

# Hilbert space representations of the annular Temperley-Lieb Algebra.

Vaughan F.R. Jones <sup>\*</sup>      Sarah A. Reznikoff <sup>†</sup>

9th October 2003

## 1 Introduction.

The annular Temperley Lieb algebra  $ATL$  has a parameter  $\delta$  and is linearly spanned by isotopy classes of  $(m, n)$  diagrams. For  $m$  and  $n$  nonnegative integers, an  $(m, n)$  diagram consists of an annulus with  $m$  marked points on the inside circle and  $n$  marked points on the outside connected to each other by a family of smooth disjoint curves, called strings, inside the annulus. There may also be (necessarily closed) curves that do not connect boundary points. If such a curve is homologically trivial in the annulus, the diagram may be replaced by the same one with the closed curve removed, but multiplied in the algebra by  $\delta$ . By definition a basis of  $ATL$  consists of such diagrams with no homologically trivial circles. Multiplication of an  $(m, p)$  diagram  $T$  by a  $(p, n)$  diagram  $S$  is by identifying the outside boundary  $T$  with the inside boundary of  $S$  in such a way that the boundary points coincide, smoothing the strings at the  $p$  common marked boundary points, and removing the common boundary to produce the annular diagram  $ST$ .

To the best of our knowledge, the first explicit investigation of  $ATL$  was in [15] where it was encountered in a concrete form as an algebra of linear transformations on the tensor powers of the  $n \times n$  matrices (and  $\delta = n^2$ ). This study was relatively simple because of the concrete situation and the fact that as soon as  $n$  is greater than 2 the algebra is "generic" and the structure of the representations does not depend on  $n$ . Also, homologically non-trivial

---

<sup>\*</sup>Research supported in part by NSF Grant DMS93-22675, the Marsden fund UOA520, and the Swiss National Science Foundation.

<sup>†</sup>Research conducted in part while the second named author was a postdoctoral fellow with John Phillips at the University of Victoria and supported by NSERC of Canada.

circles in the annulus are no different from contractible ones, which, as we shall see, is quite special.

The second explicit analysis of  $ATL$  was by Graham and Lehrer in [9], where the abstract algebra more or less as defined in the first paragraph was defined and studied in its own right. One can no longer avoid homologically non-trivial strings, and the version of  $ATL$  in [9] introduced a second parameter to account for this. Graham and Lehrer produced an impressively complete analysis and we have been greatly inspired by their results. In [12] we showed how to use  $ATL$  to obtain results about subfactors. It was recognised that, for a general planar algebra  $P$ , the operadic concept of a module over  $P$  is the same thing as an ordinary module over a canonically defined algebra spanned by annular tangles in  $P$ . This led to the perhaps confusing notation of  $TL$ -module in [12], which in fact means an ordinary module over the annular algebra. Applications to subfactors required consideration of positivity of the natural sesquilinear form on the  $ATL$  modules — which are necessarily Hilbert spaces if they occur in subfactors.

## 2 Notation

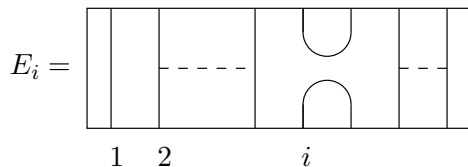
First of all let us agree on the notation  $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$  for the entirety of this paper. The Tchebychev polynomials  $P_n(x) = [n]$  with  $x = q + q^{-1}$  satisfy  $P_{n+1} = xP_n - P_{n-1}$  and we define essentially the same polynomials  $Q_n(z)$  by  $Q_0 = 0$ ,  $Q_1 = 1$  and  $Q_{n+1} = Q_n - zQ_{n-1}$ .

For the entirety of this paper, if  $\mu$  is a complex number we let  $\omega$  be such that  $\mu = \sqrt{\omega} + \sqrt{\omega^{-1}}$ . Note that  $\omega = -1$  is the same as  $\mu = 0$ .

It will be important to distinguish clearly between the abstract Temperley Lieb algebra defined by multiplication on a basis of diagrams, and a quotient of it which supports a  $C^*$ -algebra structure, and is only defined for special values of the parameter. So let  $TL_m(\delta)$  be the  $*$ -algebra over  $\mathbb{C}$  with basis formed by systems of disjoint curves (called strings) in a rectangle with  $m$  boundary points on the top and bottom as usual, with multiplication defined by stacking one rectangle on top of the other and removing closed strings with a multiplicative factor  $\delta$ . See [18] and [14] for details. (The  $*$  structure is defined by reflecting diagrams in a straight line half way between the bottom and top of a rectangle.)

It is important to note that there is a natural inclusion of  $TL_n$  in  $TL_{n+1}$ , obtained by adding a new through string to the right of a basis element of  $TL_n$ . We will often make the identification of  $TL_n$  with a subalgebra of  $TL_{n+1}$  without comment.

We include a picture of an element, called  $E_i$  for  $i = 1, \dots, n-1$ , in  $TL_n$  below for the convenience of the reader.



**Fig. 2.1**

In [13], for  $\delta \geq 2$  and  $\delta = 2 \cos \pi/a$  for each integer  $a = 3, 4, 5, \dots$  we constructed a tower of  $C^*$ -algebras, which we will call  $\mathcal{TL}_n$  for  $n = 1, 2, 3, \dots$  generated by the identity and orthogonal projections  $e_i$ ,  $i = 1, 2, \dots, n-1$  with the relations  $e_i e_{i\pm 1} e_i = \delta^{-2} e_i$  and  $e_i e_j = e_j e_i$  for  $|i - j| \geq 2$ . It is well known (see [7]) that there is a  $*$ -algebra homomorphism  $\Phi_n$  from  $TL_n$  onto  $\mathcal{TL}_n$  sending  $E_i$  onto  $\delta e_i$ . This homomorphism is compatible with the inclusions of  $TL_n \subseteq TL_{n+1}$  and  $\mathcal{TL}_n \subseteq \mathcal{TL}_{n+1}$ .  $\Phi$  is "generically" (i.e. for  $\delta \geq 2$ ) an isomorphism. When  $\Phi$  is not an isomorphism it is known (see below) that its kernel is the ideal generated by the "Jones-Wenzl" (JW) idempotent  $p_n \in TL_n$  defined in [23] by the inductive formula  $p_1 = 1$  and  $p_{n+1} = p_n - \frac{[n]}{[n+1]} p_n E_n p_n$  with  $\delta = q + q^{-1}$  as long as  $[j] \neq 0$  for  $j = 1, 2, \dots, n+1$ . As for  $TL_n \subset TL_{n+1}$ ,  $\mathcal{TL}_n$  is naturally included in  $\mathcal{TL}_{n+1}$  by adding a new vertical string to each basis diagram, and the  $\Phi_n$  are unital.

The unique irreducible representation of  $TL_n$  on which  $E_1$  (hence all  $E_i$ ) acts by zero will be called the *trivial* representation. Note that this passes to  $\mathcal{TL}_n$  exactly when  $n < a - 1$ .

For the convenience of the reader we give a proof that the kernel of  $\Phi$  is the ideal generated by the JW idempotent. Our proof will actually give a set of basis elements of  $TL_n$  that span a subalgebra mapped isomorphically onto  $\mathcal{TL}_n$  by  $\Phi$ .

**Theorem 2.2** (Goodman-Wenzl) *The kernel of the map  $\Phi : TL_n \rightarrow \mathcal{TL}_n$  is the ideal generated by  $p_{a-1}$  for  $n \geq a - 1$ .*

*Proof.* First we construct a sequence  $A_n$  of subalgebras of  $TL_n$ . For  $n < a - 1$  let  $A_n = TL_n$  and proceed inductively, setting  $A_{n+1} = A_n E_n A_n$  for  $n \geq a - 2$ . (Clearly  $A_n$  has a basis consisting of words on the  $E_i$ 's.) Although the  $A_n$  are not included in one another, each is individually an algebra. To see this use the maps ("conditional expectations")  $\mathcal{E}_n : TL_{n+1} \rightarrow TL_n$  defined on the diagram basis by connecting the rightmost top and bottom boundary points of a  $TL_{n+1}$  diagram to give a  $TL_n$  diagram. It is clear that  $\mathcal{E}_n(x E_n y) = xy$  for  $x, y \in TL_n$  and that  $E_n x E_n = \mathcal{E}_{n-1}(x) E_n$  for  $x \in TL_n$ .

One then proves inductively the following three assertions:

- (i)  $A_n$  is a subalgebra of  $TL_n$ .
- (ii)  $\mathcal{E}_n(A_n) \subseteq A_{n-1}$ .
- (iii)  $A_n$  is an  $A_{n-1}$ - $A_{n-1}$  bimodule under multiplication in  $TL_n$ .

Now let  $I_n$  be the ideal in  $TL_n$  generated by the JW idempotent  $p_{a-1}$  defined above. Observe that  $I_n \subseteq I_{n+1}$ . It follows immediately from the standard form of words on the  $E_i$ 's (see e.g. [13]) that  $TL_{n+1} = (TL_n)E_n(TL_n) \oplus \mathbb{C}id$  for all  $n$ . So since  $1 - p_{a-1} \in (TL_n)E_n(TL_n)$  for  $n \geq a - 1$ , we have  $TL_{n+1} = (TL_n)E_n(TL_n) \pmod{(I_{n+1})}$  for  $n \geq a - 1$ . Thus by induction

$$(*) \quad TL_{n+1} = A_n E_n A_n \pmod{(I_{n+1})} \text{ for } n \geq a - 1.$$

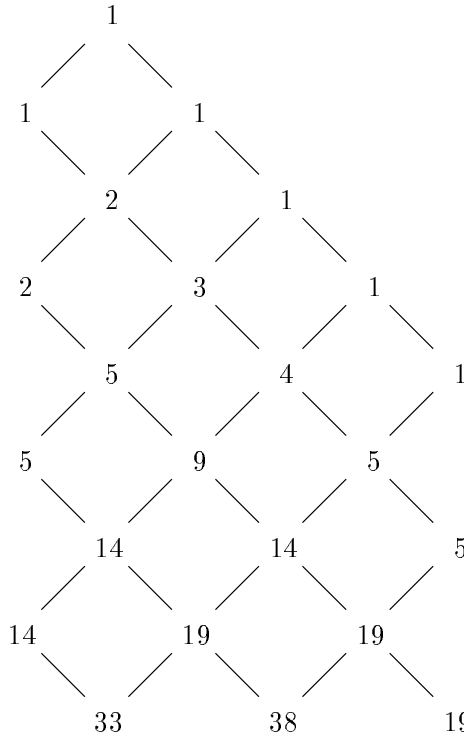
We now show, also by induction, that  $\Phi|_{A_n}$  is an isomorphism onto  $\mathcal{TL}_n$ . This assertion for  $n = a - 1$  is in some sense the main point of [13] since  $\ker(\Phi_{a-1})$  is spanned by  $1 - p_{a-1}$ . Now consider the following commutative diagram:

$$\begin{array}{ccc}
 A_n \otimes_{A_{n-1}} A_n & \xrightarrow{x \otimes y \mapsto x E_n y} & A_{n+1} \\
 \Phi \otimes \Phi \downarrow & & \downarrow \Phi \\
 \mathcal{TL}_n \otimes_{\mathcal{TL}_{n-1}} \mathcal{TL}_n & \xrightarrow{x \otimes y \mapsto w x e_n y} & \mathcal{TL}_{n+1}
 \end{array}$$

All the maps in this diagram are  $A$ -bimodule homomorphisms where  $\mathcal{TL}$  becomes an  $A$ - $A$  bimodule by transport of structure. It is shown in [7] that the bottom horizontal arrow is an isomorphism. The top horizontal arrow is surjective. It follows that the restriction of  $\Phi_{n+1}$  to  $A_{n+1}$  is an isomorphism.

Together with (\*), this proves the theorem.  $\square$

Thus the tower of algebras  $\mathcal{TL}_n$  admits a Bratteli diagram, which was shown in [13] to be as in Figure 2.3, exhibited for  $\delta = 2 \cos \pi/7$ .



**Fig. 2.3**

This Bratteli diagram can alternatively be thought of as giving the Hilbert space representations of  $TL$ , which may be obtained explicitly as follows. For each  $n = 0, 1, 2, \dots$  and each  $t \leq n$  with  $t \equiv n \pmod 2$  let  $W_{t,n}$  be the vector space whose basis is the set of rectangular  $(t, n)$  planar diagrams with  $t$  through strings.  $TL_n$  acts on  $W_{t,n}$  by concatenation in the obvious way, the answer being 0 if there are less than  $t$  through strings in the concatenated diagram. There is an invariant inner product  $\langle \cdot, \cdot \rangle$  on  $W_{t,n}$  defined by  $\langle \alpha, \beta \rangle = \beta^* \alpha$ , which is an element of the one dimensional vector space  $W_{t,n}$ . This inner product is positive semidefinite and the quotient  $\mathcal{W}_{t,n}$  is a Hilbert space affording a representation of  $\mathcal{TL}_m$ . This result is well known to the experts but probably does not appear anywhere in the literature.

Since the source of positivity for our annular Temperley-Lieb modules will be that of this inner product, we give a reasonably detailed proof here — tracking positivity down to its von Neumann algebraic origin.

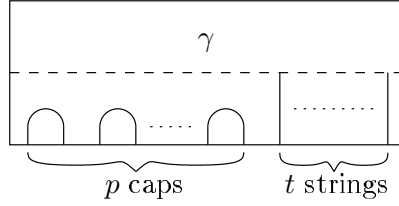
**Theorem 2.4** *For  $\delta \geq 2$  or  $\delta = 2 \cos \pi/a$ , for  $a = 3, 4, 5, \dots$ , the inner product on  $W_{t,n}$  defined in the previous paragraph is positive semidefinite; that is,  $\langle \alpha, \alpha \rangle \geq 0$  for all  $\alpha$ .*

*Proof.* We will effectively identify the representation on  $\mathcal{W}_{t,n}$  with the action on a principal left ideal in  $\mathcal{TL}_n$ , which has a positive definite inner product coming from the Markov trace of [13].

The algebra  $\mathcal{TL}_n$  was analysed in [13] using the basic construction. Adopting that technique, by induction, the irreducible representations  $\psi_t$  for  $0 \leq t \leq n$  with  $t \equiv n \pmod{2}$  so obtained are uniquely defined up to equivalence by the following property: if  $p$  is the largest integer such that  $\psi_t(e_1 e_3 e_5 \dots e_{2p-1})$  is not zero, then  $p = \frac{n-t}{2}$ . For each such  $t$  let  $q_t$  be the minimal central idempotent in  $\mathcal{TL}_n$  corresponding to  $\psi_t$  and define another inner product  $\{\cdot, \cdot\}$  on  $W_{t,n}$  by

$$\{\alpha, \beta\} = \text{tr}(\Phi(\tilde{\beta}^* \tilde{\alpha})q_t)$$

where  $\text{tr}$  denotes the Markov trace of [13] and, given a basis diagram  $\gamma \in W_{t,n}$ ,  $\tilde{\gamma}$  is the  $TL_{n,n}$  diagram obtained from  $\gamma$  as in Figure 2.5.



**Fig. 2.5**

Now observe that if  $\beta^* \alpha$  has fewer than  $t$  through strings then  $\Phi(\tilde{\beta}^* \tilde{\alpha})q_t = 0$ . This is because  $\tilde{\beta}^* \tilde{\alpha}$  may be written in the form  $\gamma_1 E_1 E_3 E_5 \dots E_{2k-1} \gamma_2$  with  $k > \frac{n-t}{2}$ . On the other hand if  $\beta^* \alpha$  has  $t$  through strings then

$$\tilde{\beta}^* \tilde{\alpha} = \langle \alpha, \beta \rangle \Phi(E_1 E_3 E_5 \dots E_{2p-1}).$$

Thus in this case the Markov trace of  $(\tilde{\beta}^* \tilde{\alpha})q_t$  is a positive non-zero multiple,  $K$ , depending only on  $n, \delta$ , and  $t$ , of  $\langle \alpha, \beta \rangle$ . Combining the two possibilities for the number of through strings we see that  $\{\alpha, \beta\} = K \langle \alpha, \beta \rangle$ . Since the trace on a  $\Pi_1$  factor gives a positive definite inner product,  $\{\cdot, \cdot\}$  is positive semidefinite and so is  $\langle \cdot, \cdot \rangle$ .  $\square$

We shall now obtain formulae for the dimensions of the individual  $\mathcal{W}_n^t$  for  $\delta = 2 \cos \pi/a$ . To this end let  $d_{t,m} = \dim(\mathcal{W}_{t+2m}^t)$  for  $t = 0, 1, 2, \dots, a-2$  and  $m = 0, 1, 2, \dots$ . Then the meaning of the Bratteli diagram is precisely that

$$d_{t,m} = d_{t-1,m} + d_{t+1,m-1},$$

with  $d_{t,-1} = 0$  for all  $t$ ,  $d_{a-1,n} = 0$  for all  $n$  and  $d_{-1,0} = 1$  but  $d_{-1,n} = 0$  for  $n > 0$ . By induction these relations uniquely determine the  $d_{t,n}$ . If we form the generating functions

$$D_t(z) = \sum_{n=0}^{\infty} d_{t,n} z^n$$

then these relations are equivalent to

$$zD_{t+1} = D_t - D_{t-1},$$

with  $D_{a-1} = 0$  and  $D_{-1} = 1$ .

Thus any power series  $D_t(z)$  satisfying these conditions must be the generating functions for the  $d_{t,n}$ . But if  $Q_r$  are the modified Tchebychev polynomials defined above then setting  $D_t = \frac{Q_{a-t-1}}{Q_a}$  we see that the relations are satisfied.

We see we have proved the following.

**Theorem 2.6** *For  $\delta = 2 \cos \pi/a$  and all integers  $t \geq 0$ , the generating function  $D_t(z) = \sum_{n=0}^{\infty} \dim \mathcal{W}_{t,t+2n} z^n$  is equal to  $\frac{Q_{a-t-1}}{Q_a}$ .*

**Remark 2.7** The ordinary Temperley-Lieb algebras may be turned into an algebroid in the obvious way with objects being the non-negative integers and morphisms from  $m$  to  $n$  being linear combinations of rectangular Temperley-Lieb diagrams with  $m$  points on the bottom boundary and  $n$  on the top. (So the morphisms are the zero vector space if  $m$  and  $n$  are different modulo 2.) It is clear that if we define for each  $t$  the vector space (graded by  $m$ ),  $\mathcal{W}^t = \{\mathcal{W}_{t,m}\}$  to be zero if  $t < m$  or  $t$  and  $m$  are not equal modulo 2, then these are the Hilbert space modules over the algebroid.

### 3 Affine Temperley-Lieb

Motivated by a conjecture of Freedman and Walker, we are going to define a slightly different version of the annular Temperley-Lieb algebra from that of [12]. It will be essentially the same as that of [9]. The difference is in how isotopies are required to act on the boundary. In order to avoid confusion with the definitions of [12], we will here call our diagrams "affine" rather than annular.

In the following definition, for a positive integer  $k$ ,  $\{k\}$  will denote the set of  $k$ th. roots of unity in  $\mathbb{C}$ . "The" annulus  $A$  will mean the set of complex numbers  $z$  with  $1 \leq |z| \leq 2$ .

**Definition 3.1** *Let  $m$  and  $n$  be two non-negative integers equal mod 2. An affine  $(m, n)$  TL diagram will be the intersection with the annulus of a system of smooth closed curves (strings) in  $\mathbb{C}$  that meet the boundary of the annulus transversally, precisely in the points  $\{m\}$  and  $2\{n\}$ . Such diagrams are considered to be the same if they differ by isotopies of the annulus which are the identity on the boundary.*

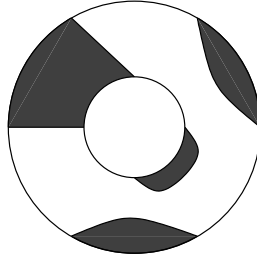
*An affine TL diagram will be called connected if it has no closed curves in the interior of the annulus.*

A through string in an affine TL diagram will be a string whose end points lie on different boundary components of  $A$ .

To make the set of all affine TL diagrams into a category we compose an  $(m, p)$  diagram  $\alpha$  with a  $(p, n)$  diagram  $\beta$  by  $\beta \circ \alpha = O(2\beta \cup \alpha)$  where we have smoothed the strings of  $\alpha$  and  $2\beta$  where they meet and  $O$  is the transformation of  $\mathbb{C}$  which sends  $re^{i\theta}$  to  $\sqrt{r}e^{i\theta}$ . (Smoothing could be avoided by requiring the isotopies to be the identity in a neighbourhood of the boundary and insisting that the strings be  $C^\infty$  perpendicular to the boundary.)

If  $m$  and  $n$  are even, an affine TL diagram admits a *shading*, i.e. a 2-colouring of the connected components of the complement of the strings in  $A$ , so that two components whose closures meet have different colours. The precise category that will interest us is the category with two objects  $(n, \pm)$  for each non-negative integer  $n$  and where the set of morphisms from  $(m, \pm)$  to  $(n, \pm)$  is the set of shaded affine  $(2m, 2n)$  TL diagrams. Shadings are determined by the following convention where "+" means shaded and "-" means unshaded — if  $\beta$  is a diagram giving a morphism from  $(m, sgn)$  to  $(n, sgn')$  then on the inner boundary of  $A$  a small region close to 1 and in the first quadrant is shaded according to  $sgn$  and a small region close to 2 and in the first quadrant is shaded according to  $sgn'$ . We illustrate in Figure 3.2 by giving an example of a morphism from  $(2, -)$  to  $(3, +)$ .

Fig. 3.2



Given an affine TL diagram  $\alpha$ ,  $\hat{\alpha}$  will denote the connected diagram formed by removing all contractible closed strings from  $\alpha$ , and  $c(\alpha)$  will be the number of contractible closed strings in  $\alpha$ .

**Definition 3.3** The affine TL algebra  $AffTL$  with parameter  $\delta \in \mathbb{C}$  will be the category whose objects are  $\mathbb{N} \cup \{0\} \times \{+, -\}$ , whose set of morphisms,  $AffTL_{(m, \pm), (n, \pm)}$ , is the vector space having as basis the set of shaded connected affine TL  $(2m, 2n)$  diagrams as above, and multiplication between composable morphisms is the linear extension of the map on basis elements given by

$$\beta\alpha = \delta^{c(\beta\circ\alpha)} \widehat{\beta\circ\alpha}.$$



A representation of  $AffTL$  will be a covariant functor from this category into the category of vector spaces.

The transformation  $z \mapsto 2/\bar{z}$  of  $\mathbb{C}$  preserves affine  $TL$  diagrams and so defines a conjugate-linear antiinvolution  $*$  of the algebroid  $AffTL$ . A representation  $\pi$  of  $AffTL$  will be called a Hilbert representation if the representing vector spaces are Hilbert spaces and  $\pi(\alpha^*) = \pi(\alpha)^*$  for all diagrams  $\alpha$ .

**Remark 3.4** *Having taken linear combinations of annular diagrams we can now give a meaning to an annular diagram which also contains a (contractible) rectangle with  $2m$  boundary points labelled by an element  $x \in TL_k$ . Such a diagram will mean the linear combination of annular diagrams obtained by writing  $x$  as a linear combination of basis elements and inserting those basis elements in the rectangle to obtain a linear combination of  $AffTL$  elements. The beginning boundary point on the rectangle would need to be marked if there were any ambiguity.*

Hilbert representations admit an obvious direct sum operation and in this paper we wish to classify all Hilbert representations into the category of finite dimensional Hilbert spaces. They will all be quotients of a universal family which we now define.

For the rest of this section we suppose that  $sgn$  is a fixed sign,  $+$  or  $-$ , and all statements are to be true for both values of  $sgn$ .

**Definition 3.5** *For any positive integer  $k$  and complex number  $\omega$  let  $V_{n,sgn}^{k,\omega}$  be the graded vector space (graded by the subscripts, in  $(\mathbb{N} \cup \{0\}) \times \{+, -\}$ ) that is the quotient of  $AffTL_{(k,+),(n,sgn)}$  by the subspace spanned by all diagrams with less than  $2k$  through strings (so that  $V_{n,sgn}^{k,\omega} = 0$  for  $n < k$ ) and all elements of the form  $\alpha\rho - \omega\alpha$  where  $\rho \in AffTL_{(k,+),(k,+)}$  is the diagram all of whose strings are through strings and for which 1 is connected to  $2e^{4\pi i/2k}$ . (We will use the notation  $\rho_k$  if we need to specify the actual number of strings  $\rho$  has. Note that  $\rho_k^k$  is the rotation by  $2\pi$ .)*

*For  $\omega \neq -1$  and  $\mu \neq \delta$  we let  $V_{n,sgn}^{0,\omega}$  be the graded vector space (graded by  $(\mathbb{N} \cup \{0\}) \times \{+, -\}$ ) that is the quotient of the vector space  $AffTL_{(0,+),(n,sgn)}$  by the linear span of elements of the form  $\alpha\sigma^*\sigma - \bar{\mu}\mu\alpha$ , where  $\sigma$  is an element of  $AffTL_{(0,+),(0,-)}$  is the diagram having exactly one closed homologically non-trivial (in  $A$ ) string.*

*For  $k = 0$  and  $\mu = \delta$  we let  $V_{n,sgn}^{0,\omega}$  be the vector space with basis the set of all ordinary Temperley-Lieb diagrams in a disc with boundary points being the  $2n$ -th. roots of unity, and having the shading determined by  $sgn$ . This is acted on in the obvious way by  $AffTL$ . Note also that it is the quotient*

of  $AffTL_{(0,sgn),(n,sgn)}$  by the relation that sets a diagram equal to any other diagram with the same system of connections between boundary points.

For  $\omega = -1$  (hence  $\mu = 0$ ) we let

(a)  $V_{n,sgn}^{0,-1,sgn}$  be the quotient of the vector space  $AffTL_{(0,sgn),(n,sgn)}$  by the linear span of elements of the form  $\alpha\sigma^*\sigma$ , and

(b)  $V_{n,sgn}^{0,-1,-sgn}$  be the quotient of the vector space  $AffTL_{(0,-sgn),(n,sgn)}$  by the linear span of elements of the form  $\alpha\sigma^*$  (or  $\alpha\sigma$  according to  $sgn$ ).

**Remark 3.6** Note that  $V_{n,sgn}^{0,\omega}$  depends only on  $\mu$  through  $\omega$  (as  $\mu = \sqrt{\omega} + \sqrt{\omega}^{-1}$ ), so using  $\omega$  in the notation is justified.

The special treatment of the case  $\omega = -1$  is unfortunate but unavoidable. If one defined two different such representations in all cases, then if  $k \neq 0$  they would be isomorphic via either  $\rho$  or a diagram with one homologically non-trivial circle, but this last map is not invertible if  $\mu = 0$ . Also of course these two representations  $V^{0,-1,\pm}$  are inequivalent since the two spaces graded by 0 have different dimensions.

**Remark 3.7** Since composition of tangles does not increase the number of through strings and the action of tangles on the inside annular boundary commutes with the action on the outside, the  $V^{k,\omega}$  become modules over  $AffTL$  by composition in that category.

**Remark 3.8** Observe that  $V_{n,\pm}^{k,\omega}$  is finite dimensional for fixed  $k$  and  $n$ . We will need their dimensions, which can be calculated by counting diagrams exactly as in [12]:

- a) For  $k > 0$  and  $n \geq k$ ,  $\dim(V_{n,\pm}^{k,\omega}) = \binom{2n}{n-k}$
- b) For  $k = 0$ ,  $\mu = 0$  and  $n > 0$ ,  $\dim(V_{n,\pm}^{0,\omega,\pm}) = \frac{1}{2} \binom{2n}{n}$ ;  
 $\dim(V_{0,sgn}^{0,-1,sgn}) = 1$ ,  $\dim(V_{0,-sgn}^{0,-1,sgn}) = 0$
- c) For  $k = 0$  and  $\mu = \delta$ ,  $\dim(V_{n,\pm}^{0,\omega}) = \frac{1}{n+1} \binom{2n}{n}$
- d) For  $k = 0$  and  $0 < \mu < \delta$ ,  $\dim(V_{n,\pm}^{0,\omega}) = \binom{2n}{n}$

For uniformity of notation, in the case  $k = 0$ ,  $\mu = 0$  we will use the superscript  $\omega$  to denote the pair  $(-1, \pm)$  in the above formulae.

We now define the key ingredient of this paper, a sesquilinear form on each  $V_{n,\pm}^{k,\omega}$ . To this end note that the quotient  $Aff_{k,sgn}$  of  $AffTL_{(k,sgn),(k,sgn)}$  by the subspace spanned by diagrams with fewer than  $k$  through strings is

a unital  $*$ -algebra freely generated by the element  $\rho$  when  $k > 0$ , and  $\sigma^*\sigma$  when  $k = 0$ . These generators are unitary and self-adjoint, respectively. Thus if  $|\omega| = 1$  and  $\mu \in \mathbb{C}$  we may define unital  $*$ -algebra homomorphisms  $\phi : Aff_{k,sgn} \rightarrow \mathbb{C}$  by  $\phi(\rho) = \omega$  and  $\phi(\sigma^*\sigma) = \bar{\mu}\mu$  respectively. We also use the letter  $\phi$  for the  $*$ -algebra homomorphism from  $AffTL_{(k,sgn),(k,sgn)}$  to  $\mathbb{C}$  obtained by composing with the quotient map.

**Definition 3.9** *With notation as in the last paragraph, define the sesquilinear forms  $\langle \cdot, \cdot \rangle$  on each  $AffTL_{(k,+),(n,sgn)}$  by  $\langle v, w \rangle = \phi(w^*v)$ .*

**Proposition 3.10** *The sesquilinear form of Definition 3.9 is invariant; that is,  $\langle \alpha v, w \rangle = \langle v, \alpha^* w \rangle$ .*

*Proof.* This follows immediately from  $w^*\alpha v = (\alpha w)^*v$  and the fact that  $\phi$  is a  $*$ -algebra homomorphism.  $\square$

**Proposition 3.11** *The sesquilinear form of 3.9 passes to the quotient  $V_{n,sgn}^{k,\omega}$ .*

*Proof.* If  $\delta \neq \mu$  it follows from the  $*$ -homomorphism property of  $\phi$  that the elements defined in Definition 3.5 spanning the subspace by which the quotient was taken are orthogonal to all diagrams in  $AffTL_{(k,+),(n,sgn)}$ . The case (b) of 3.5 requires care. One observes that if  $v = \alpha\sigma$  and  $w$  is any diagram in the same space then  $w^*v$  is actually a multiple of  $\delta$  times an element of the form  $\beta\sigma^*\sigma$ . This is because, after the removal of homologically trivial circles,  $w^*v$  has a homologically nontrivial circle, hence at least two because the shadings near the inner and outer boundaries have to match.

Finally in the case  $\mu = \delta$ , if two diagrams  $v$  and  $v'$  define the same system of connections among boundary points, the diagrams for  $w^*v$  and  $w^*v'$  are both the same system of closed curves with the inner annulus boundary possibly in different regions. The homologically non-trivial closed curves must occur in pairs for the annulus boundary shadings to match, and since  $\mu = \delta$ , such a pair will count the same if it is dealt with by  $\phi$  or if it is homologically trivial.  $\square$

The element  $1 \in V_{k,+}^{k,\omega}$  clearly generates  $V^{k,\omega}$  as a representation of  $AffTL$ . We will call it the vacuum vector and write it  $v_\omega$ . It satisfies the following properties, where the  $\epsilon_i$  are as in Definition 2.8 of [12].

- a) When  $k > 0$ ,
  - (i)  $\langle v_\omega, v_\omega \rangle = 1$
  - (ii)  $\rho(v_\omega) = \omega v_\omega$
  - (iii)  $\epsilon_i(v_\omega) = 0$  for  $0 \leq i \leq 2k - 1$
- b) When  $k = 0$ ,

- (i)  $\langle v_\omega, v_\omega \rangle = 1$
- (ii)  $\sigma^* \sigma(v_\omega) = \bar{\mu} \mu v_\omega$

The following fundamental lemma was poorly treated in [12]. This was because the conclusion was obvious from spherical invariance in the planar algebras to which it was applied. We give a careful proof here.

**Lemma 3.12** *The inner products in  $V^{k,\omega}$  can be calculated using just properties a) and b) above.*

Proof. Since any vector in  $V^{k,\omega}$  is a linear combination of affine diagrams applied to  $v_\omega$ , it suffices by invariance to show how the equations suffice to calculate  $\langle \alpha v_\omega, v_\omega \rangle$ , with  $\alpha$  connected. First consider the case  $t > 0$ . Then  $\alpha$  is a connected affine tangle with  $2k$  inner and outer boundary points. If all the strings are not through strings,  $\alpha = \alpha' \epsilon_i$  for some  $i$ , and the inner product is zero. If all the strings are through strings  $\alpha$  is necessarily some power of  $\rho$  so the inner product is determined by properties (i) and (ii).

Now suppose  $k = 0$ . Then by connectedness  $\alpha$  consists of a certain number of strings which may be isotoped into concentric circles. They must be even in number since the inner and outer boundaries have the same shading. This means precisely that  $\alpha$  is a power of  $\sigma^* \sigma$ .  $\square$

**Corollary 3.13** *If  $\omega$  is such that there is an irreducible Hilbert representation of  $AffTL$ , then it is isomorphic to a quotient of a  $V^{k,\omega}$  and the corresponding  $\langle \cdot, \cdot \rangle$  of Definition 3.9 is positive semidefinite. If  $k = 0$ ,  $0 \leq \mu \leq \delta$ .*

Proof. As in [12], if  $\mathcal{U}$  is an irreducible Hilbert space representation, all the  $\mathcal{U}_{m,sgn}$  have to be irreducible  $AffTL_{(m,sgn),(m,sgn)}$  modules. Let  $k$  be the smallest integer for which  $\mathcal{U}_{k,\pm}$  is non-zero (this  $k$  is called the lowest weight and  $\mathcal{U}_{(k,\pm)}$  is called the lowest weight space). Then  $AffTL_{(m,sgn),(m,sgn)}$  acts on the lowest weight space via the abelian quotient  $AffTL_{k,sgn}$  defined before Definition 3.9. By some version of Schur's lemma the lowest weight space is thus one dimensional and the unitary  $\rho$  must act by some  $\omega$ , with  $|\omega| = 1$ , or, if  $k = 0$ ,  $\sigma^* \sigma$  must act by some non-negative constant — choose  $\mu$  to be a non-negative square root of that constant and then choose  $\omega$  accordingly. A lowest weight vector of unit length in  $\mathcal{U}_{k,+}$  will then satisfy all the conditions of (a) or (b) above so we may define a  $\langle \cdot, \cdot \rangle$ -preserving map from the corresponding  $V^{t,\omega}$  onto  $\mathcal{U}$  by sending, for any connected  $\alpha$ ,  $\alpha(v_\omega)$  onto the  $\alpha$  applied to a lowest weight unit vector  $u \in \mathcal{U}$ . So  $\langle \cdot, \cdot \rangle$  is positive semidefinite, since the inner product in  $\mathcal{U}$  is.

The case  $k = 0$ ,  $\omega = -1$  does not quite work as above. Then either  $\mathcal{U}_{0,+}$  or  $\mathcal{U}_{0,-}$  must be non-zero; suppose that it is  $\mathcal{U}_{0,sgn}$  and choose  $u$  therein.

Then  $\|\sigma(u)\| = 0$  (or  $\|\sigma^*(u)\| = 0$ ) so by irreducibility  $\mathcal{U}_{0,-sgn} = 0$ . Then proceed as before to obtain an isomorphism between  $\mathcal{U}$  and  $V^{0,-1,sgn}$ .

That  $\mu \leq \delta$  follows as in [12].  $\square$

Conversely if  $\langle \cdot, \cdot \rangle$  is positive semidefinite on some  $V^{k,\omega}$ , the quotient by its kernel is a Hilbert space representation of  $AffTL$ , which we call  $\mathcal{V}^{k,\omega}$ .

**Lemma 3.14**  $\mathcal{V}^{k,\omega}$  is irreducible.

Proof. Suppose  $v \in \mathcal{V}^{k,\omega}$  is non-zero. Then  $\langle v, v \rangle \neq 0$ . But  $v = \sum_{\alpha} c_{\alpha} \alpha v_{\omega}$  for some affine diagrams  $\alpha$  and constants  $c_{\alpha}$ . So  $\langle \sum_{\alpha} \bar{c}_{\alpha} \alpha^* v, v_{\omega} \rangle \neq 0$  so that  $v_{\omega}$ , hence all of  $\mathcal{V}^{k,\omega}$ , is in the  $AffTL$  span of  $v$ .  $\square$

**Definition 3.15** If  $\mathcal{U}$  is an irreducible representation of  $AffTL$  isomorphic to  $\mathcal{V}^{k,\omega}$  then  $k$  will be called the lowest weight of  $\mathcal{U}$  and  $\omega$  will be called the chirality.

The determination of the set of values of  $\delta$ ,  $\omega$  and  $\mu$  for which the sesquilinear form  $\langle \cdot, \cdot \rangle$  is positive semidefinite is the subject of the next sections.

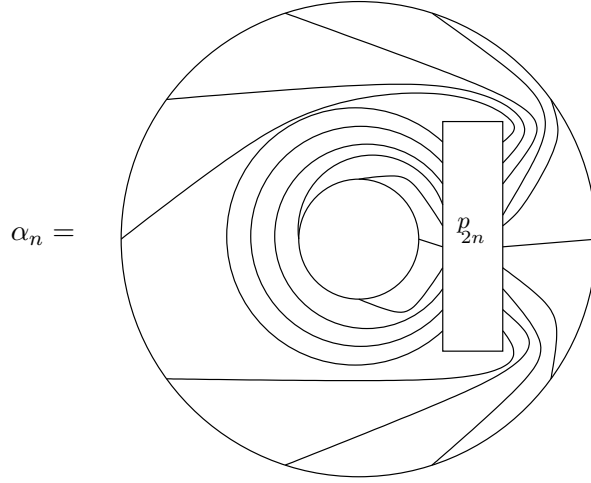
Finally remark that the representations  $V^{k,\omega}$  are all mutually inequivalent except when  $k = 0$  (when clearly  $V^{k,\omega}$  and  $V^{k,\omega^{-1}}$  are the same). Also  $V^{0,0,+}$  and  $V^{0,0,-}$  are inequivalent since the dimensions of the spaces graded by  $(0,+)$  and  $(0,-)$  are different. Thus at the end of this paper we will have obtained a complete list of irreducible Hilbert representations of  $AffTL$ .

## 4 The formula of Graham and Lehrer.

Let  $V^{k,\omega}$ , with  $|\omega| = 1$  or  $0 \leq \mu \leq \delta$ , be the affine Temperley-Lieb module constructed in the previous section and let  $v_{\omega}$  be a lowest weight unit vector therein.

**Definition 4.1** For  $k \leq n$  we will call  $\alpha_n$  the element of  $AffTL_{(k,+),(n,+)}$  containing one copy of the JW idempotent  $p_{2n}$  in a rectangle whose first boundary point is connected to  $-2$  and the next  $2n - 1$  (in cyclic order) are also connected to the outside boundary of  $A$ . The  $2k$  boundary points on the inside boundary of  $A$  are connected to the middle  $2k$  of the remaining boundary points of the rectangle and the other boundary points of the rectangle are connected to each other in the unique (planar) way so that none is connected to its nearest neighbour.

We illustrate Definition 4.1 in Figure 4.2 for  $k = 2$  and  $n = 5$ . The boundary points of the inner circle are the 4th roots of unity and the ones on the outer circle are the 10th ( $=2n$ th) roots of unity. Note that the order of shaded regions on the boundary of the rectangle containing  $p_{2n}$  will depend on the parity of  $n$ , but the ordinary  $TL$  algebra makes sense without shading the regions.



**Fig. 4.2**

Note that Figure 4.2 represents a (non-zero) linear combination of annular  $(2, 6)$  tangles obtained by expanding the J-W idempotent  $p_{2n}$  in the rectangle.

For each  $r \geq 0$  we set  $2n = 2k + 2r$  and define the vector  $w_n \in V_n^{k, \omega}$  to be the result of applying the annular element  $\alpha_n$  of Figure 4.2 to  $v_\omega$ .

Let  $C_n = \langle w_n, w_n \rangle$  for  $n > k$  and  $C_k = 1$ . Our main task in this paper will be to establish whether  $C_n$  is positive, negative or zero.

**Theorem 4.3** *Suppose  $\delta (= q + q^{-1})$  and  $n$  satisfy  $\delta > 2 \cos \pi/2n \geq 0$  (so that in particular the map  $\Phi : TL_{2n-1} \rightarrow \mathcal{TL}_{2n-1}$  is an isomorphism, and the JW idempotent  $p_{2n}$  is defined). Then with  $r = n - k$ ,*

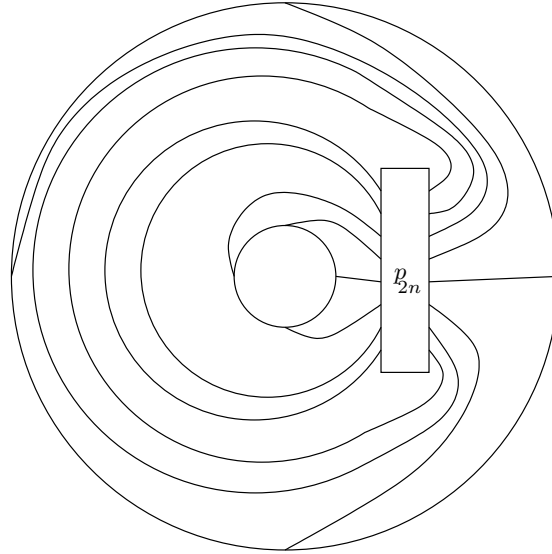
$$C_n = \frac{[r][r+k]}{[2n][2n-1]} (q^{2n} + q^{-2n} - \omega - \omega^{-1}) C_{n-1}$$

Proof. First note that neither  $[2n]$  nor  $[2n-1]$  is zero. If  $\delta > 2$  this is obvious. Otherwise write  $\delta = 2 \cos \pi/a$  ( $a$  not necessarily an integer) so that the condition becomes  $\pi/2n > \pi/a$ , or  $2n\pi/a < \pi$ . Then  $[2n] = \frac{\sin 2n\pi/a}{\sin \pi/a}$ , which is strictly positive.

We want to calculate  $C_n = \langle \alpha_n(v_\omega), \alpha_n(v_\omega) \rangle$ . By invariance it will suffice to express  $\alpha_n^* \alpha_n$  in terms of  $\alpha_{n-1}^* \alpha_{n-1}$ , which we proceed to do.

Case (i),  $k > 0$ .

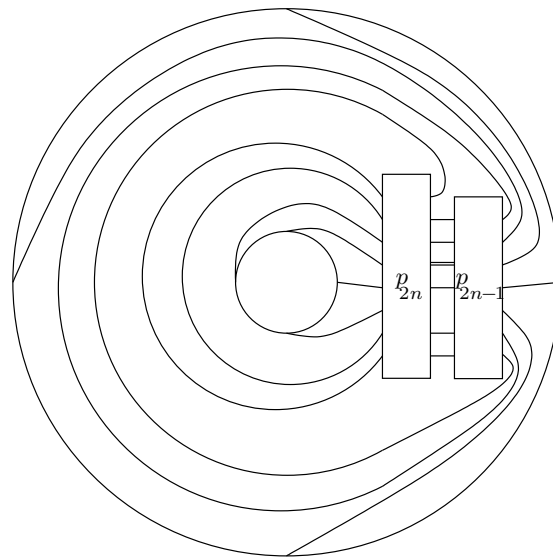
In Fig. 4.4 we have drawn  $\alpha_n^* \alpha_n$  (where for clarity we have reduced from  $n = 5$  in Fig. 4.2 to  $n = 4$ ).



**Fig. 4.4**

The first step is to introduce a JW idempotent on one less string. Because of the order on these idempotents, the *AffTL* element in Figure 4.5 is the same as in 4.4.

**Fig. 4.5**



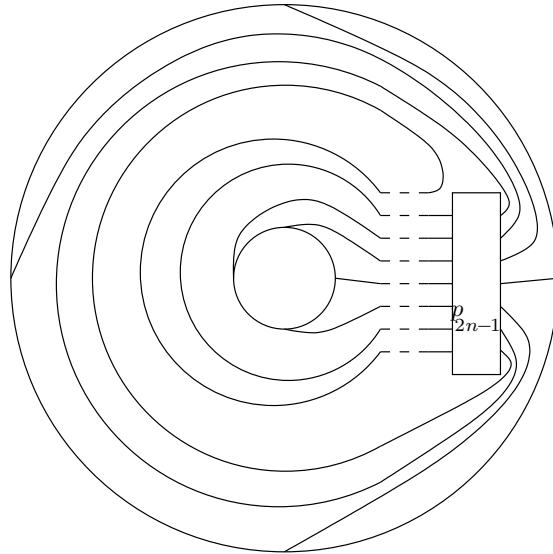
It is easily seen that there are only 3 tangles in the expansion of  $p_{2m}$  that give non-zero contributions in Fig. 4.5. There must be at least  $2n - 2$  through strings inside the rectangle, or two adjacent boundary points on the left side

of the rectangle containing  $p_{2n-1}$  would be connected — giving zero. So the only adjacent boundary points on the right side of the  $p_{2n}$  rectangle that can be connected are the top two. Then it is easy to check that the only two pairs of adjacent boundary points on the left side of the  $p_{2n}$  rectangle that can be connected are the ones which have exactly one point connected to the inner boundary circle of  $A$ . Thus we see that

$$\alpha_n^* \alpha_n = X + c_Y Y + c_Z Z$$

where  $X, Y$  and  $Z$  are given in Figures 4.6, 4.7, and 4.8, respectively.

**Fig. 4.6**



**Fig. 4.7**

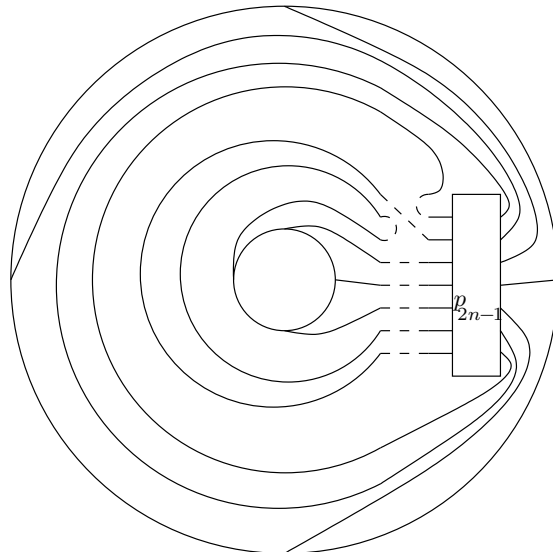
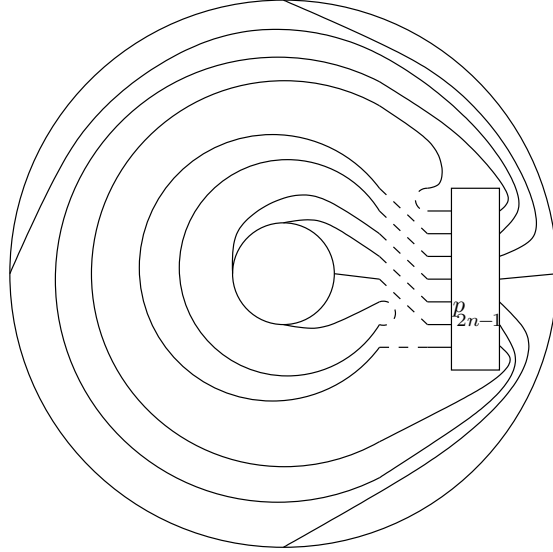




Fig. 4.8



We deal first with the situation in 4.6. Here after an isotopy we see that the bottom left and right boundary points of the  $p_{2n-1}$  rectangle are connected to each other. The result is well known to be a multiple of  $p_{2n-2}$ . By comparing the coefficient of the identity in the expansion of the idempotent, the multiple is seen to be  $\frac{[2n]}{[2n-1]}$ , so that

$$X = \frac{[2n]}{[2n-1]} \alpha_{n-1}^* \alpha_{n-1}.$$

The arguments for  $Y$  and  $Z$  are structurally identical and differ only in the constants and the direction in which the inner circle is rotated. In Figure 4.7, the the diagram inside the rectangle for  $p_{2n}$  is  $E_r E_{r-1} \cdots E_1$ , which has a coefficient of  $\frac{(-1)^r [r+2k]}{[2n]}$  in JW (see e.g. [12]). In Figure 4.8, the diagram is  $E_{r+2k} E_{r+2k-1} E_{r+2k-2} \cdots E_1$ ; this diagram's coefficient is  $(-1)^r \frac{[r]}{[2n]}$ . Since we will be doing many of these calculations, we record the relevant coefficient in the JW idempotent pictorially below:

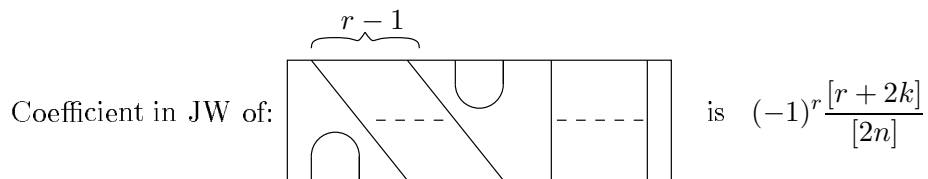


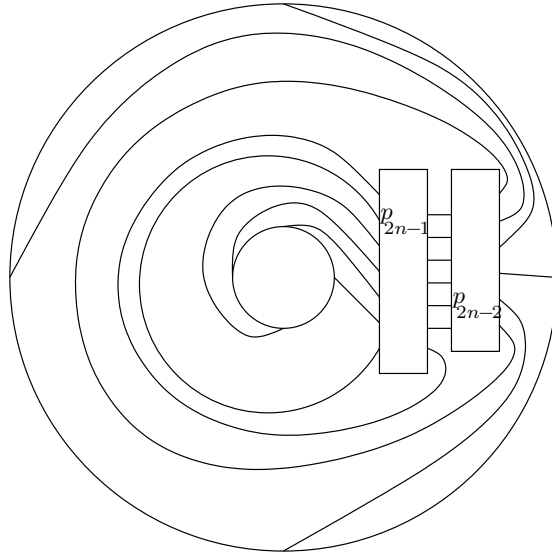
Fig. 4.9

Thus at this stage we have

$$\alpha_n^* \alpha_n = \frac{[2n]}{[2n-1]} \alpha_{n-1}^* \alpha_{n-1} + (-1)^r \left\{ \frac{[r+2k]}{[2n]} Y + \frac{[r]}{[2n]} Z \right\}.$$

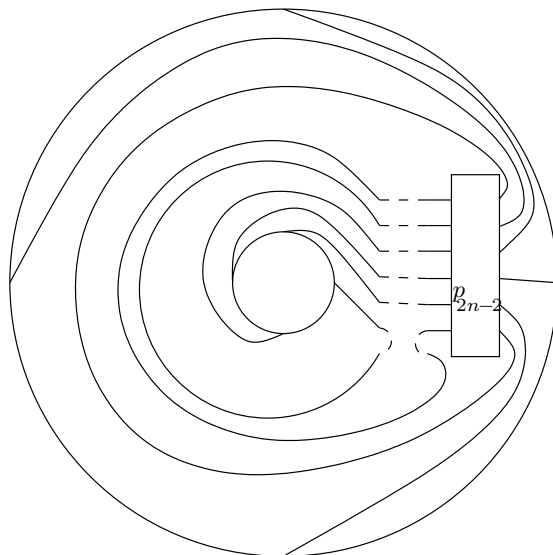
If we now start with Fig. 4.8 and insert a  $p_{2n-2}$  we obtain Fig. 4.10.

**Fig. 4.10**

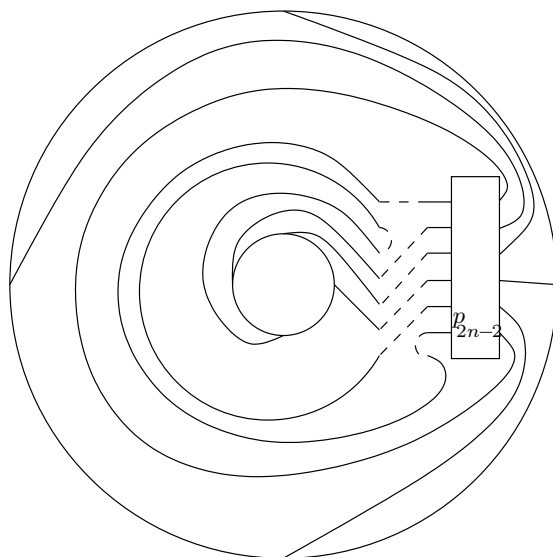


For  $Z$ , consideration of all the possible  $TL$  diagrams inside the  $p_{2n-1}$  rectangle shows that the only ones giving a non-zero contribution are ones with  $2n-2$  through strings and the top and bottom boundary points of the inner annulus boundary are connected down and up respectively to their nearest neighbours as shown in Fig. 4.11 and Fig. 4.12.

**Fig. 4.11**



**Fig. 4.12**



The coefficient of the  $TL$  diagram from Fig. 4.11 is, again by 4.9,

$$\frac{(-1)^{r+1}[r + 2k]}{[2n - 1]}$$

and the coefficient of the  $TL$  diagram from Fig 4.12 is

$$\frac{(-1)^{r+1}[r]}{[2n - 1]}.$$

But notice that Figure 4.11 is just  $\alpha_{n-1}^* \alpha_{n-1}$  composed with  $\rho$  and Fig. 4.12 is just  $\alpha_{n-1}^* \alpha_{n-1}$ . So we have

$$Z = (-1)^{r+1} \alpha_{n-1}^* \alpha_{n-1} \left( \frac{[r+2k]}{[2n-1]} \rho + \frac{[r]}{[2n-1]} \right).$$

Doing the corresponding calculation for Fig. 4.7 we obtain that

$$Y = (-1)^{r+1} \alpha_{n-1}^* \alpha_{n-1} \left( \frac{[r+2k]}{[2n-1]} + \frac{[r]}{[2n-1]} \rho^{-1} \right).$$

So

$$C_Y Y + C_Z Z = \frac{-1}{[2n][2n-1]} \alpha_{n-1}^* \alpha_{n-1} ([r]^2 + [2k+r]^2 - [r][r+2k](\rho + \rho^{-1})).$$

Altogether,

$$\alpha_n^* \alpha_n = \frac{[r][r+2k]}{[2n][2n-1]} \alpha_{n-1}^* \alpha_{n-1} \left\{ \frac{[2n]^2}{[r][r+2k]} - \frac{[r]}{[r+2k]} - \frac{[r+2k]}{[r]} - \rho - \rho^{-1} \right\}.$$

But we have the identity  $[2n]^2 - [r]^2 - [2n-r]^2 = (q^{2n} + q^{-2n})[r][2n-r]$ , and on  $V_k^{k,\omega}$ ,  $\rho = \omega$  so that

$$C_n = \frac{[n-k][n+k]}{[2n][2n-1]} \left\{ q^{2n} + q^{-2n} - \omega - \omega^{-1} \right\} C_{n-1}$$

This proves the theorem when  $k > 0$ .

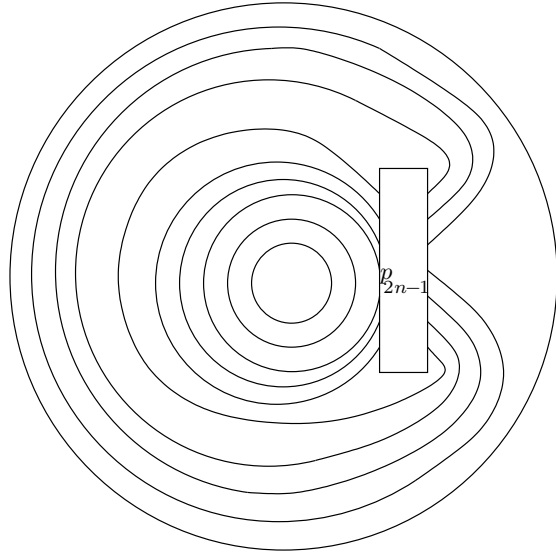
The case  $\mu = \delta$  needs no consideration since in this case ordinary  $TL$  diagrams inside a disc provide a Hilbert representation.

Case (ii);  $0 < \mu < \delta$ .

We may consider Figure 4.5 when  $k = 0$ . In this case there are only two ways to fill in the  $p_{2n}$  rectangle to obtain non-zero diagrams. The first is with the identity which gives  $\frac{[2n]}{[2n-1]}$  as before, and the second, which is the common case  $r = n$  of the terms  $Y$  and  $Z$  in the previous argument, so we have:

$$\alpha_n^* \alpha_n = \frac{[2n]}{[2n-1]} \alpha_{n-1}^* \alpha_{n-1} + (-1)^n \frac{[n]}{[2n]} Y'$$

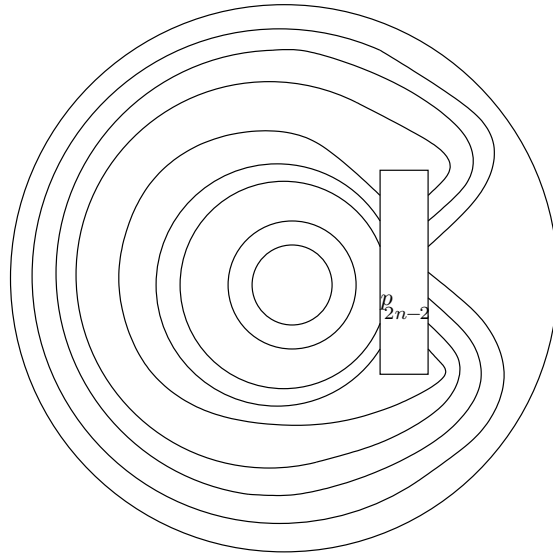
where  $Y'$  is the tangle with the inner annulus boundary surrounded by a homologically nontrivial circle as illustrated in Fig. 4.13.



**Fig. 4.13**

Introducing a  $p_{2n-2}$  as before there is only one contributing diagram that can be put in the  $p_{2n-1}$  rectangle and its coefficient is  $\frac{(-1)^{n-1}[n]}{[2n-1]}$ . The resulting annular diagram is Fig 4.14:

**Fig. 4.14**



Note that the innermost circle is an annulus boundary and the next two are strings — which contribute precisely  $\sigma^* \sigma$ . Thus we have

$$\alpha_n^* \alpha_n = \alpha_{n-1}^* \alpha_{n-1} \left\{ \frac{[2n]}{[2n-1]} - \frac{[n]^2}{[2n][2n-1]} \sigma^* \sigma \right\}.$$

But  $\sigma^*\sigma$  acts as  $\mu^2$  on  $V_0^{0,\omega}$  so that

$$C_n = \left( \frac{[2n]}{[2n-1]} - \frac{\mu^2[n]^2}{[2n][2n-1]} \right) C_{n-1}.$$

Using  $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ , and  $\mu^2 = 2 + \omega + \omega^{-1}$  we get:

$$C_n = \frac{[n]^2}{[2n][2n-1]} \left( q^{2n} + q^{-2n} - \omega - \omega^{-1} \right) C_{n-1}.$$

This proves the theorem in case (ii).

The only remaining case is  $k = 0, \omega = 0$  where of course " $\omega + \omega^{-1}$ " is taken to mean zero. In this case the argument is extremely simple as the term  $Y$  in case (ii) already acts by zero. Note that in fact there are two cases for  $w_n$  in this situation according to the shading on the inside annulus boundary. This does not change the argument in any way.

□

## 5 Restrictions on $\delta$ , $k$ and $\omega$ when $\delta < 2$ .

**Theorem 5.1** *Suppose  $\mathcal{U}$  is an irreducible Hilbert space AffTL module with lowest weight  $k$  and chirality  $\omega$ . Suppose  $\delta = 2 \cos \pi/a$  for  $a = 3, 4, \dots$  (choose  $q = e^{\frac{\pi i}{a}}$ ). Then*

$$\omega = q^{\pm 2r} \quad \text{for some integer } r \quad \text{with } k < r \leq a/2.$$

Proof. Our first job is to show that  $k < a/2$ . Suppose  $2k \geq a$  and let  $u$  be a unit vector spanning  $\mathcal{U}_k$ . Then consider the  $a$  annular tangles  $\nu_l$ ,  $l = 1, \dots, a$  with  $2k$  inner boundary points and  $2k + 2$  outer ones,  $2k$  through strings, with  $2e^{\frac{l\pi i}{k+1}}$  connected to  $2e^{\frac{(l+1)\pi i}{k+1}}$  and 1 connected to 2. The matrix of inner products of the vectors  $\nu_l(u)$  is the  $a \times a$  matrix with  $\delta$  on the diagonal, one on the first off-diagonals and 0 elsewhere. The determinant of this matrix is well known to be  $[a+1] = \frac{\sin(a+1)\pi/a}{\sin \pi/a}$ , which is negative. This is impossible in a Hilbert space, so  $k < a/2$ .

Since the rotation is unitary and  $0 \leq \mu \leq \delta$  we know that  $|\omega| = 1$  and we may suppose, by taking the complex conjugate if necessary, that  $Im(\omega) \geq 0$ . Let  $\theta = arg(\omega)$ . We know from Corollary 3.13 that  $\mathcal{U}$  is a quotient of a  $V^{k,\omega}$  so that it makes sense to talk about  $v_k, w_n$  etc. Then if  $\theta$  is not  $2r\pi/a$  for some  $r$  with  $k < r \leq a/2$  then let  $r_0$  be the largest value of  $r$  with  $\theta \geq 2r\pi/a$ . Suppose first that  $2(r_0 + 1) < a$ . Then by Theorem 4.3 we have  $C_m > 0$  for

$k \leq m \leq r_0$  but  $C_m < 0$  for  $m = r_0 + 1$ , which is disallowed by positivity. So we may suppose  $2r_0 < a$  but  $2(r_0 + 1) \geq a$ . We will divide the proof into two cases.

Case (i)  $a$  odd so that  $2r_0 + 1 = a$ .

The difficulty is clear: using Theorem 4.3 we get  $C_m > 0$  for  $1 \leq m \leq r_0$  but we cannot apply Theorem 4.3 for  $r_0 + 1$  as the hypotheses of the theorem are no longer valid.

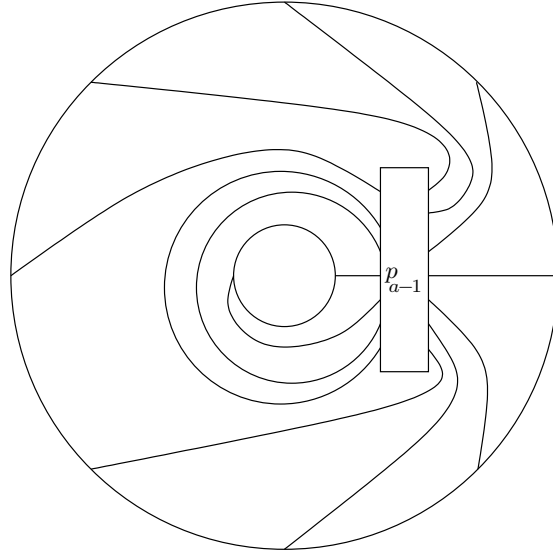
But the vector  $w_{r_0}$  still exists and by Theorem 4.3 it is non-zero. Now form, as above,  $2r_0 + 1$  vectors in  $\mathcal{U}_{r_0+1}$  from  $w_{r_0}$  by applying  $2r_0 + 1$  annular  $(2r_0, 2r_0+2)$  diagrams in which one pair of outer boundary points is connected to its nearest neighbour and all other strings are through strings, excluding the one in which  $-2$  is connected to the neighbouring boundary point with negative imaginary part. If the vector  $w_{r_0}$  is normalised to be a unit vector, we get vectors whose matrix of inner products is the  $(2r_0 + 1) \times (2r_0 + 1)$  matrix with  $\delta$  on the diagonal, one on the first off-diagonals and 0 elsewhere. (Careless choice of how the inside annulus boundary is connected to the outside will lead to powers of  $\omega$  on the off-diagonal but they can be removed by renormalizing the vectors one after another.) The determinant of this matrix is well known to be  $[2r_0 + 2] = [a + 1] = \frac{\sin(a + 1)\pi/a}{\sin \pi/a}$ , which is negative. This is impossible in a Hilbert space.

Case (ii)  $a$  even so that  $2r_0 = a - 2$ .

We will suppose that  $k > 0$ . The case  $k = 0$  goes in exactly the same way but the diagrams need to be modified a little. We leave the details to the reader.

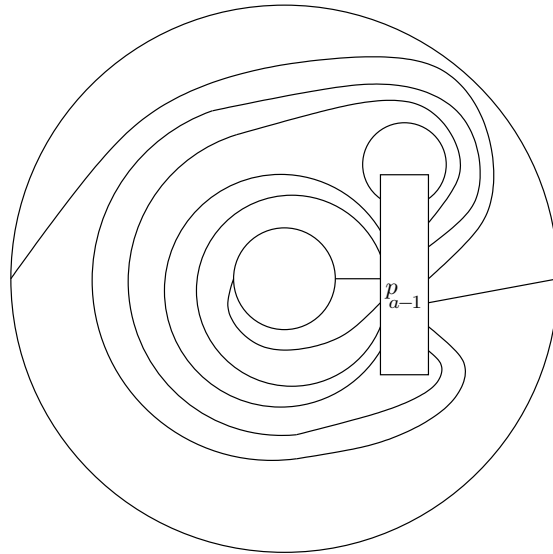
Here we will use the tangle encountered midway through the proof of Theorem 4.3. First let  $\beta$  be the  $(2k, a)$  annular tangle with the  $2k$  internal boundary points connected to a JW idempotent on  $a - 1$  strings (the last one for which the inductive definition works) in a rectangle,  $a - 2k - 2$  boundary points of the rectangle connected pairwise by strings that go around the internal annulus boundary and one rectangle boundary point connected to  $-2$ , as shown in Figure 5.2 for  $a = 8$  and  $k = 1$ . The other  $a - 1$  rectangle boundary points are connected to the outer annulus boundary points.

**Fig. 5.2**



The first thing we want to show is that  $\beta(v_\omega) = 0$ . We do this by calculating  $\langle \beta^* \beta(v_\omega), v_\omega \rangle$ . Since  $p_{a-1}$  is a projection,  $\beta^* \beta$  is as in Figure 5.3.

**Fig. 5.3**



But in Figure 5.3 we see the JW idempotent with two boundary points capped off. In general this would be non-zero since the boundary points are not on the same side of the rectangle, but since this JW idempotent is the last one to exist, it spans the kernel of the natural inner product on  $TL$  diagrams so is invariant (at least up to a scalar) under the rotation. Thus if

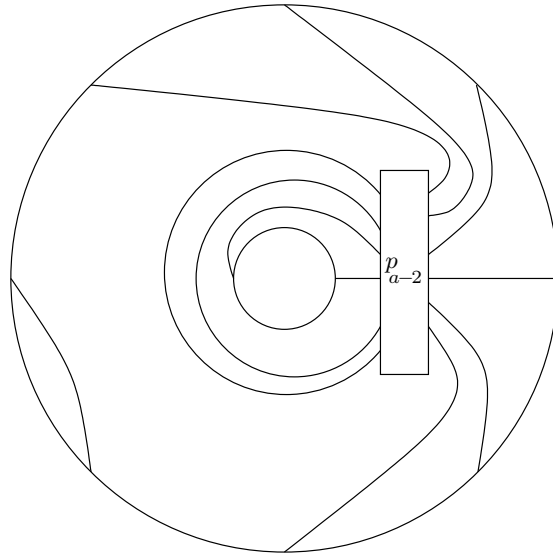


any two adjacent boundary points are connected the result is zero. (One can also show this by Wenzl's inductive formula.)

Thus  $\beta(v_\omega) = 0$ .

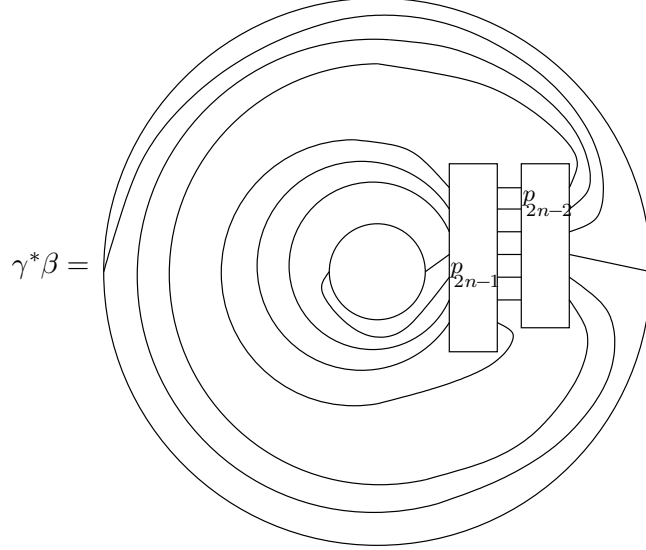
We will now derive a contradiction by showing that the inner product of  $\beta(v_\omega)$  with another vector is non-zero. This vector will be obtained from applying a  $(2k, a)$  tangle called  $\gamma$  to  $v_\omega$  where  $\gamma$  is obtained from the  $(2k, a-2)$  tangle  $\alpha_{a-2}$  of definition 4.1 by connecting  $-2$  to  $-2e^{(a+1)\pi i/a}$  and the other outer boundary points to the boundary points of the rectangle as indicated in Figure 5.4.

**Fig. 5.4**



To calculate the inner product  $\langle \beta(v_\omega), \gamma(v_\omega) \rangle$  we use the tangle  $\gamma^*\beta$ , which we have drawn in Figure 5.5.

**Fig. 5.5**



As in the proof of Theorem 4.3, there are only two diagrams that can be put in the rectangle which give non-zero contributions- those in which the boundary points connected to the first and last internal annular boundary points are connected to their neighbours (which are not connected to the inner annulus boundary). The coefficients of these diagrams are (from figure 4.9),  $\frac{[a/2 - k]}{[a - 1]}$  and  $\frac{[a/2 + k]}{[a - 1]}$ . But these are both equal to  $\frac{\cos k\pi/a}{\sin(a - 1)\pi/a} \neq 0$ . On the other hand the two resulting tangles are ( $\delta$  times)  $\alpha_{a-2}^*\alpha_{a-2}\rho$  and  $\alpha_{a-2}^*\alpha_{a-2}$ . Since  $\omega \neq -1$  (by the assumption on  $\theta$ ), the sum of these two tangles applied to  $v_\omega$  is non-zero. Hence by Theorem 4.3,  $\langle \beta(v_\omega), \gamma(v_\omega) \rangle \neq 0$ , a contradiction, so we are done.  $\square$

We would like to point out two corollaries of Theorem 5.1. The first is immediate but somehow surprising.

**Corollary 5.6** *Let  $\mathcal{U}$  be a Hilbert space representation of  $AffTL$ . Then none of the rotations  $\rho_t$  for  $t \geq 1$  acts by the identity.*

*Proof.* One may reduce to the irreducible case by using a maximal abelian subalgebra in the commutant of the algebra acting on the lowest weight space. Then the result follows from Theorem 5.1.  $\square$

In [12] we studied representations of the quotient  $AnnTL$  of our  $AffTL$  in which the rotations by  $2\pi$ ,  $\rho_t^t$ , act by the identity. We know that any Hilbert space representation of  $AnnTL$  will give one of  $AffTL$ , so we now identify those ones allowed by Theorem 5.1.

**Corollary 5.7** *Let  $\mathcal{U}$  be an irreducible Hilbert space representation of  $AffTL$  with chirality  $\omega$  and lowest weight  $k > 0$ . Then  $\mathcal{U}$  passes to  $AnnTL$  iff there is an integer  $b$  so that  $a = \frac{rk}{b}$ .*

Proof. The main thing to observe is that  $\rho_k^k = 1$  on  $\mathcal{U}_{k,\pm}$  implies that  $\rho_n^n = 1$  on  $\mathcal{U}_{n,\pm}$  for all  $n \geq k$ . This is because  $\mathcal{U}$  is a quotient of  $V^{k,\omega}$  and it is clear that  $\rho_n^n \alpha = \alpha \rho_k^k$  for any  $\alpha \in AffTL_{(k,sgn),(n,\pm sgn)}$ .

So whenever  $\omega$  is a  $k$ th. root of unity,  $\mathcal{U}$  passes to  $AnnTL$ . This is the condition  $a = \frac{rk}{b}$  combined with the conclusion of Theorem 5.1.

□

## 6 Construction of the allowed Hilbert space representations.

In this section we will undertake the most difficult part of this paper, namely the explicit construction of a representation  $\mathcal{V}^{k,\omega}$  for each pair  $(k, \omega)$  allowed by Theorem 5.1.

**Theorem 6.1** *Let  $(k, \omega)$  be a pair where  $k$  is a non-negative integer and  $\omega$  is a complex number (or  $\omega = (-1, \pm)$ , if  $k = 0$ ). Then  $\mathcal{V}^{k,\omega}$  exists if*

- (i)  $\delta \geq 2$  and either  $k = 0$  and  $0 \leq \mu \leq \delta$ , or  $k > 0$  and  $|\omega| = 1$ , or
- (ii)  $\delta = 2 \cos \pi/a$  for  $a = 3, 4, 5, \dots$ , and  $\omega = q^{\pm 2r}$  for some integer  $r$  with  $k < r \leq a/2$  (where by  $-1$  we mean  $(-1, \pm)$  if  $k=0$ ).

Proof. As observed in Section 3, it suffices to show that the sesquilinear forms  $\langle \cdot, \cdot \rangle$  of 3.9 are positive semidefinite on  $V^{k,\omega}$ . The method, as in [12], where it is done for the generic case, is to inductively decompose the representation  $V_{(n,\pm)}^{k,\omega}$  with respect to a large ordinary TL subalgebra, which we will soon define. When  $q = 1$  (so that  $\delta = 2$ ) and  $\omega = 1$  the argument is different from the generic case because the form is only positive semidefinite and we must identify the kernel. In this case, it follows from Theorem 4.3 that  $C_n = 0$  for all  $n \geq k + 1$ , so that in fact  $\mathcal{V}^{k,\omega}$  is generated by  $V_k^{k,1}$ . When  $\delta < 2$ , the large ordinary TL subalgebra will eventually fail to be semisimple. Semisimplicity is so important to our analysis that we begin our proof by taking a quotient of  $V^{k,\omega}$  on which, by Theorem 2.2, the action of TL passes to the  $C^*$  quotient,  $\mathcal{TL}_n$ .

It will be convenient to consider only the spaces  $V_{n,+}^{k,\omega}$ . The rotation provides an isometry between this and  $V_{n,-}^{k,\omega}$  in all but a special case when  $k = 0$ , where the situation is clear.

Thus let  $\delta = 2 \cos \pi/a$  for  $a = 3, 4, 5, \dots$  and let  $\omega$  and  $k$  be as in the statement of the theorem. Define the subspace  $\mathcal{JW}_{n,\pm}^{k,\omega}$  of  $V_{n,\pm}^{k,\omega}$  to be the span of the image of  $v_\omega$  under annular diagrams containing a rectangle labelled by the JW idempotent  $p_{a-1}$  (see Remark 3.4). The subspaces  $\mathcal{JW}_{n,\pm}^{k,\omega}$ , as  $n$  varies, are clearly invariant under the action of *AffTL*. We claim that they are in the kernel of  $\langle \cdot, \cdot \rangle$ . To see this, it suffices to show that an annular  $(k, k)$  diagram containing the JW idempotent  $p_{a-1}$  in a rectangle is zero modulo diagrams with fewer than  $k$  through strings. Let  $\alpha$  be such a diagram. We shall show that it has the same form as the element  $\alpha_n^* \alpha_n$  of Theorem 4.3 and then apply the Graham-Lehrer formula.

If  $\alpha$  had a string connecting the inside annulus boundary to the outside, then, cutting along that string, the rectangle would lie in a disc (after isotopy) with  $4k - 2$  boundary points. The rectangle itself has  $2a - 2$  boundary points and  $k < a/2$  so some boundary point on the rectangle must be connected to its nearest neighbour, which gives zero. So all of the strings from the inner annulus boundary are connected to the rectangle as are all those from the exterior annulus boundary. In fact  $k \leq a/2 - 1$  so that  $4k \leq 2a - 4$  so there are boundary points on the rectangle that are not connected to the annulus boundary. If these points were not connected symmetrically around the interior of the annulus there would be a rectangle boundary point connected to its nearest neighbour.

Let us first treat the case where  $a$  is odd so the rectangle has an even number of boundary points at the top and at the bottom. Since the JW idempotent is rotationally invariant (up to a scalar, as observed in the proof of Theorem 5.1), and  $v_\omega$  is invariant up to a scalar, we may suppose that  $\alpha$  is as in Figure 4.4 (which illustrates the case  $a = 9, n = 3$ ). Thus  $\langle \alpha(v_\omega), v_\omega \rangle$  is equal to  $C_{\frac{a-1}{2}}$  and thus, by 4.3, proportional to  $C_r$  since  $r \leq \frac{a-1}{2}$ . But, again by 4.3,  $C_r = 0$  since  $\omega = q^{2r}$ .

Now turn to the case where  $a$  is even. Then by rotational invariance as above we may suppose that  $\alpha$  is as in Figure 5.5.

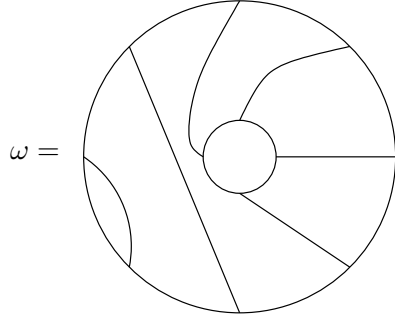
First suppose  $r = a/2$  so  $\omega = -1$ . Then by the argument after Figure 5.5 we get that  $\langle \alpha(v_\omega), v_\omega \rangle = 0$  precisely because  $\omega = -1$  (which is the value it did not take there). The case  $k = 0$  is slightly different. In this case there is only one diagram, shown in Fig. 6.2, that can be put into the rectangle to give a non-zero contribution, and that results in a homologically non-trivial circle surrounding the inner annulus boundary. This will give zero for  $\langle \alpha(v_\omega), v_\omega \rangle$ .



that the ideal  $\mathcal{I}_{2n}$  preserves  $\mathcal{JW}_{(n,+)}^{k,\omega}$  so that the quotients  $\mathcal{X}_n^{k,\omega} = \frac{V_{(n,+)}^{k,\omega}}{\mathcal{JW}_{(n,+)}^{k,\omega}}$

become modules over the quotients  $\mathcal{T}\ell_{2n} = \frac{t\ell_{2n}}{\mathcal{I}_{2n}}$ . But by Theorem 2.2,  $\mathcal{T}\ell_{2n}$  is a finite-dimensional  $C^*$ -algebra (with irreducible representations as described in Section 2). Note further that, by using the isomorphisms established above, we see that for every value of  $t$   $F_n\mathcal{W}_{t,2n}$  is isomorphic to  $\mathcal{W}_{t,2n-2}$  as a  $\mathcal{T}\ell_{2n-2}$ -module.

Thus  $\mathcal{X}_n^{k,\omega}$  as a  $\mathcal{T}\ell_{2n}$ -module is a direct sum of as many copies of each  $\mathcal{W}_{t,2n}$  as  $\mathcal{X}_{n-1}^{k,\omega}$  is of  $\mathcal{W}_{t,2n-2}$ , plus a certain number of copies of the trivial representation if  $2n < a - 1$  (and none if  $2n \geq a - 1$ ). The form  $\langle \cdot, \cdot \rangle$  is invariant under  $t\ell$  and there is only one such form (up to a scalar) on  $\mathcal{W}$ , so positive semidefiniteness on the span of the non-trivial representations follows by induction as soon as it is established on the trivial ones as they appear. But the JW idempotent  $p_{2n}$  projects on to the trivial representation for  $2n < a - 1$ , and annihilates all vectors in  $V_{n,+}^{k,\omega}$  except for the  $w$  depicted in Fig. 6.3, where all strings that connect the outside boundary to itself go through the wedge defined above.



**Fig. 6.3**

Thus there is exactly one copy of the trivial representation for  $2n < a - 1$  and none for  $2n \geq a - 1$ . Inspection of Fig. 6.3 shows that  $p_{2n}(w) = \alpha_n(v_\omega)$  so that positive semidefiniteness of  $\langle \cdot, \cdot \rangle$  follows from  $C_n \geq 0$ . This follows from Theorem 4.3 and the choice of  $k, \omega$  and  $\delta$ , which force  $C_n$  to be zero before it has a chance to be negative.  $\square$

We see in the proof of the theorem that we have in fact determined the structure of the Hilbert space representations as modules over the large Temperley-Lieb subalgebra. In fact the  $\mathcal{V}_{n,+}^{k,\omega}$  become modules over the TL algebra explained in Remark 2.7. The ordinary TL algebra is spanned by all diagrams in  $AffTL_{(m,+),(n,+)}$  that do not intersect the wedge  $\{re^{i\theta} \in \mathbb{C} : (1 + \frac{1}{r+2})\pi < \theta < (1 + \frac{1}{r+1})\pi\}$  where  $r = \max(m, n)$ . The Hilbert space representations of this algebra are precisely the modules  $\mathcal{W}^t$ .

**Scholium 6.4** Let  $\delta = 2 \cos \pi/a$  for  $a = 3, 4, 5, \dots$ . Suppose  $\omega$  and  $k$  satisfy the conditions of Theorem 6.1. Then as a module over the ordinary TL algebroid,  $\mathcal{V}_{(n,+)}^{k,\omega}$  is

$$\begin{aligned}
(i) \quad & \bigoplus_{j=k}^{r-1} \mathcal{W}_{2n}^{2j} \quad \text{if } \omega = q^{\pm 2r}, \\
(ii) \quad & \bigoplus_{\substack{j=0 \\ n-j \equiv 0 \pmod{2}}}^{a-1} \mathcal{W}_{2n}^{2j} \quad \text{if } k = 0 \text{ and } \omega = (-1, +), \\
(iii) \quad & \bigoplus_{\substack{j=0 \\ n-j \equiv 1 \pmod{2}}}^{a-1} \mathcal{W}_{2n}^{2j} \quad \text{if } k = 0 \text{ and } \omega = (-1, -).
\end{aligned}$$

**Corollary 6.5** The dimension of  $\mathcal{V}^{k,\omega}$  is

$$\begin{aligned}
(i) \quad & \frac{1}{Q_a(z)} \sum_{j=k}^{r-1} z^j Q_{a-2j-1}(z) \quad \text{if } \omega = q^{\pm 2r}, \\
(ii) \quad & \frac{1}{2Q_a(z)} \sum_{j=0}^{a-1} z^j Q_{a-2j-1}(z) \quad \text{if } k = 0 \text{ and } \omega = (-1, +), \\
(iii) \quad & \frac{1}{2Q_a(z)} \sum_{j=1}^{a-1} z^j Q_{a-2j-1}(z) \quad \text{if } k = 0 \text{ and } \omega = (-1, -).
\end{aligned}$$

## References

- [1] J. Bion-Nadal, *Subfactor of the hyperfinite  $II_1$  factor with Coxeter graph  $E_6$  as invariant*. J. Operator Theory **28** (1992), 27-50.
- [2] D. Bisch and V.F.R. Jones, *Singly generated planar algebras of small dimension*, Duke Math. Journal **101** (2000), 41-75
- [3] D. Bisch and V. Jones, *Algebras associated to intermediate subfactors*. Inventiones Math. **128** (1997), 89-157.
- [4] O. Bratteli, *Inductive limits of finite dimensional  $C^*$ -algebras*. Transactions AMS **171** (1972), 195-234.
- [5] J.H. Conway, *An enumeration of knots and links, and some of their algebraic properties*. Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967) (1970), 329-358

- [6] D. Evans and Y. Kawahigashi, *Quantum symmetries on operator algebras* Clarendon Press, Oxford (1998).
- [7] F. M. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter graphs and towers of algebras* Springer-Verlag, 1989.
- [8] R. Graham, D. Knuth and O. Patashnik, *Concrete Mathematics*(second edition) , Addison Wesley (1994).
- [9] J. J. Graham and G.I. Lehrer, *The representation theory of affine Temperley Lieb algebras*, L'Enseignement Mathématique **44** (1998), 1–44.
- [10] U. Haagerup, *Principal graphs of subfactors in the index range  $4 < [M : N] < 3 + \sqrt{2}$*  in: "Subfactors", World Scientific, Singapore-New Jersey-London-Hong Kong (1994) 1–39.
- [11] M. Izumi, *On flatness of the Coxeter graph  $E_8$ .*, Pacific Math. Journal **166** (1994), 305-327.
- [12] V.F.R. Jones, *The annular structure of subfactors*, to appear.
- [13] ———, *Index for subfactors*, Invent. Math **72** (1983), 1–25.
- [14] ———, *Planar Algebras, I*, NZ Journal Math, to appear.
- [15] ———, *A quotient of the affine Hecke algebra in the Brauer algebra.*, L'Enseignement Mathématique **40** (1994), 313-344.
- [16] ———, *The planar algebra of a bipartite graph.* In *Knots in Hellas '98* World Scientific (2000), 94–117.
- [17] V.F.R. Jones and V.S. Sunder, *Introduction to Subfactors* LMS lecture note series **234** (1997)
- [18] L. Kauffman, *State models and the Jones polynomial.* Topology **26** (1987), 395-407.
- [19] G. Kuperberg, *Spiders for rank 2 Lie algebras*, Commun. Math. Phys. **180** (1996), 109–151.
- [20] J.P. May, *Definitions: operads, algebras and modules*, Contemporary Mathematics **202** (1997), 1–7.
- [21] A. Ocneanu, *Quantized group, string algebras and Galois theory for algebras* in **Operator algebras and applications, vol. 2** L.M.S lecture note series, **136** , (1987), 119–172.



- [22] S. Popa, *An axiomatization of the lattice of higher relative commutants*, Invent. Math **120** (1995), 427–445.
- [23] H. Wenzl, *On sequences of projections*, C. R. Math. Acad. Sci. Soc. R. Can. **9** (1) (1987), 5–9.