frac{z}{\delta}(\delta^2 - 3 + \delta)$

$2\frac{z}{\delta}(\delta - 1) \quad + \quad \cancel{\frac{z}{\delta}\left((\delta^2 - \delta - 1)(\delta - 1)\right) - 1}$ $\quad \cancel{= \frac{\delta}{\delta}\left((\delta+3)\right)}$

coeff of A: $\quad (\delta-1)^2\frac{z}{\delta} + \frac{z}{\delta}(\delta^2-\delta-1)(\delta-1) \quad -1$

$= \cancel{(\delta-1)(\delta^2-\delta)\frac{z}{\delta}} \quad \cancel{-1 = (\delta-1)^2 z} - 1$

-1

$= \frac{z}{\delta}(\delta-1)\left(\delta - 1 + \delta^2 - \delta - 1\right) \quad -1$

$= z/\delta\, (\delta-1)(\delta^2 - 2)$

$\sqrt{13}$

$\left(\frac{-1+\sqrt{13}}{2}\right)^2$

$\frac{7+\sqrt{13}}{2} \quad \frac{1-2\sqrt{13}+13}{4} \quad \frac{7+13}{2}$

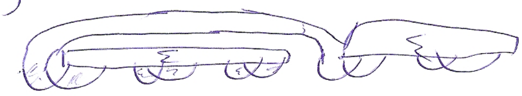
⑤

Now the easy way (for some bizarre reason)

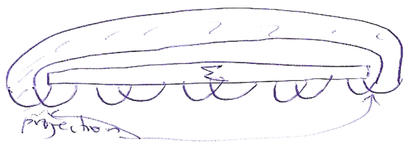
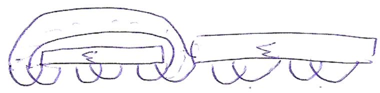
recall $a_n =$  we will argue on where the

second boundary point from left is connected to

The parity mod 4 is eliminated by the following picture e connected sum

 $=0$

so

$a_n =$  $+\sum_{j=1}^{n-2}$ 

projection

$+$  $n \geq 1$

$$= a_{n-1} + \frac{1}{\delta}\sum_{j=1}^{n-2} a_{n-j-1}\, d_j + \frac{1}{\delta}(\delta^2-1)\, d_{n-1} \qquad (d_n = b_{0,n})$$

$a_{n-1} = \frac{1}{\delta} a_{n-1} d_0$

$$= \frac{1}{\delta}\sum_{j=0}^{n-2} a_{n-j-1} d_j + \frac{1}{\delta}(\delta^2-1)\, d_{n-1}$$

$$= \frac{1}{\delta}\sum_{j=0}^{n-1} a_{n-j-1} d_j + \frac{1}{\delta}(\delta^2-1) d_{n-1} - \frac{1}{\delta} d_{n-1}$$

so $$z^n a_n = \frac{z}{\delta}\left(\sum_{j=0}^{n-1} a_{n-j-1} d_j\right) z^{n-1} + \frac{z}{\delta}(\delta^2-2)\, d_{n-1} z^{n-1}$$

so $$A(z) - 1 = \frac{z}{\delta} DA + \frac{z}{\delta}(\delta^2-2) D$$

and we've done $D = A^2 + (\delta-1)A$

$$A - 1 = \frac{z}{\delta}(A^3 + (\delta-1)A^2) + \frac{z}{\delta}(\delta^2-2)(A^2+(\delta-1)A)$$

coeff of A^2: $\frac{z}{\delta}((\delta-1)+\delta^2-2) = \frac{z}{\delta}(\delta^2+\delta-3)$ | coeff. of A: $-1 + \frac{z}{\delta}(\delta^2-2)(\delta-1)$

References

[1] Popa, Sorin *The relative Dixmier property for inclusions of von Neumann algebras of finite index.* Ann. Sci. École Norm. Sup. (4) 32 (1999), no. 6, 743–767.

[2] Connes, Alain *Noncommutative geometry*, Academic Press, 1994.