The Planar Algebra of a bipartite graph.

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Abstract

We review the definition of a general planar algebra $V = \cup V_k$. We show how to associate a general planar algebra with a bipartite graph by creating a specific model using statistical mechanical sums defined by a labelled tangle. It supports a partition function for a closed tangle which is spherically invariant and defines a positive definite inner product on each V_k. We then describe how any planar algebra is naturally a cylic module in the sense of Connes and do some computations.

1 Introduction

The concept of planar algebra was introduced in [10] for many reasons, the most important of which was to help in the calculation of subfactors. As observed in [10], it is hardly surprising that such a natural concept arises in many other situations - notably in [1, 16, 14]. Our axioms for a planar algebra are in this respect rather special and could be criticised for being somewhat narrow because of the restrictions imposed by the shadings, but we would argue that our structure is to the more general ones as a group is to a semigroup. Indeed this could be made precise in the $C*$-tensor category context, but the justification with the most content is that provided by the results of Popa. In a series of papers, [18, 19, 20] he came quite independently across axioms (shown in [10] to be equivalent to planar algebras with positivity) which guarantee the existence of a subfactor of a II_1 factor whose associated invariant is the λ-lattice or planar algebra one began with! The most recent paper in the series is a universal construction which should allow one to control the isomorphism classes of the II_1 factors in question. Popa's results should be viewed as a co-ordinatizaion theorem analogous to the fundamental theorem of projective geometry, in which the II_1 factors give the (non-commutative) co-ordinate ring associated to the combinatorical- geometrical structure defined by the planar algebra. In fact such a co-ordinatization will exist, and be much easier to prove, over an arbitrary field. It is the completeness properties associated with the II_1 factors that make Popa's results so compelling.

The planar algebra of a subfactor was approached in [10] in a somewhat abstract way as invariant vectors in the tensor powers of a bimodule. It would have also been possible to do it via an explicit model in a way similar in spirit to Ocneanu's paragroup approach. The first step in such a program would have been to construct a rather general kind of planar algebra using statistical mechanical sums on the principal graph of the subfactor. Then the actual planar algebra would be obtained as a planar subalgebra formed by certain "flat" elements. This approach was deliberately avoided as being long, clumsy and inelegant. However in more recent work the idea of constructing interesting planar algebras from the rather general ones based on graphs has turned out to be extremely useful and we find it necessary to give these general models. We will present such an application in a forthcoming paper. These general planar algebras may not turn out to be of much interest for their own sake as they can be formed from very general bipartite graphs. Their detailed construction does bring to light several interesting points however, such as the role of the choice of an eigenvector of the adjacency matrix of the graph in the statistical mechanical sums. In fact an arbitrary choice of weights would give a planar algebra but it would not in general have the property that a closed circle in a diagram contributes a simple multiplicative constant. We have presented the construction here for arbitrary weights-what we will call the "spin vector" later on.

One feature of these general planar algebras is that there are no obstructions to obtaining graphs as "principal graphs" or at least connected components thereof. It is known ([8]) that graphs occuring as principal graphs of subfactors are quite rare so the role of the factor/connectedness condition becomes clear. This suggests the study of planar algebras intermediate between the rather simple kind constructed here and the restricted subfactor kind with connected principal graphs. We will present further results on this question, inculding an ABCDEFGHI classification for modulus less than 2, in a forthcoming paper.

We noticed some time ago that Connes' cyclic category appears in the annular or "affine" Temperley Lieb category. (Composition of morphisms in the Temperley Lieb category in general leads to closed circles, but they do not occur if one restricts to the annular tangles of Connes' category.) This means that any planar algebra is in fact a cyclic module in the sense of Connes. In fact the natural adjoint map in the Temperley Lieb category defines a second copy of the cyclic category. Together with the first, they generate the Temperley Lieb category. If the planar algebra has the property that isolated circles in a tangle contribute a non-zero scalar multiplicative factor, the face maps of the second category provide homotopy contractions of the first so that the cyclic homology of such a planar algebra will always be zero. There are however many interesting planar algebras that do not have this property. We simply present these observations in this paper as we do not yet know how to use cyclic homology in planar algebras.

2 Definition of a general planar algebra.

A $k-$tangle for $k \geq 0$ is the unit disc D with $2k$ marked and numbered (clockwise) points on its boundary, containing a finite number of internal discs each with an even (possibly zero) number of marked and numbered points on their boundaries. All the marked points of all the discs are connected by smooth disjoint curves called the *strings* of the tangle. The strings lie between the internal discs and D. The strings must connect even-numbered boundary points to even-numbered ones and odd to odd. There may also be a finite number of closed strings (not connecting any discs) in the subset $\mathcal{I}$ of the large disc between the internal discs and the external one. The connected components of $\mathcal{I}$ minus the strings are called the *regions* of the tangle and may be shaded black and white in a unique way with the convention that the region whose closure contains the interval on the boundary of D between the first and second marked points is shaded black. If necessary the strings of the tangle will be oriented so that black regions are on the left as one moves along strings. To indicate the first point on the boundary of a disc in a picture we will select the unshaded region immediately preceding the first point(in clockwise order) and place a * in that region near the relevant boundary component.

Tangles are considered up to smooth isotopy. An example of a 4-tangle,with

7 internal discs is given in figure 1.1 below:

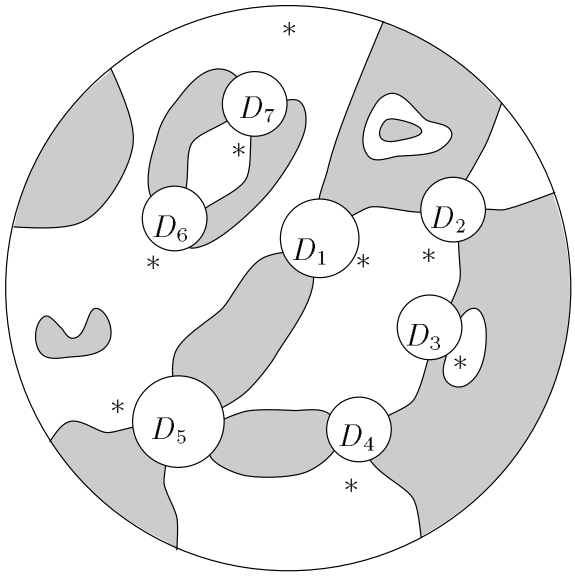


fig. 1.1

Tangles with the appropriate number of boundary points can be glued into the internal discs of another tangle making the set of all tangles into a coloured operad, the colour of an element being the number of boundary points (and the colour of the region near the boundary for 0-tangles). To perform the gluing operation, the tangle T to be glued to an internal disc(with the same colour as T) of another tangle S is first isotoped so that its boundary coincides with the boundary of the chosen internal disc $\mathcal{D}$, the marked points on each boundary disc being also made coincident by the isotpy. Some smoothing may need to be done near the marked points so that the strings of T and S meet smoothly. Finally the common boundary is removed. The result of the gluing is another tangle $T \circ_{\mathcal{D}} S$ with the same number of external marked points as S and having $n_T + n_S - 1$ internal discs, n_T and n_S being the numbers of internal discs of T and S respectively. It is clear that the isotopy class of $T \circ_{\mathcal{D}} S$ depends only on the isotopy classes of S and T and the choice of $\mathcal{D}$. An example of the gluing operation is depicted in figure 1.2:

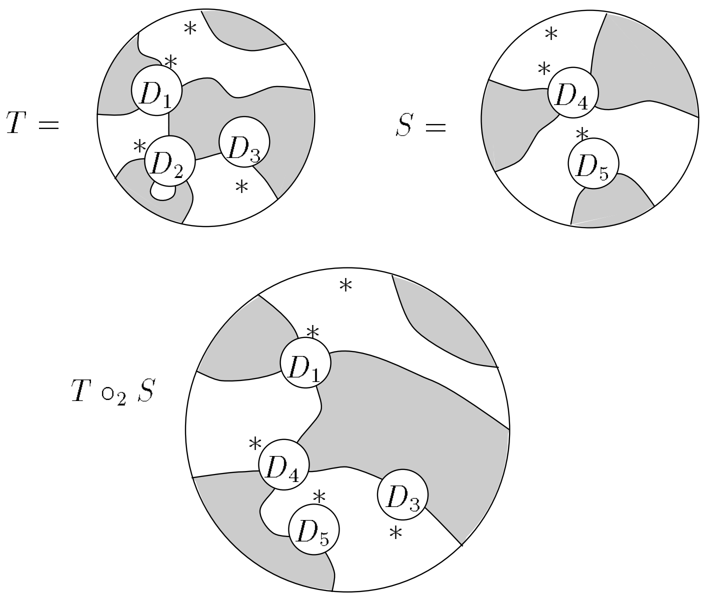


fig. 1.2

See [17] for the definition of an operad. Slight modifications need to be made to handle the colours.

Definition 2.1 *The planar operad $\mathcal{P}$ is the set of isotopy classes of planar tangles with colours and compositions defined above.*

One may construct another coloured operad with the same colours as $\mathcal{P}$ from vector spaces. Operad elements are then multilinear maps from "input" vector spaces, each vector space having a colour as in $\mathcal{P}$ to an "output" vector space. Composition is only permitted when the appropriate vector spaces have the same colour. One obtains another coloured operad $Vect$. See [17] for a preciese definition(without colours).

Definition 2.2 *A planar algebra is an operad homomorphism from $\mathcal{P}$ to $Vect$.*

What this means in more concrete terms is this: a planar algebra is a graded vector space V_k for $k > 0$ and two vector spaces V^+ and V^- so that every element T of $\mathcal{P}$ determines a multilinear map from vector spaces, one for each internal disc of T to the vector space of the boundary of T, vector spaces being required to have the same colour as the discs they are assigned to. Composition in $\mathcal{P}$ and $Vect$ correspond in the following sense (S (an r-tangle) and T as above): by singling out an internal $k-$disc $\mathcal{D}$, S determines a linear map from V_k to $Hom(W, V_r)$, W being the tensor product of the vector spaces corresponding to the internal discs of S other than $\mathcal{D}$. Composing this map with the multilinear map determined by T one obtains a multilinear map from the vector spaces of all the internal tangles of $T \circ_{\mathcal{D}} S$. This multilinear map must be the same as the one the planar algebra structure assigns to $T \circ_{\mathcal{D}} S$.

Here are three good exercises to help understand this homomorphism property of a planar algebra.

(i) Show that V^+ and V^- are both commutative associative algebras.

(ii) Show that each V_k becomes an associative algebra with multiplication being the bilinear map defined by the tangle below:

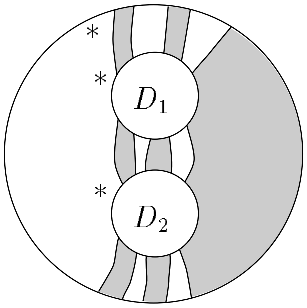


fig. 1.3 multiplication

(iii) Show that the V_k's for $k > 0$ and V^+ for V_0 become an assoicative graded algebra over V_0 with multiplication being the bilinear map defined by tangle below: (we will not show the shading any more-it is determined as soon as we know a region with a *, and given near the boundary if $k = 0$.)

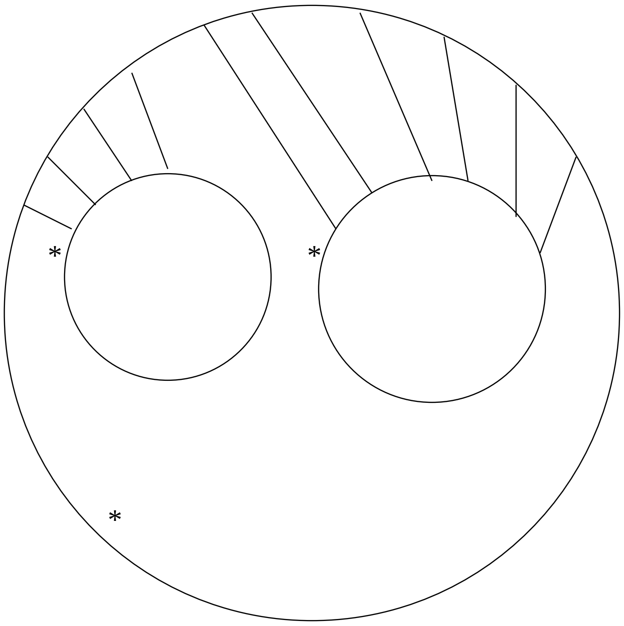


fig. 1.4 graded multiplication

Tangles without internal discs are required to give linear maps from the field into the output vector space. Thus the image of 1 under the 0-tangle with nothing inside is thus the identity for the algebras V_0. And in general the vector space spanned by k-tangles with no internal discs is a subalgebra of V_k.

The above definition of a planar algebra is quite general and one might be especially interested in many special cases. The following is rather commonly satisfied:

Definition 2.3 *The planar algebra will be said to have modulus δ if inserting a contractible circle inside a tangle causes its multilinear map to be multiplied by δ.*

There are maps from $\iota_k : V_k \to V_{k+1}$ defined by the "inclusion" tangle below:

Proposition 2.4 *If V is a planar algebra with modulus $\delta \neq 0$, the ι_k are injective.*

Proof. Connecting the middle two boundary points and dividing by δ gives an inverse to ι_k. $\Box$.

There may be examples of planar algebras where the ι_k are not injections but we have not looked in that direction. We will think of the V_k as being embedded one in the next via the ι_k.

Planar tangles possess the following involution: call the region preceeding the first boundary point of any disc the first region. Now reflect the tangle in the a diameter passing through the first region on the boundary. Number all boundary points of all discs of the reflected tangle counting clockwise so that the image under the refletion of the first region becomes again the first region. The involution applied to the original tangle is the one obtained by this process.

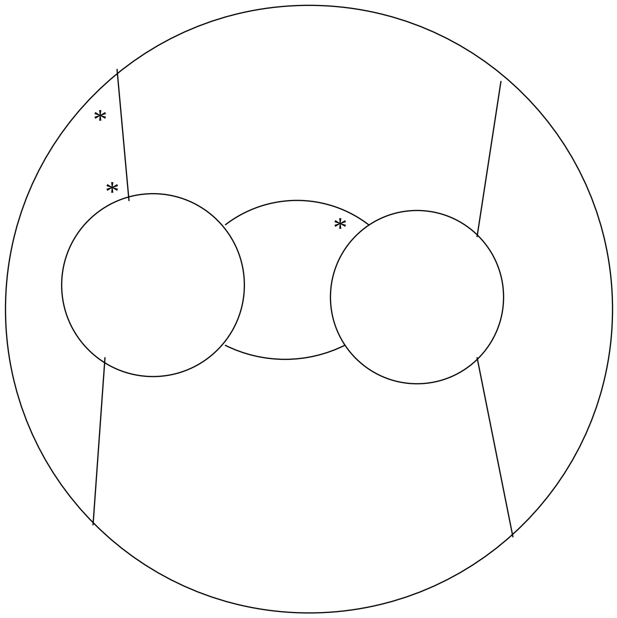
If the field K possesses a conjugation and each V_k has a conjugate linear involution "$*$" we will say that V is a *planar *-algebra* if the involution on tangles and the involution on V commute in the obvious sense. If $K = \mathbb{R}$ or $\mathbb{C}$, V will be called a C^*-planar algebra if it is a planar *-algebra and each V_k becomes a C^*-algebra under its involution.

Planar algebras V and W are isomorphic if ther are vector space isomorphisms $\theta_k : V_k \to W_k$ intertwining the actions of the planar operad. The isomorphism are required to be *-isomorphisms in the planar *-algebra case.

There is a "duality" automorphism of the planar operad defined on a tangle by moving the first boundary point by one in a clockwise direction on every disc in the tangle and reversing the shading. Call this map $\Delta : \mathcal{P} \to \mathcal{P}$. It is clear that Δ preserves the composition of tangles. If V has modulus δ, so does $\tilde{V}$.

Definition 2.5 *The dual $\hat{V}$ of the planar algebra V will be the planar algebra whose underlying vector spaces for $k > 0$ are those of V, $\hat{V}_0^{\pm} = V_0^{\mp}$, but for which the multilinear linear map corresponding to the tangle T is that of $\Delta(T)$.*

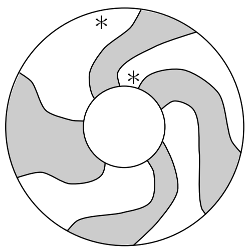
In general V is not isomorphic to $\tilde{V}$. One may check for instance that the algbebra structure induced on V_2 induced by multiplication in $\tilde{V_2}$ is that defined by the tangle below:



Comultiplication

We have somewhat abusively called this second multiplication "comultiplication" in [2]. For the planar algebra of a finite groups as in [10], multiplication is that of the group algebra and comultiplication that of functions on the group.

On the other hand $\tilde{\tilde{V}}$ is isomorphic to V via the linear maps defined by the " rotation" tangles below:



The rotation tangle ρ.

3 The planar algebra of a bipartite graph.

Let Γ be a locally finite connected bipartite graph (possibly with multiple edges) with edge set $\mathcal{E}$, vertex set $\mathcal{U} = \mathcal{U}^+ \cup \mathcal{U}^-$, $\sharp(\mathcal{U}^+) = n_1$ and $\sharp(\mathcal{U}^-) = n_2$,

$n_1 + n_2 = n = \sharp(\mathcal{U})$. No edges connect $\mathcal{U}_1$ to itself nor $\mathcal{U}_2$ to itself. The adjacency matrix of Γ is of the form $\begin{pmatrix} 0 & \Lambda \\ \Lambda^t & 0 \end{pmatrix}$ where there are Λ_{v^+,v^-} edges connecting v^+ to v^-.

The other piece of data we suppose given is a function $\mu : \mathcal{U} \to K$, $a \to \mu_a$ where K is the underlying field and μ_a is required to be different from zero for all a. In the cases of most interest so far (μ_a^2) has been an eigenvector for the adjacency matrix of Γ but that is only needed to guarantee that contractible circles inside pictures count as scalars.

The function μ will be called the *vector* and its value at a will be called the *spin* of a.

For each $k > 0$ let V_k be the vector space whose basis consists of loops of length $2k$ on Γ starting and ending at a point in $\mathcal{U}^+$. Such a loop will be represented by the pair (π, ϵ) of functions from $\{0, 1, 2, ..., 2k-1\}$ to $\Gamma \cup \mathcal{E}$ where the i-th. step in the loop goes from π_i to $\pi(i-1)$ along the edge $\epsilon(i)$ (the i's being counted modulo $2k$). Recording the vertices π of a path is redundant since the edges contain that information but in many examples of most interest Γ will have no multiple edges in which case we would suppress the function ϵ. For $k = 0$ a loop of length 0 is just an element of $\mathcal{U}$, so we define ${V_0}^+$ (resp. ${V_0}^-$) to be the vector space with basis $\mathcal{U}^+$ (resp. $\mathcal{U}^-$).

We will make the V's into a general planar algebra in the sense of [10]. One may do this for discs and arbitrary smooth isotopies in the plane as in section 1 but consideration must be given to the angles at which the strings meet the boundaries of the discs. We prefer to use the equivalent picture where the discs are replaced by rectangles or "boxes" as in the second section of [10]. If we use the V's as a labelling set what we have to do is, given a k_0-tangle T in the sense of section 2 of [10], with an element of V_k assigned to each internal $k-$box of T, construct an "output" element $Z(T)$ in V_{k_0}, in such a way as to respect the compostion of tangles, be independent of isotopy and be multilinear in the "input" V elements. Many ingredients of the construction below were present in [13].

Definition of $Z(T)$. A *state* of the (unlabelled) tangle T will be a function σ: (regions of T)$\cup$ (strings of T) $\to \mathcal{U} \cup \mathcal{E}$ such that

(1) $\sigma(\{\text{shaded regions}\}) \subseteq \mathcal{U}^+$ and $\sigma(\{\text{unshaded regions}\}) \subseteq \mathcal{U}^-$.

(2) $\sigma(\{\text{strings}\}) \subseteq \mathcal{E}$.

(3) If the closure of the regions r_1 and r_2 both contain the string t then $\sigma(t)$ is an edge joining $\sigma(r_1)$ and $\sigma(r_2)$.

Now suppose T is labelled. Let $\{b\}$ be the set of internal boxes of T and let v_b be the vector in V_k be the vector assigned to the internal k-box b by the labelling.

To define the vector $Z(T)$ in V_{k_0} we must give the coefficient of a loop (π, ϵ) of length $2k_0$. Say that the state σ of T is *compatible* with (π, ϵ) in the following way. Consider the ith boundary segment of the external box of T (the one between the ith and (i+1)th boundary points according to the numbering convention of [10]). Then this segment is part of the boundary of a region r of T. The compatibility requirement of (π, ϵ) with σ is that $\pi_i = \sigma(r)$, $\epsilon_i = \sigma(s)$, $\epsilon_{i+1} = \sigma(s')$ where s and s' are the strings of T meeting the boundary at the ith and $(i+1)$th points respectively. Compatibility of a state with a loop at an internal box of T is defined in the same way (with inside replaced by outside) so that every state assigns a loop (π_b, ϵ_b) to an internal box b of T as the only loop compatible with the state at b.

In the case $k = 0$ the whole boundary is a single segment and the loop is just a single vertex of $\mathcal{U}$. A state is compatible with the loop a on the boundary if it assigns a to the region near the boundary.

Now rotate the internal boxes of T so that they are all horizontal with the first boundary point at the top left. Isotope the strings if necessary so that any singularities of the y-coordinate function are local maxima or minima.

We define the vector $Z(T)$ in V_{k_0} by its coefficient of the basis element (π, ϵ) which is:

$$(Z(T))_{(\pi,\epsilon)} = \sum_{\{\sigma \text{ compatible with}(\pi,\epsilon)\}} \prod_b (v_b)_{(\pi_b,\epsilon_b)} \prod_{\{\text{singularities } \alpha \text{ of y on strings}\}} \mu_\alpha$$

where (i) $(v)_{(\pi_b,\epsilon)}$ denotes the coefficient of the vector v in the basis $\{(\pi, \epsilon)\}$. ii The spin factor μ is the ratio $\frac{\mu_x}{\mu_y}$ where the state σ assigns x to the concave region near α and y to the region on the other side of the curve (regardless of the shading) as below:

This ends the definition of $Z(T)$. Note that the sum is finite even though Γ may be infinite because we only consider states compatible with the boundary state.

Note that the spin term

$$\prod_{\{\text{singularities } \alpha \text{ of y on strings}\}} \mu_\alpha$$

can be replaced by $\int \mu(\sigma)d(\theta)$ where σ denotes the locally constant function on the strings defined by σ and the spin vector whose value on a string segment is determined by the ratio of the spin values to the left and right of that string. This would allow one to define the "singularity" factor without considering singularities or arranging the tangle so that the boxes are all horizontal. See [12] for this in a simpler knot-theoretical context.

Theorem 3.1 *The above definition of $Z(T)$ for any planar tangle T makes the vector spaces of linear combinations of loops on Γ into a planar algebra with $dim(V_0^+) = n_1$ and $dim(V_0^-) = n_2$. This planar algebra will have modulus δ if (μ_a^2) is an eigenvector of the adjacency matrix with eigenvalue δ. It will be a planar *-algebra if K has a conjugation, the spin values are fixed by conjugation, and each V_k is equipped with the involution defined on the basis by reading loops backwards, and a C^*-planar algebra if, as well, $K = \mathbb{R}$ or $\mathbb{C}$.*

Proof. The first thing to show is that $Z(T)$ does not change if the labelled tangle is changed by isotopy of the disc. As in section 4 of [10] this follows from the fact that isotopy is generated by simple moves. Isotopies that do not change the maxima and minima are irrelevant. The first move is a cup-cap simplification as below which leaves each term in the sum invariant by the definition of the spin term. The second move required to generate isotopy is a 360° rotation of an internal box, since two ways of making the boxes horizontal with the first boundary point at the top left must differ by such rotations. To see that such rotations do not affect Z, consider the spin term for a given state on a tangle that has a box surrounded by strings effecting a 360° rotation in the most obvious way with the maxima above the box and the minima below. Reading from the top down, cancellation occurs in the factors $\frac{\mu_x}{\mu_y}$ for all the maxima, and similarly for all the minima. So the overall spin term is exactly as if the rotation were not there.

Multilinearity is obvious, as is the compatibility of the operad gluing with $Z(T)$-just do the gluing operation with the inputs as horizontal boxes and the spin factors for the singularities of the y coordinate behave in the right way.

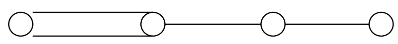
That contractible circles contribute a multiplicative factor of δ follows from the eigenvalue condition and the choice of the spin factors at local maxima and minima.

To see the *-algebra property, note that in the definition of the spin term, the thing that determines the numerator and denominator is the concavity of the region near the vertex. This is unaltered under reflection. The C^* property is clear since the * is just the transpose on a basis of matrix units. $\square$

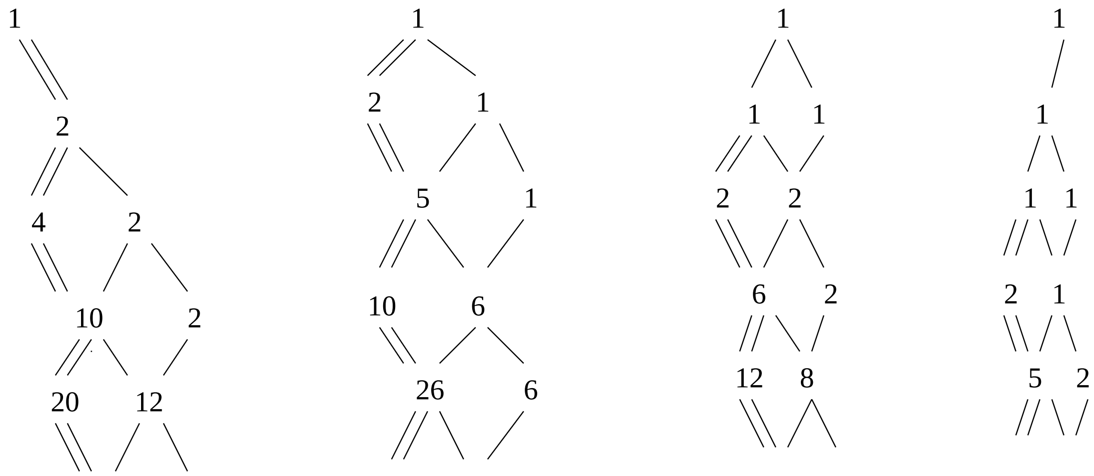
Definition 3.2 *With notation as in the previous theorem, we will call the planar algebra V (or $V(\Gamma)$ if necessary) the planar algebra of the bipartite graph Γ with respect to the spin vector μ.*

Notes. (i) The algebra structure on V_0^+ and V_0^- is such that the basis paths of length zero are idempotents, self adjoint in the *-case.

(ii) In general the loops with given first point and midway point form matrix units for a simple summand of V_k which is thus a multimatrix algebra with matrix summands indexed by pairs of vertices. Any pair of vertices will occur for appropriate k. The Bratteli diagram for the increasing sequence of multimatrix algebras V_k thus consists of a connected component for each vertex in $\mathcal{U}^+$ and each component is erected on a "principal graph" equal to Γ with starting vertex "*" being the element of $\mathcal{U}^+$. See ([9, 6]) for a detailed treatment of such Bratteli diagrams. We give an example below of a graph Γ and the Bratteli diagram of its planar algebra:



The graph Γ



The Bratteli diagram of V(Γ)

(iii) The sum over all basis loops is the identity of V_k.

We shall now define linear functions on $V_0^{\pm}$. When applied to a labelled tangle with no boundary points, it will be called the *partition function* of the labelled tangle.

Definition 3.3 *The linear functions from $Z : V_0^{\pm} \to K$ are defined as the linear extensions of the function which takes the basis path a to μ_a^4.*

Thus for the partition function of a closed tangle one sums over states taking all possible values in all the regions, including the external one. There is an extra multiplicative spin factor of μ_a^4 for the external region.

Proposition 3.4 *The partition function of a closed labelled tangle T depends only on T up to isotopies of the 2-sphere.*

Proof. Spherical isotopy is generated by planar isotopy and isotopies that change a closed tangle by sending a string that meets the external region to a string that encloses the whole tangle(thus changing the shading of the external region). Invariance of the partition function under this move is easy to check. □

Up to this point the normalisation of the spin vector has been irrelevant. It is desirable that the partition function of an empty closed tangle be equal to 1. This suggests the following.

Definition 3.5 *We will say that the planar algebra of a graph is normalised if*

$$\sum_{a \in \mathcal{U}^+} \mu_a^4 = 1.$$

Note that this is the same as requiring $\sum_{a \in \mathcal{U}^-} \mu_a^4 = 1$.

Theorem 3.6 *Let K be $\mathbb{R}$ or $\mathbb{C}$ and let V be the normalised planar algebra of the finite graph Γ with respect to the appropriately normalised Perron Frobenius eigenvector of the adjacency matrix of Γ. Then $tr(x) = \delta^{-n+1} Z(\hat{x})$ defines a normalised trace on the union of the V's (with inclusion of V_k in V_{k+1} by adding a string to the right as usual-see [10]) where $\hat{x}$ is any 0-tangle obtained from x by connecting the first $k/2$ boundary points to the last $k/2$. The scalar product $< x, y >= tr(x^* y)$ is positive definite.*

Proof. Normalisation is a simple calculation which also shows that the definition of the trace is consistent with the inclusions. The property $tr(ab) = tr(ba)$ is a consequence of planar isotopy when all the strings added to x to get $\hat{x}$ go round x in the same direction, and spherical invariance reduces the general case to this one.

Positive definiteness follows from the fact that the loops, which form a basis of the V_k, are mutually orthogonal elements of positive length. In fact the square of the norm of a loop $(\pi, \epsilon) \in V_k$ is $\mu^2_{\pi(0)} \mu^2_{\pi(k)}$. □

Proposition 3.7 *The rotation tangle ρ is an isometry for the Hilbert space structure defined above by the trace and it acts on a basis path in V_k by:*

$$\rho(\pi,\epsilon)=\frac{\mu_{\pi(0)}\mu_{\pi(k)}}{\mu_{\pi(2k-2)}\mu_{\pi(k-2)}}(\alpha,\beta)$$

where $\alpha(i)=\pi(i-2)$ and $\beta(i)=\epsilon(i-2)$, with indices mod $2k$.

Proof. This is an exercise in using the definition of $Z(T)$. □

4 Examples

Some of the simplest examples were already present in [10].

Example 4.1 *Tensors.*

The tensor planar algebra of [10] is just the planar algebra of the bipartite graph with 2 vertices and n edges.

Example 4.2 *Spin models and discrete string theory.*

A so-called spin model of [10] is the planar algebra P^σ for the bipartite graph Γ with $\#(\mathcal{U}^+)=1$ and $\#(\mathcal{U}^-)=n$. The somewhat mysterious normalisations of [10] are due to the spin vector which is the Perron Frobenius eigenvector of the adjacency matrix. This spin vector is particularly simple in this case which is why it was possible to complete the discussion of spin models in [10] without using the formalism developed here.

Observe that action of a permutation of the elements of $\mathcal{U}^-$ on loops preserves the planar algebra structure so that any group of such permutations defines a group of automorphisms of P^σ.

In [10] we considered the planar subalgebra $P^\mathcal{G}$ of P^σ generated by the single element of V_2 defined by the adjacency matrix of some arbitrary graph $\mathcal{G}$ with n vertices. The vertices of $\mathcal{G}$ form the set $\mathcal{U}^-$ in the spin model. The partition function of a labelled k- tangle T is then the number of graph homomorphisms from $col(T)$ to $\mathcal{G}$ where $col(T)$ is the planar graph obtained from T by taking as vertices the shaded regions of T and as edges the 2-boxes. The path (π,ϵ) is just a choice of vertices of $\mathcal{G}$ which specifies where these boundary vertices are sent by the graph homomorphism. As the tangle T becomes bigger we could imagine it filling up the inside of the disc so that we are exploring the graph $\mathcal{G}$ by counting larger and larger (singular) discs inside it. If the adjacency matrix were replaced by another symmetric matrix with the same pattern of zero entries we could think of its entries as interaction coefficients so we would have some kind of metric on the discs inside $\mathcal{G}$. Of course all this is the genus zero case. One might want to consider higher genus "surfaces" inside $\mathcal{G}$. For this one could try to extend the planar operad to allow systems of strings and internal discs on a surface of higher genus. It is not at all clear when this is possible. For vertex and spin models it should be straightforward enough though the spin term in the partition function will already cause a problem for spin models on higher genus surfaces.

It remains to be seen just how useful it is to explore a graph by counting homomorphisms of planar subgraphs into it. For a random graph one would expect that $P^{\mathcal{G}}$ is all of P^{σ}. But if $\mathcal{G}$ has automorphisms it is obvious that $P^{\mathcal{G}}$ will be contained in the planar subalgebra of P^{σ} given by fixed points for the action of the automorphism group of $\mathcal{G}$.

We record the following result giving a sufficient condition for any planar $*-$subalgebra of P^{σ} to be the fixed point algebra for the automorphism group coming from a group of permutations of the spins. We offer a proof that is curious in that the result is finite dimensional but we shall use type II_1 factors in our argument. Recall from [10] that the algebra V_k of P^{σ} is faithfully represented on the vector space $\otimes^{[\frac{k+1}{2}]}W$ where W is a vector space with basis equal to $\mathcal{U}^-$. We call an element of P^{σ} a *transposition* if it acts on $\otimes^{[\frac{k+1}{2}]}W$ as a transposition between adjacent tensor product components.

Theorem 4.3 *A planar *-subalgebra $P = \{P_k\}$ of P^{σ} which contains a transposition is equal to the fixed points of P^{σ} under the action of some group of permutations of $\mathcal{U}^-$.*

Proof. The union of the increasing sequence of finite dimensional C^*-algebras P_k^{σ} admits a faithful trace. Complete the algebra using the GNS consruction ([21],page 41) to obtain the hyperfinite type II_1 factor R. The group S_n of all permutations of $\mathcal{U}^-$ acts on R so that every non-trivial permuation is an outer automorphism. It was shown in [11] that the fixed point algebra for the action of S_n is generated by the Temperley Lieb algebra and the "other" symmetric group-the representation of $S_{[k+1/over 2]}$ on $\otimes^{[\frac{k+1}{2}]}W$ which is obviously generated in P^{σ} by the transpostions. We see that the algebra R_0 generated in R by the P_k contains the fixed point algebra for the S_n action. Thus by the Galois theory for type II_1 factors R_0 is the fixed point algebra for some subgroup G of S_n.

The only thing left to show is that each individual P_k is the set of fixed points for the action of G. It is obvious that P_k is pointwise fixed. If x is in P_k^{σ} and is fixed by G and $\varepsilon > 0$ is given, there is a y in $P_{\hat{k}}$ for some sufficiently large $\hat{k}$ with $\|x-y\|_2 < \varepsilon$. But the conditional expectation $E_{P_k^{\sigma}}(y)$ is defined by a multiple of a tangle applied to y and so belongs to P_k. But x is in P_k^{σ}

and so is fixed by the condtional expectation. We thus get an element of P_k within ε of x for every x, and P_k is finite dimensional. $\Box$

The above result can give an extremely rapid calculation of $P^{\mathcal{G}}$. In joint work with Curtin ([5]) we have shown that the planar algbebra thus associated to the Petersen graph is in fact the fixed point algebra for the automorphism group of that graph. It would be desirable to have an effective way of deciding if the first transposition is in $P^{\mathcal{G}}$.

Example 4.4 *Graphs with* $\delta < 2$

It is well known (see for instance [6]) that graphs the norm of whose adjacency matrix is < 2 are given by the Coxeter graphs A, D and E. These are the simplest examples where the formalism we have developed is necessary. The Perron Frobenius eigenvectors of the adjacency can be found in [6]. It is of course possible to use an eigenvector other than the one of largest eigenvalue to obtain planar algebras with $\delta < 2$ from most graphs but the partition function will only give a positive definite form is one uses A, D or E.

The A, D, E graphs were used in the first construction of irreducible subfactors of non-integer index by finding what we would recognize as a connected planar algebra inside the general one defined above. In a future paper we will present a construction which gives all connected planar algebras with positivity and $\delta < 2$ in this way.

5 Towers of Algebras

An inclusion $A_0 \subseteq A_1$ of finite dimensional multimatrix algebras with the same identity defines a bipartite graph Γ with $\mathcal{U}^+$ and $\mathcal{U}^-$ being the sets of irreducible representations of A_0 and A_1 respectively. If $a \in \mathcal{U}^+$ and $b \in \mathcal{U}^-$ there are $n(a,b)$ edges between a and b where the restriction of b to A_0 contains a $n(a,b)$ times. We say the inclusion is *connected* if Γ is. Choose an eigenvector of the adjacency matrix of Γ with all entries non-zero and with non-zero eigenvalue (if there is one), and suppose the a entry of the eigenvector is a square, say ν_a^2. We can then define, as in [9] and [6], the Markov trace Tr of modulus δ^2 on A_1 by setting the trace of a minimal idempotent p_b in the matrix algebra direct summand of A_1 corresponding to $b \in \mathcal{U}^-$ to be ν_b^2, normalising the eigenvector so that

$$\sum_{b \in \mathcal{U}^-} dim(p_b A_1)\nu_b^2 = 1.$$

Performing the "basic construction" of [9] we obtain $A_2 =< A_1, e_1 >$, the algebra of linear endomorphisms of A_1 generated by left multiplication by A_1 and the "conditional expectation" $e_1 : A_1 \to A_0$ defined by $Tr(e_1(y)x) = Tr(yx)$ for $y \in A_1, x \in A_0$. The centre of A_2 is naturally identified with that of A_0 and the bipartite graph for $A_1 \subseteq A_2$ is the transpose of that for $A_0 \subseteq A_1$. If we use the same eigenvector of the adjacency matrix as for $A_0 \subseteq A_1$ to define a trace on A_2, the restriction of that trace to A_1 is Tr. Continuing in this way one gets a tower A_n for $n \geq 0$ with $A_{n+1} =< A_n, e_n >$ and a coherent trace Tr on $\cup_n A_n$ defined by $Tr(xe_n) = \delta^{-2}Tr(x)$ for $x \in A_n$.

On general principles we expect the centralisers $A_0' \cap A_k = B_k$ to form a planar algebra and this could no doubt be done by the method of section 4 of [10]. We give an alternative, explicit realisation of B_k as the planar algebra of Γ with spin vector equal to (ν_a). Before proceeding to construct the linear isomorphism between B and $V(\Gamma)$ we give two warnings.

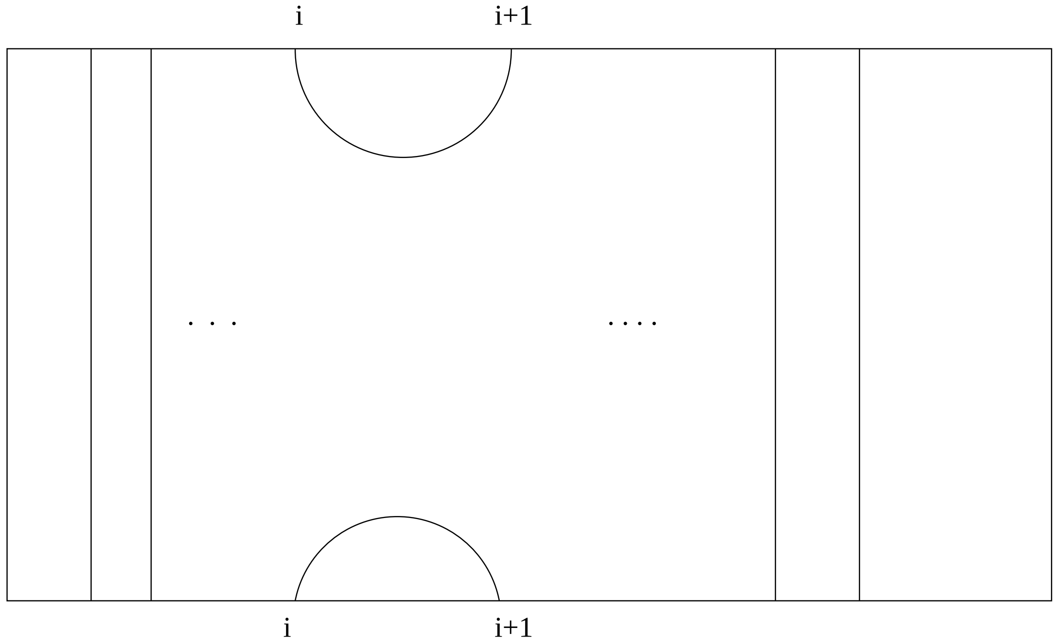
(i) The centraliser inclusions $B_k \subseteq B_{k+1}$ are not connected - the planar algebra is not "connected" in the sense of [10].

(ii) The trace Tr on $V(\Gamma)$ does *not* correspond to the restriction of Tr to B.

The isomorphism will give an "intrinsic" meaning to the planar algebra of a bipartite graph though it is not quite canonical since there are many pairs $A_0 \subseteq A_1$ which realise a give Γ. To make the choice unique we shall now assume that $dim(pA_0) = 1$ for all minimal central idempotents p in A_0. This has the desirable consequence that the algebra A_1 itself can be realised as having a basis consisting of paths (π, ϵ) of length 2 on Γ starting at a point in $\mathcal{U}^+$. In this model the centre of A_1 can be identified with elements b of $\mathcal{U}^-$ thought of as sums of all loops from points in $\mathcal{U}^+$ to b and back.

So with the data Γ, ν form the planar algebra $V(\Gamma)$ and the pair $A_0 \subseteq A_1$ as above. Define the maps $\Theta : V_0^{\pm} \to A_1$ in the obvious way - a loop (π, ϵ) is a special kind of path so automatically defines an element of A_1.

In the following theorem E_i will denote the tangle having all strings vertically connecting the top to the bottom except two which connect the boundary points numbered i and $i+1$, $2k-i$ and $2k-i+1$ as below:



Theorem 5.1 *There is a linear extension of* Θ *to all of* $V(\Gamma)$ *such that:*
(i) Θ *is an injective algebra homorphism.*
(ii) $\Theta(E_i) = \delta e_i$.
(iii) $\Theta(V_k) = B_k$.

Proof. Paths of length $2k$ will be given by pairs (π, ϵ) of functions where $\pi : \{0, 1, ..., 2k\} \to \mathcal{U}$ and $\epsilon : \{0, 1, ..., 2k-1\} \to \mathcal{E}$ and the path goes from $\pi(i)$ to $\pi(i+1)$ along $\epsilon(i)$. The multiplication of paths is

$$(\pi_1, \epsilon_1)(\pi_2, \epsilon_2) = (\pi, \epsilon)$$

if $\epsilon_1(k+i) = \epsilon_2(k-i-1)$ for $0 \leq i < k$ in which case $\pi(i) = \pi_1(i)$ for $0 \leq i \leq k$ and $\epsilon(i) = \epsilon_1(i)$ for $0 \leq i < k$, $\pi(i) = \pi_2(i)$ $k \leq i \leq 2k$ and $\epsilon(i) = \epsilon_2(i)$ for $k \leq i < 2k$.

If the conditions are not satisfied, i.e. the "returning" path of (π_1, ϵ_1) is different from the "outgoing" path of (π_2, ϵ_2), the product is zero.

It follows from [6] that there is a unique algebra isomorphism Π between A_k and the algebra of paths of length $2k$ on Γ with initial vertex in $\mathcal{U}^+$ which is the extension of an identification of A_1 with linear combinations of paths of length two, and such that $\Pi(Z(E_i)) = \delta e_i$. Note that $Z(E_i)$ is a linear combination of loops which are also paths so $Z(E_i)$ can be viewed as an element of A_k. The formula at the top of page 86 of [6] is exactly what Z

does to E_i if we choose ν as the square roots of the entries in the eigenvector of the adjacency matrix of Γ.

That multiplication of loops in V corresponds to multiplication of paths described above is just the definition of $Z(T)$ when T is the multiplication tangle in section 2 drawn with rectangular boxes rather than discs - all strings are vertical so there are no singularities and no spin term.

It is simple to check that a path is a loop if and only if the corresponding element of A_k is in the commutant B_k of A_0. Thus for Θ we may just take the restriction of Π to loops.$\square$

Using Θ we can thus transport all the planar algebra structure of V, and the spherically invariant partition function, to B which becomes a planar algebra with $B_0^+ = A_0$ and $B_0^- = A_0' \cap A_1$. The normalised trace thus defined on B_k will be written tr and as we have pointed out, it is not the restriction of the Markov trace ot B_k. In particular if a is a minimal idempotent in A_0, identified with an element of $\mathcal{U}^+$, $tr(a) = \mu_a^4$ where μ is the multiple of ν with $\sum_{a\in\mathcal{U}^+} \mu_a^4 = 1$. But $Tr(a) = \nu_a^2$ where $\sum_{a\in\mathcal{U}^+} \nu_a^2 = 1$.

6 Connes' cyclic category

Connes' cyclic category is defined in ([3, 4, 15]) alternatively as (i) the category with objects C_i, for $i = 0, 1, 2, ...$ and generated by morphisms $d_i : C_{n-1} \to C_n$, for $i = 0, 1, ..., n$, $s_i : C_{n+1} \to C_n$ for $i = 0, 1, ..., n$ and $t_i : C_n \to C_n$ subject to the relations

$$
\begin{aligned}
&d_i d_j = d_{j-1} d_i \ \text{ for } \ i < j \\
&s_i s_j = s_{j+1} s_i \ \text{ for } \ i \leq j \\
&d_i s_j = \begin{cases} s_{j-1} d_i, & \text{for } \ i < j \\ id & \text{for } \ i = j, i = j+1 \\ s_j d_{i-1} & \text{for } \ i > j+1. \end{cases} \\
&d_i t_n = t_{n-1} d_{i-1} \ \text{ for } \ 1 \leq i \leq n, \qquad d_0 t_n = d_n \\
&s_i t_n = t_{n+1} s_{i-1} \ \text{ for } \ 1 \leq i \leq n, \qquad s_0 t_n = t_{n+1}^2 s_n \\
&t_n^{n+1} = id.
\end{aligned}
$$

(ii) the category whose objects are the sets of $ith.$ roots of unity and whose morphisms are homotopy classes of monotone degree one maps from the unit circle to itself sending roots of unity to roots of unity.

Fix an $n > 0$ and define elements d_i,s_i for $i = 0, 1, ..., n$ and t_n of the planar operad P as follows:

d_i is the tangle having one internal disc with $2n + 2$ boundary points, and

$2n$ boundary points on the outside disc. Internal boundary points numbered $2i+1$ and $2i+2$ are joined by a string. All internal boundary points are connected to external ones, with the first internal one connected to the first external one except when $i=0$ when the first external point is connected to the third internal boundary point. These conditions uniquely determine d_i since the strings do not cross. .

For $0 \leq i \leq n$, s_i is the tangle having one internal disc with $2n+2$ internal boundary points, and $2n+4$ boundary points on the outside disc. External boundary points numbered $2i+2$ and $2i+3$ are connected and all other external ones are connected to internal ones with the first external point connected to the first internal point.

The tangle t_n is the clockwise rotation by two: there are $2n+2$ boundary points on both the internal and external discs, all strings connect inside points to outside ones and the first outside one is connected to the third inside one, as below:

Note that these tangles d_i,s_i and t_n have exactly one input and one output so their composition makes perfect sense provided the d's and s's have the right numbers of internal and external boundary points.

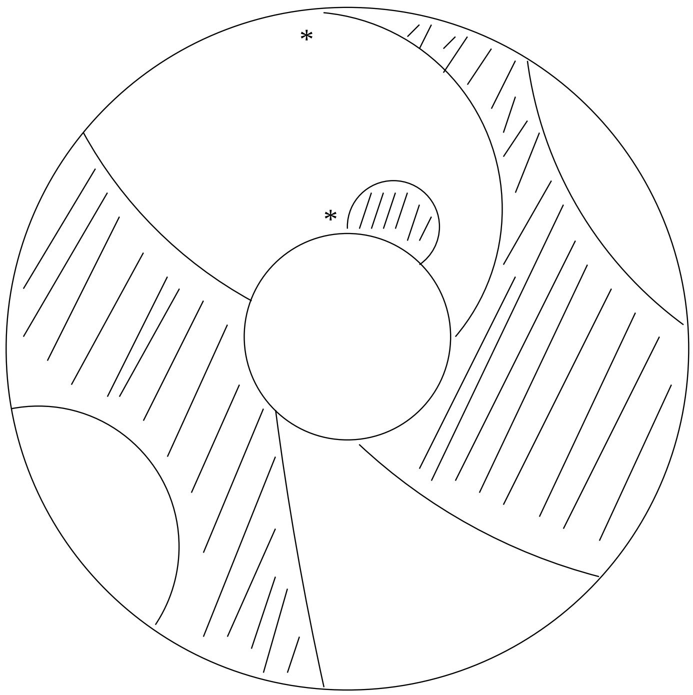
Theorem 6.1 *The operad element tangles defined above satisfy the relations of the generators of (the opposite of)Connes' cyclic category. The map they define from the cyclic category to annula tangles (with one input and one output) is injective.*

Proof. Verification of the relations is simply a matter of drawing pictures. Injectivity is more interesting. We prove it by providing a concrete realisation of the cyclic category which is arguably simpler than the one provided in [4]: for integers m and $n \geq 0$ let $\mathcal{C}_{m,n}$ be the set of all annular tangles (modulo a twist of 2π near the boundary) with $2m+2$ internal points and $2n+2$ external ones, with the following properties:

(i) All strings either connect an internal boundary point to an external one or a boundary point to one of it's neighbours.

(ii)If a string connects an internal boundary point to its neighbour the region between the string and the internal boundary is shaded, if it connects external boundary points the region between the string and the external boundary is unshaded.

A typical element in $\mathcal{C}_{2,3}$ is depicted below:



A morphism in the Cyclic Category

Construct the category with one object for each $n \geq 0$, the objects being the $2n+2$ boundary points up to isotopy and morphisms being the $\mathcal{C}_{m,n}$. One may check that composition of tangles makes this set into a small category (no closed loops are formed in composing such tangles) and that it is generated by the d_i,s_i and t_n . But to each element of this category there is a well defined homotopy class of degree one monotone maps from the circle to itself defined by isotoping the tangle so that the each root of unity is in exactly one unshaded region on the inside and outside. The degree one map can then be constructed as follows. First contract any regions enclosed by non through-strings to their segments on the internal or external boundary so that the shading near that boundary interval will reverse. Each unshaded interval on the outside boundary then contains exactly one root of unity and all the regions are topologically rectangles with internal and external boundary segments as opposite edges. These rectangles determine mappings from the inner circle to the outer one and after a litlle isotopy each internal root of unity may be sent onto the unique external root of unity in the same shaded region. Tangle composition obviously corresponds to composition of homotopy classes so we have a section of the map from the cyclic categroy to the $\mathcal{C}_{m,n}$'s which sends our d_i,s_i and t_n to the corresponding generators of the cyclic category. This is more than enough to show injectivity (and identify the image).$\square$

Now define the tangles $\partial_i = \Delta(d_i)$, $\sigma_i = \Delta(s_i)$ where Δ is the duality map of the first secion. It follows from the previous result that ∂_i , σ_i and t_n $(= \Delta(t_n))$ define another copy of Connes' cyclic category in annular tangles. Note also that there is a natural involutory antiautomorphism of annular tangles (reflect with respect to a circle half way between the internal and external boundaries) which exchanges ∂ with s and d with σ.

Thus the vector spaces V_{n+1} become a cyclic module in two different ways. The two structures interact as follows.

Proposition 6.2 *If V has modulus δ then $D_n\sigma_0 - \sigma_0 D_{n+1} = \delta id$ where $D_n = \sum_{i=0}^{n}(-1)^i d_i$ is the Hochschild boundary*

Corollary 6.3 *If V has modulus $\delta \neq 0$ then it is acyclic so the cyclic homology is that of the ground field.*

The same result holds for any representations of the category of annular tangles as in [7].

Thus the cyclic homology is of no immediate interest in planar algebras coming from subfactors. On the other hand there are very interesting planar algebras of modulus 0. The first is the one coming from the skein theory of the Alexander polynomial. It can be obtained by specialising the planar algebra of example 2.5 in [10] to the values $x = \sqrt{t} - \frac{1}{\sqrt{t}}$. The planar algebra involved in a conjecture of .. concerning asymptotics of the $sl(2)$ knot polynomials also has modulus 0. At this stage we have no results concerning the cyclic homology of these planar algebras. It is also true that the planar algebras of section 2 do not have any modulus at all if the spin vector is not an eigenvector for the adjacency matrix. Once again we have no results.

The appearance of cyclic homology in this context is not understood. One should note that, in the isomorphism of section 2 between the planar algebra of a finite graph and the centraliser tower of a finite dimensional multimatrix inculsion the cyclic module structure defined above corresponds to the usual one used in calculation the homology of the $A-A$ bimodule B. Since the category or annular tangles is generated by the two copies of the cyclic category we have defined, and one of those categories is typically the adjoint of the other for some invariant inner productIt might be interesting to see if there are known natural examples of cyclic modules for which the cyclic category extends to an action of the whole annular structure and even to a planar algebra.

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