Two subfactors and the algebraic decompsition of bimodules over II_1 factors.

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1 Introduction

Ocneanu noted that the data for what is now known as a Turaev-Viro type TQFT is supplied by a subfactor N of finite index and depth of a II_1 factor M. One takes all the irreducible bimodules arising in the decomposition of the tensor powers, in the sense of Connes ([3]), of the Hilbert space $L^2(M)$ viewed as an $N-N$ bimodule, or, in Connes' terminology, a correspondence. (We will use the term correspondence systematically to differentiate between these and purely algebraic bimodules over algebras.) These can be $N-N, N-M, M-N$ and $M-M$ correspondences. To get a 3-manifold invariant one starts with a triangulation and assigns in an arbitrary way M or N to the vertices, appropriate correspondences to the edges connecting vertices and intertwiners between the three correspondences around a face. Each 3-simplex then defines a scalar by a clever composition of the intertwiners on its boundary. One then multiplies these scalars over all the simplices and sums over all ways of assigning M, N, bimodules and intertwiners to obtain the invariant of the three manifold.

It is natural to ask the following questions about this procedure:
(i) Does one have to introduce Hilbert spaces and the Connes tensor product or is the purely algebraic decomposition of tensor powers of M over N enough?
(ii) Might there not be more than 2 factors involved, with correspondences between all of them?

Problem (i) is a subtle problem and easily overlooked since intertwiners between bimodules preserve bounded vectors. In the simplest case of $L^2(M)$

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it is of course equivalent to the algebraic simplicity of a II_1 factor. In this paper we shall answer question (i) by showing that the purely algebraic decomposition is exactly the same as the L^2 one. The main technical result will be Popa's theorem on relative Dixmier averaging - [10]. In recent work on the structure of intermediate subfactors of finite index of a II_1 factor ([6]) we repeatedly used the decomposition of a II_1 factor as a bimodule over a finite index subfactor. Having to go back and forward between these bimodules and their L^2-closures was awkward and left open such questions as the strong closedness of products of intermediate subfactors. These difficulties are removed by the results of this paper.

Thus unitary TQFT's can be obtained as categories of bimodules no more exotic than both left and right) finitely generated projective bimodules over a fixed algebra discovered by Murray and von Neumann in the 1930's!

Question (ii) was well understood and investigated by Ocneanu who introduced the concept of a "maximal atlas" for some kind of largest compatible collection of factors and bimodules. One can also ask if the introduction of another subfactor of M can lead to a larger system of bimodules. This leads directly to the study of two subfactors which we begin here.

2 Preliminaries

Let P and Q be II_1 factors. A $\underline{P-Q \text{ correspondence}}$ $\ _P\mathcal{H}_Q$ is a Hilbert space $\mathcal{H}$ togehter with normal commuting left and right actions of P and Q respectively (see [3]). If $\xi \in \mathcal{H}, p \in P, q \in Q$, $p\xi q$ will be the result of p and q acting on ξ. The correspondence will be called bifinite if both the left and right Murray -von Neumann dimensions $dim_P\mathcal{H}$ and $dim_Q\mathcal{H}$ are finite.

For a II_1 factor P on a Hilbert space $\mathcal{H}$, a vector $\xi \in \mathcal{H}$ is called bounded if the map $p \mapsto p\xi$ from P to $\mathcal{H}$ is bounded when P is given its L^2 norm. We will write $\mathcal{H}^0$ for the subspace of bounded vectors. The vector subspace $\mathcal{H}^0$ is always dense in $\mathcal{H}$ and is obviously invariant under M and M'.

It is well known and we prove it below that for a bifinite correspondence the bounded vectors for P and Q coincide. Thus in this case $\mathcal{H}^0$ is a $P-Q$ bimodule which we may consider purely algebraically. The notions of direct sum and irreducibility of correspondences are obvious. Our main task will be to show that a bifinite correspondence $\mathcal{H}$ is irreducible, in the sense that there are no *closed* invariant subspaces iff $\mathcal{H}^0$ is irreducible in the sense that there are no invariant subspaces whatsoever.

We will begin with some simple material on bounded vectors which we include for the convenience of the reader. See [9].

Lemma 2.0.1. *The bounded vectors for M (or by symmetry) M' on $L^2(M)$*

are M.

Proof. A bounded vector defines a bounded linear operator on M which extends to L^2. □

Lemma 2.0.2. *Let $\mathcal{H}$ be a Hilbert space with a (normal) action of the II_1 factor M. Let e be a projection in M and f be a projection in M'. Then*
(i)The bounded vectors for the II_1 factor pMp acting on $p\mathcal{H}$ are $p\mathcal{H}^0$.
(ii) The bounded vectors for M on $q\mathcal{H}$ are $q\mathcal{H}^0$.
(iii) If $\mathcal{K}$ is another Hilbert space on which M acts then $(\mathcal{H}\oplus\mathcal{K})^0 = \mathcal{H}^0\oplus\mathcal{H}^0$.

Proof. (i)Let $\xi \in p\mathcal{H}$ be a bounded vector for the action of pMp. It suffices to show that ξ is bounded for M. If $m \in M$ then $||m\xi||^2 = ||pmp\xi||^2 + ||(1-p)m\xi||^2$. Writing $1-p$ as a sum of projections e_i of trace less than that of p we see that $||m\xi||^2 = ||pmp\xi||^2 + \sum_i ||u_i e_i m\xi||^2$ for unitaries u_i with $u_i e_i u_i^* < p$. But then $u_i e_i m\xi = pu_i e_i mp\xi$ and we see that there is a constant $K > 0$ such that $||m\xi|| < K||m||_2$.
(ii) Obvious.
(iii) Obvious.

□

Corollary 2.0.3. *If M and H are as in the previous lemma and* $\dim \mathcal{H} < \infty$ *then the bounded vectors for M and M' coincide.*

Proof. Any such $\mathcal{H}$ can be obtained from $L^2(M)$ by a combination of the operations in 2.0.2. The bounded vectors and the commutants behave in compatible ways. (Note that the bounded vectors for $M \otimes M_n(\mathbb{C})$,acting on the direct sum of n copies of $L^2(M)$ is the direct sum of the M's.) □

Corollary 2.0.4. *If $N \subseteq M$ is a finite index subfactor and M acts on $\mathcal{H}$ with* $\dim_M \mathcal{H} < \infty$*,then the bounded vectors for M and N coincide.*

Proof. If ξ is bounded for N it is bounded for N', hence it is bounded for M' which means it is bounded for M. □

Corollary 2.0.5. *If $\mathcal{H}$ is a bifinite $P-Q$ correspondence the bounded vectors for the actions of P and Q^{opp} are the same.*

Proof. Q defines a finite index subfactor of P'. □

Lemma 2.0.6. *Let $\mathcal{H}$ be a bifinite $P - Q$ correspondence and $q \in Q$ be a projection. Then*
(i) The $P - qQq$ correspondence $\mathcal{H}q$ is irreducible iff $\mathcal{H}$ is.
(ii) The $P - qQq$ bimodule $\mathcal{H}^0 q$ is irreducible iff $\mathcal{H}^0$ is.

Proof. Property (i) follows by standard arguments on the behaviour of commutants of factors under reduction by projections, or follows from the proof of (ii).

For (ii), first suppose $\mathcal{H}^0 q$ is irreducible and choose non-zero elements v and w in $\mathcal{H}^0$. We must show how to obtain w from v by applying finitely many elements of P and Q. Note that since Q is a factor we may suppose that $v \in \mathcal{H}^0 q$. Now choose a partition of the identity of the form $u_i q u_i^*$ for $u_i \in Q$. For each i choose $a_i^k \in P$ and $b_i^k \in qQq$ with

$$wu_i q = \sum_k a_i^k v b_i^k$$

then

$$w = \sum_i wu_i q u_i^* = \sum_{i,k} a_i^k v b_i^k q u_i^*$$

.

Now suppose $\mathcal{H}^0$ is irreducible. Then if v and w are in $\mathcal{H}^0 q$, they are in $\mathcal{H}^0$ so there are $a_i \in P$ and $b_i \in Q$ with $v = \sum_i a_i w b_i$. But since $vq = v$ and $wq = w$ this gives $v = \sum_i a_i w (q b_i q)$.

□

Lemma 2.0.7. *If $N \subseteq M$ is an irreducible subfactor then M is an irreducible $N - M$ bimodule.*

Proof. Suppose $x \neq 0$ is an element of a sub $N - M$ bimodule V of M. Then xx^* is in V. So by Popa's result in [10], for every $\epsilon > 0$ there are unitaries $u_i \in N$ and constants λ_i with $|| \sum \lambda_i u_i x x^* u_i^* - tr(xx^*)|| < \epsilon$. For ϵ sufficiently small this means that there is an invertible element in M from which it immediately follows that $V = M$. □

Putting it all together we have the following result:

Theorem 2.0.8. *A bifinite $P - Q$ correspondence $\mathcal{H}$ is irreducible iff $\mathcal{H}^0$ is an irreducible $P - Q$ bimodule.*

Proof. ($\implies$) Since $\dim_P(\mathcal{H}) \dim_Q(\mathcal{H}) = [P' : Q]$, at least one of $\dim_P(\mathcal{H})$ and $\dim_Q(\mathcal{H})$ is ≥ 1. By passing to the opposite correspondence if necessary we may suppose $\dim_P(\mathcal{H}) \geq 1$. Choosing a projection q of the appropriate trace in Q we have that $\mathcal{H}q \cong L^2(P)$ as a P-module. Then pQp may be identified with a subfactor M of JPJ, and $\mathcal{H}q$ is a $P - JMJ$ correspondence which is irreducible by 2.0.6. This means that the subfactor $JMJ \subseteq P$ is irreducible and so by 2.0.7 is $(\mathcal{H}q)^0$. Hence by 2.0.6 $\mathcal{H}^0$ is irreducible as a $P-Q$ bimodule.

($<=$) The bounded vectors in a proper subcorrespondence of $\mathcal{H}$ would be a proper sub-bimodule of $\mathcal{H}^0$. □

Corollary 2.0.9. *If $\mathcal{H}$ is a bifinite $P-Q$ correspondence and $e_i, i=1,2,...,n$ is a partition of unity consisting of minimal projections in the (finite dimensional) von Neumann algebra $P'\cap Q'$, then each $e_i(\mathcal{H}^0)$ is an irreducible $P-Q$ bimodule and $\mathcal{H}^0 = \oplus_i e_i(\mathcal{H}^0)$.*

Corollary 2.0.10. *Let $\mathcal{H}$ be a bifinite $P-Q$ correspondence and V be any sub $P-Q$ bimodule of $\mathcal{H}^0$. Then the Hilbert space closure $\overline{V}$ of V is a $P-Q$ correspondence whose bounded vectors are precisely V.*

Proof. Since $\overline{V}$ is a bifinite correspondence, decompose it according to the previous corollary and observe that for each i, $e_i(V)$ is a sub-bimodule of the irreducible bimodule $e_i(\overline{V})^0$. If any of these sub-bimodules were zero, V would not be dense in $\overline{V}$. □

Proposition 2.0.11. *If P and Q are finite index subactors of M then the bounded vectors in the $P-Q$ correspondence $L^2(M)$ are M.*

Proof. Follows from 2.0.4 □

Corollary 2.0.12. *Let P and Q be finite index subfactors of the II_1 factor M. Then any sub $P-Q$ bimodule of M is weakly closed and equal to the bounded vectors in the bifinite correspondence given by its L^2 closure.*

Proof. Follows immediately from previous results. □

The next result is relevant to [6].

Corollary 2.0.13. *Let P and Q be finite index subfactors of the II_1 factor M. Then if $V \subseteq M$ and $W \subseteq M$ are $P-Q$ bimodules then VW is weakly (hence strongly) closed.*

3 Pairs of subfactors.

A pair P, Q of subfactors of a II_1 factor M can be thought of as a "quantisation" of the notion of a pair of closed subspaces of a Hilbert space. The complete classification of such subspaces is well known and has two parts-one combinatorial and one spectral. Let us briefly review it (for a complete reference see [13]). Suppose V and W are closed subspaces of $\mathcal{H}$ with orthogonal projections e_V and e_W respectively.

Combinatorial part:
Some of the closed subspaces

$$V \cap W, V \cap W^{\perp}, V^{\perp} \cap W, V^{\perp} \cap W^{\perp}$$

are zero. Since they are mutually orthogonal the problem breaks up into each subspace and the orthogonal complement $\mathcal{H}_{\mathcal{G}}$ of all of them. The combinatorial part of the invariant is just the set of these subspaces which are zero.

On $\mathcal{H}_{\mathcal{G}}$ the two closed subspaces are said to be in "general position" and using the polar decomposition of the operator $e_V e_W$ we find a direct sum decomposition of $\mathcal{H}_{\mathcal{G}}$ for which, as matrices, $e_V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_W = \begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix}$ for some selfadjoint a, $0 \leq a \leq 1$ with neither 0 nor 1 as an eigenvalue. (By analogy with the case of one-dimensional subspaces the angle operator between V and W is that positive operator θ such that $cos^2\theta = e_V e_F e_V$.

Knowledge of a is equivalent to knowledge of V and W so the spectral part of the classification of pairs of closed subspaces is equivalent to the classification of bounded self-adjoint operators on Hilbert space.

Second quantisation is a functor taking Hilbert spaces to algebras of operators on Hilbert spaces thus the (second) quantisation of subspaces will give certain subalgebras of operator algebras. The simplest of operator algebras are factors so the simplest quantised subspace problem is the study of subfactors. In general subfactors do not arise as quantised subspaces so we can think of the study of subfactors as a (second) quantisation of the study of subspaces. The simplest non-trivial situation for subfactors is when both factors are II_1 factors and (rather simple) examples arise immediately from fermionic second quantisation of subspaces.

Here we propose the study of pairs of subfactors of a II_1 factor as a (second) quantisation of the study of pairs of closed subspaces. The problem will once again break up into a "combinatorial" and a "spectral" part but both are considerably more complicated.

Several papers can be considered precursors to this idea. First is the paper [14] where the angle between subfactors is introduced as the spectrum of the angle operator of the subfactors viewed as Hilbert subspaces and several examples are calculated including those from fermionic second quantisation. If finiteness of the index of a subfactor is taken as the analogue of finite codimension of a subspace then an entirely new situation arises in the subfactor setting, namely the intersection of two finite index subfactors can be of infinite index. It was shown in [8] that this is the case iff the spectrum of the angle operator is infinite. If we suppose that the intersection of two subfactors is a finite index irreducible subfactor then we are faced with many constraints. This was first looked at in [15] where the interesting question of the possibilities for the lattice of intermediate subfactors was posed (see also [1]). More general rigidity results were obtained in [6],[5].

Here we assume only that P and Q are finite index subfactors of the II_1 factor M.

Definition 3.0.14. *The* combinatorial invariant *of P and Q is the isomorphism class of the chain of finite index subfactors $P \subseteq M \subseteq < M, e_Q >$ (where $< M, e_Q >$ is the basic construction of [7] for the subfactor $Q \subseteq M$.*

Notes:

i) For simplicity we have chosen to include the whole subfactor chain as the invariant although the isomorphism class of M itself for instance makes it not entirely combinatorial. We are really thinking of the standard invariant/planar algbera for the subfactor $P \subseteq < M, e_Q >$, with the privileged biprojection onto M, as the combinatorial part, but this would take a little formalising, and the connection between this and the spectral invariant is more clear with the above definition.

ii) The simplest interesting instance of such a pair is when both P and Q are fixed point algebras for actions of finite groups on M. The combinatorial invariant is then a Bisch-Haagerup subfactor as in [2].

The difference between isomorphism of the pair (P, Q) and equality of their combinatorial invariants is made clear by the following:

Lemma 3.0.15. *Let P and Q be von Neumann subalgebras of M. If $u \in M$ is a unitary then $x \mapsto JuJxJu^*J$ from $\mathfrak{B}(L^2(M))$ to itself restricts to an isomorphism from the basic construction $< M, e_Q >$ to $< M, e_{uQu^*} >$ which is the identity on M.*

Proof. We have $JuJ(JQJ)'Ju^*J = (JuQu^*J)'$, and JuJ commutes with M. $\square$

Another way of seeing the same thing is:

Lemma 3.0.16. *Let P and Q be von Neumann subalgebras of M. Then if $u \in M$ is a unitary, the map $x \mapsto xu$ defines a unitary on $L^2(M)$ which intertwines the $P - Q$ and $P - u^*Qu$ bimodule structures given by left and right multiplication.*

Thus the discrete invariant for (P, Q) is invariant under arbitrary inner perturbations of Q, with P fixed.

Moreover we have the following:

Lemma 3.0.17. *If (P, Q) and (P_1, Q_1) have the same combinatorial invariant then (P_1, Q_1) is isomorphic to (P, uQu^*) for some unitary $u \in M$.*

Proof. We may identify $P \subseteq M \subseteq < M, e_Q >$ with $P_1 \subseteq M \subseteq < M, e_{Q_1} >$ so that Q_1 is the intersection with M of the commutant in $< M, e_Q >$ of e_{Q_1}. But there is a unitary in $< M, e_Q >$ with $ue_Qu^* = e_{Q_1}$. By [11] we may suppose there is a v in M with $ve_Qv^* = e_{Q_1}$ and taking conditional expectations onto M we see that v is unitary. But if $x \in M$ then x commutes with ve_Qv^* iff v^*xv commutes with e_Q and so iff $v^*xv \in Q$. Thus $vQv^* = Q_1$. □

The proof of 3.0.17 in fact proves the following:

Lemma 3.0.18. *Let $N \subseteq M \subseteq < M, e_N >$ be a basic construction of finite factors. Then if $\alpha \in Aut(< M, e_N >, M)$, there is a unitary $u_\alpha \in M$ such that $\alpha(N) = u_\alpha N u_\alpha^*$.*

Let $\mathcal{N}(N)$ be the normaliser of N in M. The space of all unitary conjugates of N in M is obviously $U(M)/\mathcal{N}(N)$. The action of $Aut(< M, e_N >, M)$ on this space is clearly

$$\alpha(v\mathcal{N}(N)) = \alpha(v)u_\alpha\mathcal{N}(N)$$

(with u_α as above).

Returning to a pair (P, Q) of subfactors we see that another isomorphic pair determines and is determined by an element in $U(M)/\mathcal{N}(Q)$ every time its combinatorial invariant is identified with that of (P, Q). Since two such identifications differ by an element of $Aut(< M, e_Q >, M, P)$ we obtain the following:

Theorem 3.0.19. *Given finite index subfactors (P, Q) of M, the set of all other such pairs with the same combinatorial invariant is given by the set of orbits for the action of $Aut(< M, e_Q >, M, P)$ on $U(M)/\mathcal{N}(Q)$.*

If the element $\alpha \in Aut(< M, e_Q >, M, P)$ is inner when restricted to M then it is given by conjugation by an element $u_\alpha \in \mathcal{N}(P)$. The action of α on $U(M)/\mathcal{N}(Q)$ is then given by left multiplication by u_α. In particular of $\mathcal{N}(P)$ defines a normal subgroup of $Aut(< M, e_Q >, M, P)$ acting by left multiplication so the set of isomorphism classes of pairs with given combinatorial invariant is given by the orbits of $Aut(< M, e_Q >, M, P)$ on the double coset space

$$\mathcal{N}(P)\backslash U(M)/\mathcal{N}(Q).$$

Should M not have any outer automorphisms this double coset space is thus a complete description of pairs of subfactors with given combinatorial invariant. However although theoretically nice, one does not necessarilly know much about even this double coset space. The very simplest case is in finite dimensions with $P = Q$ and M are all matrix algebras. The normaliser of P is then the cartesian product $U(m) \times U(n)$ for suitable m and n and we are looking at its double coset space. A II_1 factor may also fail to have outer automorphisms ([12]) and if so $\mathcal{N}(P)\backslash U(M)/\mathcal{N}(Q)$ may be considered in the simplest possible situation when $P = Q$ and M is just the 2x2 matrices over P. Not much seems to be known about the double coset space even in that case. Note that the spectrum of the angle operator between P and Q is clearly an isomorphism invariant.

Thus the "spectral" side of the classification of pairs of subfactors looks rather difficult to analyse. There are, however many simple questions one may ask, especially about the influence of the combinatorial invariant on the spectral one. The following is perhaps the most obvious:

"If $P \subseteq JQ'J$ is of finite depth does there exist a unitary $u \in M$ such that $P \cap uQu^*$ is of finite index?"

It is even possible that the converse is true also if we suppose that the combinatorial invariant is an irreducible subfactor. Some evidence for a positve answer is given by the (obstructionless) Bisch-Haagerup subfactors for which it is indeed if and only if:

Theorem 3.0.20. *Let M be a II_1 factor with G and H finite groups of outer automorphisms. Suppose the group generated by G and H in $Out(M)$ lifts to $Aut(M)$. Then there is a unitary $u \in M$ such that $M^G \cap u(M^H)u^*$ is of finite index (in the sense of Pimsner-Popa, [11]) iff the subfactor $M^G \subseteq< M, e_{M^H} >$ is of finite depth.*

Proof. By [2] the subfactor is of fnite depth iff the group K generated by G and H in the quotient $Aut(M)/Int(M)$ is finite.

Suppose first that such a u exists. Then the group generated by G and $(Adu)H(Adu^*)$ is necessarily finite modulo inner automorphisms since the fixed point algebra for a group which is infinite modulo inner automorphism is necessarily of infinite index.

To prove the converse we will use the following result from nonabelian cohomology which is no doubt well known. We give a proof here for the sake of the reader.

Lemma 3.0.21. *Let G be a finite group of outer automorphisms of a II_1 factor M. If $g \mapsto u_g$ and $g \mapsto v_g$ are unitaries satisifying $u_g g(u_h) = \mu(g,h)u_{gh}$ and $v_g g(v_h) = \mu(g,h)v_{gh}$ for some circle-valued 2-cocyle μ (so that $g \mapsto Ad\, u_g g$ and $g \mapsto Ad\, v_g g$ are also outer actions of G), then there is a unitary $w \in M$ with $(Ad\, w)Ad\, u_g g(Ad\, w^*) = Ad\, v_g g$ for all $g \in G$.*

Proof. We will use Connes' 2x2 matrix trick. For each $g \in G$ let $U_g = \begin{pmatrix} u_g & 0 \\ 0 & v_g \end{pmatrix}$ be unitaries in the II_1 factor $M \otimes M_2(\mathbb{C})$. Then $(Ad\, U_g)g \otimes id$ is an outer action of G on $M \otimes M_2(\mathbb{C})$. The projections $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are of equal trace in the fixed point algebra for $(Ad\, U_g)g \otimes id$, which is a II_1 factor. Thus there is a unitary $w \in M$ so that $\begin{pmatrix} 0 & w^* \\ 0 & 0 \end{pmatrix}$ is fixed by $(Ad\, U_g)g \otimes id$. But this means $u_g\, g(w^*)v_g^* = w^*$ for all $g \in G$. This inmediately implies the result.

$\square$

We now return to the proof of theorem 3.0.20. Finite depth of the Bisch-Haagerup subfactor implies by [2] that the group K generated by G and H in $Out(M)$ is finite. By our assumption there is an action $\alpha : K \to Aut(M)$ and unitaries u_g for $g \in G$ and v_h for $h \in H$ with $g = Ad\, u_g \alpha_g$ and $h = Ad\, u_h \alpha_h$.Thus there are two-cocycles $\mu(g_1, g_2)$ and $\nu(h_1, h_2)$ with $u_{g_1} g_1(u_{g_2}) = \mu(g_1, g_2)u_{g_1 g_2}$ and $v_{h_1} h_1(v_{h_2}) = \nu(h_1, h_2)v_{h_1 h_2}$. The fixed point algebra M^K for the action α is a type II_1 factor so we may choose, in M^K, commuting projective unitary representations w_g and t_h of G and H with cocycles μ and ν respectively. By the previous lemma we may conjugate G and H by an inner automorphism $Ad\, x$ so that $(Ad\, x)g(Ad\, x^*) = (Ad\, w_g)g$. This conjugation does not change the 2-cocycle for the inner perturbation of H so we may now conjugate just H by some Ady so that $(Ad\, y)h(Ad\, y^*) = (Ad\, t_h)h$. But then the fixed point algebra for $(Ad\, w_g)g$ <u>and</u> $(Ad\, t_h)h$ contains the relative commutant in M^K of the finite dimensional algebra generated by the w_g and the t_h. It is thus of finite index in M.

$\square$

The question above of when one subfactor can be perturbed by an inner automorphism so that the intersection of both is of finite index leads to a little insight into the spectral invariant in some cases, the easiest with any bite being the case where M is the crossed product of P by a period 2 outer automorphism and $P = Q$. In this case we use the notation that elements of M are of the form $a + bu$, with $a, b \in P$ and u being a self-adjoint unitary in

$\mathcal{N}(P)$ orthogonal to P. Then the conditional expectation E of M onto P is given by $E(a+bu)=a$.

Proposition 3.0.22. *With $P \subset M$ of index 2 as above and $w = a + bu$ a unitary in M, $Q = uPu^*$, the angle operator EE_QE is*

$$(EAd(w)EAd(w^*)E)(x+yu) = aa^*xaa^* + b\alpha(a^*xa)b^* + a\alpha(b^*xb)a^* + bb^*xbb^*$$

Proof. A simple calculation. □

Lemma 3.0.23. *If a, b as above are in a finite dimensional *-subalgebra D of P with $\alpha(D) = D$ then $[M : P \cap Q] < \infty$.*

Proof. Let $\mathfrak{D}$ be the crossed product of $D \otimes D^{opp}$ by $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ acting by $\alpha \otimes \alpha^{opp}$. Then the algebra generated by $\{$(left multiplication by d_1)$\otimes$(right multiplication by d_2)$\}$ for $d_i \in D$, and $\alpha \otimes \alpha^{opp}$ is a finite dimensional *-subalgebra of $End(P)$. This algebra contains EE_QE by the previous calculation so it must have finite spectrum. By [8] we are done. □

Corollary 3.0.24. *If P is hyperfinite, the set of all unitaries $u \in M$ such that $[M : P \cap uPu^*] < \infty$ is strongly dense in the unitary group of M.*

Proof. By [4] any two outer actions of $\mathbb{Z}/2\mathbb{Z}$ are conjugate and it is a simple matter to build one with an increasing dense sequence D_n of invariant finite dimensional *-subalgebras of P. Then the sequence of algebras generated by D_n and u are dense in in M and hence their unitary groups are dense in that of M. □

Similar results hold when M is the algebra of $n \times n$ matrices over P.

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