1 The definition of a planar algebra.

By "smooth disc" we will mean the image of the closed unit ball under a C^∞ diffeomorphism of $\mathbb{R}^2$. By "smooth curve" we will mean the image of the unit circle or a closed interval under a C^∞ diffeomorphism of $\mathbb{R}^2$.

1.1 Planar tangles

Definition 1.1.1. *A* planar tangle *T consists of the following data:*
i) A smooth disc $D^T \subset \mathbb{R}^2$
ii) A certain finite set $\mathfrak{D}_T$ of disjoint smooth discs in the interior of D^T
iii) A finite number of disjoint smooth curves in D^T (called the strings $\mathfrak{S}(T)$ *of T)which do not meet the interiors of the D in $\mathfrak{D}_T$. The boundary points of a string of T (if it has any) lie in the boundaries of either D^T or the discs in $\mathfrak{D}_T$. The strings meet the boundaries of the discs transversally if they meet them at all.*

The subset of $\mathbb{R}^2$ obtained by taking away from D^T the strings of T and the discs in $\mathfrak{D}_T$ is called the set subjacent to T *and the connected components of the set subjacent to T are called the* regions *of T.*

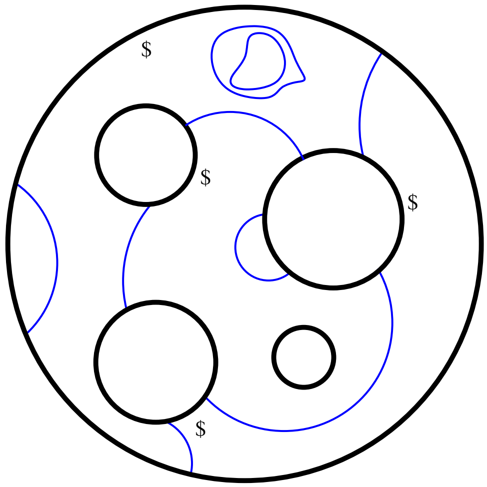
The points at which a string meets a disc will be called the boundary points *of that disc. To each disc D of a planar tangle let n_D be the number of boundary points of D.*

The boundary of a disc D of T consists of disjoint open curve segments together with the boundary points of D. These open curve segments will be called the intervals *of D (if D does not meet the strings of T, its whole boundary will be the (only) interval of D).*

For each disc $D \in \mathfrak{D} \cup \{D^T\}$ there will be chosen one of its intervals, called the marked interval *of D. The boundary points of D are then numbered $1, 2, \cdots n_D$ in clockwise order starting from the first one encountered after the marked interval.*

If $n_{D^T} = n$, T is called a "planar n-tangle".

Here is a picture of a planar 4-tangle. We have drawn the discs as round circles to clearly distinguish them from the strings of the tangle, and the marked intervals for each disc have been indicated by placing a $ near them in the region whose boundary they meet (a disc with one boundary interval needs no $):



If θ is a diffeomorphism of $\mathbb{R}^2$ and T is a planar tangle then $\theta(T)$ is also a planar tangle where the marked intervals of $\theta(T)$ are the images under θ of those of T.

Under certain special circumstances tangles may be "glued". For the following definition note that a planar tangle is determined by its subjacent set and the marked intervals. Any choice of marked intervals is allowed and different choices give different tangles.

Definition 1.1.2. *Let T and S be planar tangles. Suppose that the outer boundary disc D^S of S is the same set as some disc $D_S \in \mathfrak{D}_T$, and that the marked intervals of this disc coming from S and T are the same.*
Then if the union W of the subjacent sets of T and S is the set subjacent to a planar tangle, the discs of such a tangle are D^T and $\{\mathfrak{D}_T \cup \mathfrak{D}_S\} \setminus \{D_S\}$. We call $T \circ S$ that planar tangle whose subjacent set is W and whose intervals are those of T and S (except the intervals in D_S).

The regions of $T \circ S$ are unions of regions of T and regions of S.

Note that the disk D_S is not part of the glued tangle so that

$$D^{T\circ S} = D^T \text{ and } \mathfrak{D}_{T\circ S} = (\mathfrak{D}_T \setminus \{D_S\}) \cup \mathfrak{D}_S.$$

Remark 1.1.3. *More general notions.*

The notion of planar tangle defined above could be altered/generalised in several ways by adding structure, for instance:
i) The regions of the tangle could be labelled (“coloured").
ii) The strings of the tangle could be labelled.
iii) The strings of the tangle could be oriented.

Composition of tangles in all these cases would require also that the extra structure on the boundary of the disc D_S as above be the same for both T and S. The function n_D should be modified so as to contain the information which the added structure gives to the boundary. We will call this the *boundary condition* of D and write it ∂_D. This extra structure is part of the tangle so tangles can only be composed if $\partial(D^S) = \partial(D_S)$ with notation as above, in which case the extra structure of T and S should define extra structure on $T \circ S$.

All these notions would lead to systems that should be called planar algebras.

We will treat explicitly the cases of *shaded* and *oriented* planar tangles.

Definition 1.1.4. *Shaded planar tangle.* A planar tangle T will be called *shaded* if its regions are shaded with two colours so that if the closures of two regions meet, then they are shaded differently. The shading is part of the data of the tangle. Note that for a planar tangle to admit a shading all its discs must meet an even number $2n$ of strings, and discs will be of two kinds, $+$ and $-$ when the distinguished interval meets the closure of an unshaded or shaded region respectively. Since the shadings of the intervals on the boundary of a disc simply alternate, the extra boundary data for the function ∂ is just the kind of disc it is. Thus for a disc D of kind $\pm$ with $n_D = 2n$ we will write $\partial(D) = (n, \pm)$.

Definition 1.1.5. *Oriented planar tangle.* A planar tangle T will be called *oriented* if all the strings of T are oriented. Then the boundary points of each disc inherit orientations. So for each n we define $\mathfrak{B}_n$ to be the set of all functions from $\{1, 2, ..., n\} \to \{\uparrow, \downarrow\}$. Then each disc D (with n boundary points) of T defines an element $\partial(D) \in \mathfrak{B}_n$ according to:

$\partial(D)(k) = \uparrow$ if the string meeting the kth boundary point of D exits D and $\downarrow$ otherwise.

Remark 1.1.6. Observe that orientation-preserving diffeomorphisms of the plane map shaded (oriented) tangles to shaded(oriented) tangles in the obvious way. It is clear how orientation reversing diffeomorphisms should act on shaded tangles but not entirely clear for oriented tangles.

Definition 1.1.7. *If θ is an* orientation-reversing *diffeomorphism of the plane and T is an oriented planar tangle with underlying unoriented tangle $\overset{\circ}{T}$, then $\theta(T)$ is the oriented tangle whose underlying non-oriented tangle is $\theta(\overset{\circ}{T})$ but whose strings are oriented in the* opposite *way from their orientation as oriented images of the strings of T.*

1.2 Planar algebras.

For the definition of a planar algebra recall that if S is a set and V_s is a vector space for each $s \in S$, the cartesian product $\underset{s\in S}{\times} V_s$ is the vector space of functions f from S to $\bigcup_{s\in S} V_s$ with $f(s) \in V_s \ \forall \ s \in S$.

Definition 1.2.1. *Planar algebra.*

A planar algebra P *will be a family* P_n *of vector spaces indexed by* $\mathbb{N} \cup \{0\}$ *together with multilinear maps*

$$Z_T : \bigtimes_{D \in \mathfrak{D}_T} P_{\partial(D)} \to P_{\partial(D^T)}$$

for every planar tangle T *with* $\mathfrak{D}_T$ *non-empty, satisfying the following two axioms.*

1) If θ *is an* orientation preserving *diffeomorphism of* $\mathbb{R}^2$*, then*

$$Z_{\theta(T)}(f) = Z_T(f \circ \theta).$$

2)(Naturality)

$$Z_{T \circ S} = Z_T \circ Z_S$$

Where the right hand side of the equation is defined as follows: first recall that $\mathfrak{D}_{T \circ S}$ *is* $(\mathfrak{D}_T \setminus \{D^S\}) \cup \mathfrak{D}_S$ *. Thus given a function* f *on* $\mathfrak{D}_{T \circ S}$ *to the appropriate vector spaces, we may define a function* $\tilde{f}$ *on* $\mathfrak{D}_T$ *by*

$$\tilde{f}(D) = \begin{cases} f(D) & \text{if } D \neq D^S \\ Z_S(f|_{\mathfrak{D}_S}) & \text{if } D = D^S \end{cases}$$

then the formula $Z_T \circ Z_S(f) = Z_T(\tilde{f})$ *defines the right hand side.*

Lemma 1.2.2. *Let* T *be the tangle with no strings* $D^T =$ *the unit circle and* $\mathfrak{D}_T = \{A, B\}$ *where* $A = \{(x,y) | (y + 1/2)^2 + x^2 \leq 0.1\}$ *and* $B = \{(x,y) | (y - 1/2)^2 + x^2 \leq 0.1\}$*. If* P *is a planar algebra show that* P_0 *becomes a commutative associative algebra under the multiplication*

$$ab = Z_T(f) \text{ where } f(A) = a \text{ and } f(B) = b.$$

Proof. This is an *important* exercise in the definitions of naturality and diffeomorphism invariance. □

Definition 1.2.3. *(i) A* sub planar algebra Q *of a planar algebra* P *will be a family* Q_n *of subspaces of* P_n *such that* $Z_T(f) \in Q_{\partial(D^T)}$ *whenever* $f(D) \in Q_{\partial(D)}$ *for all* $D \in \mathfrak{D}_T$*.*
(ii) An ideal I *of a planar algebra* P *will be a family* I_n *of subspaces of* P_n *such that* $Z_T(f) \in Q_{\partial(D^T)}$ *whenever* $f(D) \in I_{\partial(D)}$ *for some* $D \in \mathfrak{D}_T$*.*
(iii) A homomorphism $\theta : P \to Q$ *between planar algebras will be family* $\theta_n : P_n \to Q_n$ *of linear maps such that* $\theta(Z_T(f)) = Z_T(\theta \circ f)$*. An* isomorphism *is a bijective homomorphism.*

Exercise 1.2.4. *If I is an ideal in P, show that the quotient P/I with $(P/I)_n = P_n/I_n$ may be endowed with a planar algebra structure in the obvious way. If $\theta : P \to Q$ is a homomorphism then $ker\theta$ is an ideal, image(θ) is a subalgebra of Q and image(θ) $\cong P/ker\theta$.*

Remark 1.2.5. It may on occasion be convenient to refer to a planar algebra as above as an *unoriented* planar algebra.

Definition 1.2.6. *A* shaded planar algebra *will be a family $P_{n,\pm}$ of vector spaces indexed by $\big(\mathbb{N} \cup \{0\}\big) \times \{+,-\}$ together an action of shaded planar tangles as in 1.2.1*

Definition 1.2.7. *An* oriented *planar algebra will be a family P_α of vector spaces where $\alpha \in \mathfrak{B}_n$, for all $n \geq 0$, together with an action of oriented planar tangles as in 1.2.1.*

The notions of isomorphism, automorphism, subalgebra and ideal of oriented and shaded planar algebras are the obvious extensions of 1.2.3.

Remark 1.2.8. *Is this really true?* Observe that a planar algebra defines a shaded planar algebra by setting $P_{n,\pm} = P_{2n}$ and considering shaded planar tangles just as planar tangles by forgetting the shading. Similarly a planar algebra defines an oriented planar algebra.

An oriented planar algebra also defines a shaded planar algebra by orienting the strings of a shaded tangle as the boundary of the shaded regions which are oriented as subsets of $\mathbb{R}^2$. The $P_{n,\pm}$ are then P_α and $P_{\alpha'}$ where for $i = 1, 2, \cdots 2n$,

$$\alpha(i) = \begin{cases} \uparrow & \text{if } i \text{ is odd} \\ \downarrow & \text{if } i \text{ is even} \end{cases} \qquad \text{and} \qquad \alpha'(i) = \begin{cases} \downarrow & \text{if } i \text{ is odd} \\ \uparrow & \text{if } i \text{ is even} \end{cases}$$

This shaded planar algebra actually forms a sub-planar algbera of $\overrightarrow{P}$.

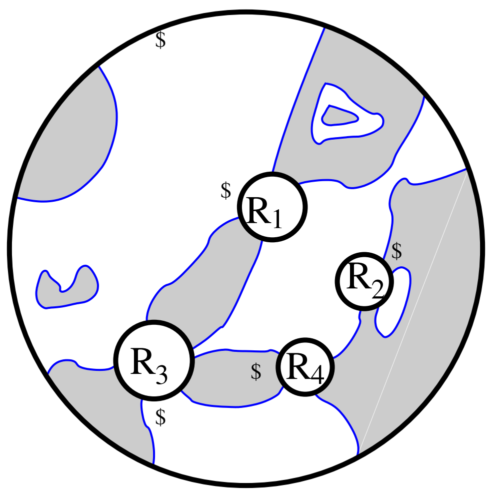
Moreover a *central*-(see 1.5.6) shaded planar algebra defines an oriented one by setting

$$\overrightarrow{P}_0 = P_{0,+} = P_{0,-} \qquad \text{and}$$

$$\overrightarrow{P}_\beta = \begin{cases} P_{n,+} & \text{if } \beta = \alpha \text{ as above} \\ P_{n,-} & \text{if } \beta = \alpha' \text{ as above} \\ 0 & \text{otherwise.} \end{cases}$$

Note that this procedure does not work if the shaded planar algebra is not a central one as we cannot identify $P_{0,+}$ and $P_{0,-}$.

A natural notation for $Z_T(f)$ is to place $f(D)$ in D for each $D \in \mathfrak{D}_T$ in D. This is just like the notation "$y(x_1, x_2, ..., x_n)$" for a function of several variables where the $f(D)$ correspond to the x_i and the internal discs correspond to the spaces in between the commas. (We also call the internal discs "input discs".) Thus if we are dealing with a shaded planar algebra and if R_1 is in $P_{2,+}$, R_2 and R_4 are in $P_{2,-}$ and R_3 is in $P_{3,+}$ then the following picture is an element of $P_{4,-}$.

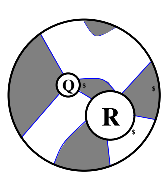


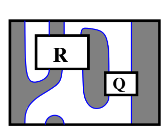
With this in mind the following definition is natural:

Definition 1.2.9.

A constant tangle *is a planar n-tangle with no input discs,*
a linear tangle *is a planar n-tangle with one input disc and*
a quadratic tangle *is a planar n-tangle with two input discs.*
And in general the degree *of a planar tangle is the number of input discs.*

Remark 1.2.10. *A useful convention for shaded planar algebras.* For a shaded planar algebra all discs in all relevant tangles have an even number of boundary points. Thus the strings at each disc can canonically be split into two equal sets and the diagrams isotoped into ones where the discs are visually indistinguishable from horizontal rectangles, with the strings meeting the edges of the rectangle orthogonally and half attached to the top and half to the bottom directly below the strings at the top. The distinguished boundary interval is the one containing the left edge of the rectangle.

Thus we replace 

with 

There are many variations on the definition of planar tangles and planar algebra.

Since the action of a tangle depends only on the tangle up to isotopy/diffeomorphism it is possible to use tangles defined up to isotopy. But then one must keep track of the input discs and choose representatives and the definition of gluing must be done much more carefully. We have chosen the definition we have given to avoid these problems and because we foresee a more general structure where the action of a tangle is not simply invariant under isotopy. For instance the angles made by the strings where they meet the boundary disc could play a role. In fact there is already a relevant toy version of non-invariance under diffeomorphisms which is rather important, and that is for *-structure.

Definition 1.2.11. *We will say that a planar algebra P over $\mathbb{C}$(oriented or shaded planar algebra) is a* planar *-algebra *if each P_n (P_α or $P_{n,\pm}$) possesses a conjugate linear involution * so that if θ is an* orientation reversing *diffeomorphism of $\mathbb{R}^2$, then*

$$Z_{\theta(T)}(f)^* = Z_T((f \circ \theta)^*).$$

Note that any two orientation reversing diffeomorphisms differ by an orientation preserving one so it would suffice to take *any* orientation-reversing θ in the above definition.

1.3 Unital Planar algebras.

The mathematical structure which a planar algebra seems to most strongly resemble is that of an algebra over an operad. According to [], given a monoidal symmetric category with product $\otimes$ and unit object κ an operad $\mathfrak{C}$ is a collection of objects $\mathfrak{C}(j)$ for $j = 0, 1, 2, 3, ...$, a unit map $\eta : \kappa \to \mathfrak{C}(1)$ and product maps

$$\gamma : \mathfrak{C}(k) \otimes \mathfrak{C}(j_1) \otimes \mathfrak{C}(j_2) \otimes \cdots \otimes \mathfrak{C}(j_k) \to \mathfrak{C}(\sum_{i=1}^{k} j_i)$$

for $k \geq 1$ and $j_i \geq 0$. Satisfying a bunch of axioms. The idea is that the elements of $\mathfrak{C}(k)$ will paramatrise $k-$ary operations on objects of the category so that an algebra over an operad is an object A together with maps

$$\theta : \mathfrak{C}(j) \otimes A^j \to A$$

that satisfy a bunch of axioms similar to those of γ.

There are also representations of the symmetric group to keep track of which input goes where.

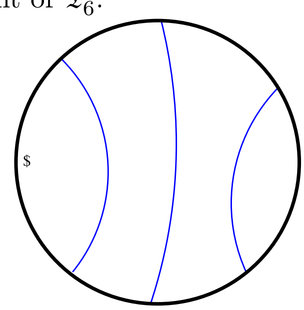
Since we have an explicit operad-like object the detailed axioms of an operad need not concern us, but it is of considerable interest to investigate the meaning of operadic notions in our context. First we describe how the ingredients of May's definition line up with planar algbebras.

The planar tangles of course correspond to the elements of an operad. To get the category stuff right we could easily linearise and consider linear combinations of tangles with identical boundary disc structure. The underlying category would then be vector spaces under tensor product with the unit object being the field itself. The map γ in May's definition corresponds to the gluing operation on tangles. In the definition we have given of an operad all the internal discs would be glued at once but May points out that one can also use individual $\circ_i$ operations to define an operad. The main thing preventing the planar tangles from being an operad on the nose is the fact that not any tangle can be glued into any other. This is rather extreme in our definition of tangles as subsets of the plane but could be allevitated a little by considering tangles up to isotopy. Even so one could only glue one tangle into another if the numbers of intersections of the boundaries with the strings line up and the marked intervals have the same shading. So we have what should be (and no doubt is) called an example of a "partial operad". It is now clear how the definition of an algebra over an operad corresponds to our definition of planar algebra. The map θ in operad theory is nothing but the partition function Z_T (once we have linearised the multilinear maps to the tensor product).

So what do the various bits and pieces of operad theory correspond to? The identity κ in May's definition would be a linear map from the ground field to $\mathfrak{C}(1)$. But $\mathfrak{C}(1)$ corresponds to linear tangles (one input disc) which we will treat later but we already have a lot to say about discs with no input discs which correspond to elements of $\mathfrak{C}(0)$. So let us pass to the next notion which is that of a *unital* operad. Here May makes the assumption that $\mathfrak{C}(0) = \kappa$. For our planar operad nothing like this can be true.

Definition 1.3.1. *Let $\check{\mathfrak{T}}^0_n$ be the set of all planar n-tangles T (unoriented, oriented, shaded) with $\mathfrak{D}_T = \emptyset$, and $\check{\mathfrak{T}}^0 = \bigcup_n \check{\mathfrak{T}}^0_n$.*

Here is a picture of an element of $\mathfrak{T}^0_6$:



The set $\check{\mathfrak{T}}^0$ has a lot of structure.

Just what might correpsond to May's identity axiom is unclear but we would surely be unwise to try to eliminate the richness of these input-free tangles.

Looking at the role of the identity for algebras over operads, the first thing we encounter in [] is that of a *unital* algebra over an operad. This involves extending the action of the operad to $\mathfrak{C}(0)$. The only thing that makes sense for a unital operad is to suppose that there is a map from $\kappa = \mathfrak{C}_0$ to the algebra A satisfying the obvious axioms extending those of θ. In particular if κ is a field and algebras over the unital operad $\mathfrak{A}ss$ whose algebras are precisely the associative algebras, unital algebras over the operad are unital associative algebras in the usual sense. For planar tangles the unital structure has been enriched by all TL diagrams so we see that the notion of a unital planar algebra will be correspondingly enriched.

Definition 1.3.2. *We say the planar algebra P is* unital *if for each $S \in \check{\mathfrak{T}}_n^0$ there is an element $Z(S) \in P_partial(D^T)$ such that*
(i) If θ is an orientation preserving diffeomorphism of $\mathbb{R}^2$ then

$$Z(\theta(S)) = Z(S)$$

(ii) (naturality)

$$Z_T \circ S = Z_T \circ Z_S$$

where $Z_T \circ Z_S(f)$ is defined to be $Z_T(\tilde{f})$ with

$$\tilde{f}(D) = \begin{cases} f(D) & \text{if } D \neq D^S \\ Z(S) & \text{if } D = D^S \end{cases}$$

Thus in a unital planar algebra the isotopy class of every such picture defines an element of the planar algebra.

Definition 1.3.3. *Let $\mathfrak{T}_n^0$ be the set of all isotopy classes of planar n-tangles T (unoriented, oriented, shaded) with $\mathfrak{D}_T = \emptyset$, and $\mathfrak{T}^0 = \bigcup_n \mathfrak{T}_n^0$.*

The set of all such diagrams is infinite because of the presence of an arbitrary number of closed strings. But there are exactly $\frac{1}{n+1}\binom{2n}{n}$ connected such diagrams in $\mathfrak{T}_{2n}^0$ (and none in $\mathfrak{T}_{2n+1}^0$). So if we want P_0 to be as close as possible to a unital algebra over the operad of planar $0-tangles$ we would require that $\{Z(S)|S \in \mathfrak{T}_0^0\}$ be all linearly dependent.

Definition 1.3.4.
(i) The connected elements of T_{2n}^0 will be called the Temperley-Lieb diagrams or TL diagrams for short. Their images in a unital planar algebra will be called the TL elements.
(ii) O will denote the unique connected element of $\mathfrak{T}_0^0$ with one string.
(iii) Ω will denote the unique element of $\mathfrak{T}_0^0$ with a single closed string.

We will often leave out the output disc for a 0-tangle.

Proposition 1.3.5. *Let P be a planar algebra and suppose that $Z(\Omega) = \delta Z(O)$ for some scalar δ. Then all the $\{Z(S)|S \in \mathfrak{T}_0^0\}$ are linearly dependent.*

Proof. This follows immediately by using naturality to remove the closed strings of S one at a time. □

Remark 1.3.6. The proof actually shows that if a planar tangle T contains k closed strings which are contractible in $D^T \setminus \bigcup_{D \in \mathfrak{D}_T} D$, Z_T is the same as $\delta^k Z_{\check{T}}$ where $\check{T}$ is T from which those k closed strings have been removed.

Definition 1.3.7. *A planar algebra satisfying $Z(\Omega) = \delta Z(O)$ for some scalar δ will be called a* reduced *(temporary terminology) planar algebra with (loop) parameter δ*

Remark 1.3.8. Note that reduced oriented planar algebras will require two δ's, one for each orientation of the closed string in Ω and reduced shaded planar algebras will require two δ's according to the shading (δ_+ for a closed string enclosing a shaded region and δ_- for the other shading). However we have the following:

Lemma 1.3.9. *If P is a shaded reduced planar with non-zero loop parameters δ_+^0 and δ_-^0 we may alter the action of planar tangles on P by scalars (multiplicatively) to obtain a new planar algebra with $\delta_+ = \delta_- = \sqrt{\delta_+^0 \delta_-^0}$.*

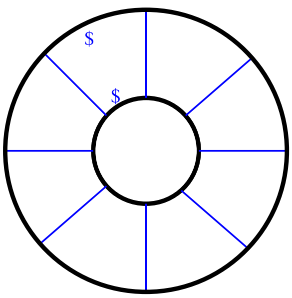
Proof. Define a function $\nu(T)$ on shaded planar tangles as follows. Construct a (not necessarily connected) TL tangle from T by "smoothing" all the internal discs, that is smoothly joining the string meeting boundary point j to the string meeting boundary point $2n-j+1$ for $j = 1, 2, \cdots n$. Orient the strings so that shaded regions are always on the left. Let $k+$ be the number of closed positively oriented strings and $k-$ be the number of closed negatively oriented strings. Now throw away "through" strings that connect the first n boundary points (of D^T) to the last n boundary points. Form closed loops with the remaining strings and the part of the boundary of D^T joining their ends. Let ℓ_+ be the number of positively oriented such loops and ℓ_- be the number of negatively oriented ones.Then set $\nu(T) = 2(k_+ - k_-) + \ell_+ - \ell_-$. $\nu(T)$ is obviously an isotopy invariant of T. I further claim that $\nu(T \circ S) = \nu(T) + \nu(S)$. This is readily seen by isotoping T to the "boxes" form of 1.2.10. For then $\nu(T)$ is nothing but $\frac{1}{2\pi} \int_{\text{strings of T}} d\theta$ where $d\theta$ is the change of angle form. This is manifestly additive under gluing. With these properties it is clear that renormalising Z by $Z_T^r = r^{\nu(T)} Z_T$ (for any invertible r in the ground field) defines a planar algebra structure with the same vector spaces as P, which is reduced if P is. The effect on the loop parameters is to change δ_+ to $r\delta_+$ and δ_- to $r^{-1}\delta_-$. Choosing $r = \sqrt{\frac{\delta_-}{\delta_+}}$ gives the conclusion. □

Thus each P_n in a unital planar algebra will contain a quotient of the vector space of linear combinations of TL diagrams. This quotient can be strict - consider the trivial planar algebra or for a (much) more interesting example the spin model planar algebra of 2.9 when $n = 2$ and $n = 3$. The dimension growth of these algebras is as

$2^{n/2}$ and $3^{n/2}$ respectively whereas the growth of the Catalan numbers is something like 2^n so there are linear dependences between the various Temperley-Lieb diagrams. These are very interesting relations.

1.4 More operadic considerations.

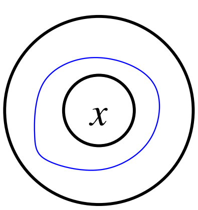
(i) May's definition in [] requires an operad to have an "identity". This is a map from κ to $\mathfrak{C}(1)$ for which the image ι of 1 acts by the identity on operad elements. In the axioms for an algebra over an operad ι is also required to act by the identity. There is a very natural analogue of ι in the planar operad and that is the element:

$\iota_n =$ 

In a planar algebra there is no particular reason why these elements should act by the identity. For instance in zero planar algebras it does not. On the other hand we can take the subspaces $\iota_n(P_n)$ and observed that they form a planar algebra on which ι_n is the identity. Hence the following.

Definition 1.4.1. *A planar algebra P will be called* nondegenerate *if Z_{ι_n} is the identity map for all $n \geq 0$.*

(ii) If one considers planar tanges with only closed strings, i.e. no disc has boundary points, one is very close to an operad, on the nose. If, instead of our concrete tangles where the input discs label themselves, we choose isotopy classes of tangles with labelled internal discs, and define gluing in the obvious way, we obtain a non-Σ operad in the sense of []. We have seen that a unital algebra over this operad is a commutative associative unital algebra A. The extral structure imposed by the closed loops is a linear map $L : A \to A$ defined by the formula below.

$L(x) =$ 

Closed contractible loops may be removed provided we multiply by $L(1)$. A and L completely define the action of planar tangles and conversely any such A and L

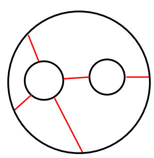
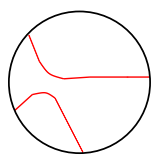
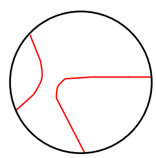
can be used to construct an algebra over this operad. It is not entirely clear that algebra over this operad can be extended to a planar algebra.

Note the subtle difference here between the oriented and shaded versions of this structure. The shaded version will have two algebras A_+ and A_- for the two shadings and L will be a map between them (the operad will still be partial), whereas in the oriented case there is one algebra A but two maps L according to the orientations on the string in the above figure.

(iii) One of the uses of the unital structure in [] is to provide "augmentations". Given and element of $\mathfrak{C}_j$, and the identification of $\mathfrak{C}_0$ with κ, the structural map γ for an operad gives a map from each $\mathfrak{C}(j)$ ($\cong \mathfrak{C}(j) \otimes \mathfrak{C}_0 \otimes \mathfrak{C}_0 \otimes \cdots \otimes \mathfrak{C}_0$) to κ.

It is not so clear how one should augment planar tangles. Any input discs with no boundary points can be augmented as for operads but what should one do with a disc with lots of boundary points. I propose the following definition:

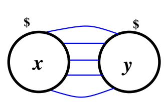
Definition 1.4.2. *If T is a planar n-tangle define the augmentation $\epsilon(T)$ to be the linear combination of constant n-tangles obtained by summing over all ways of inserting Temperley-Lieb diagrams into the internal discs of T.*

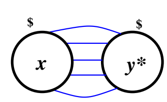
Thus for instance $\epsilon($  $) =$  $+$  .

1.5 Measured planar algebras.

Definition 1.5.1. *A planar algebra P with boundary data $\mathfrak{B}^P$ will be called* measured *if there is a non-zero linear function $\mu : P_\sigma \to \mathbb{F}$ (called the measure) for each $\alpha \in \mathfrak{B}_0^P$, which is compatible with the gluing in the obvious way.*

Definition 1.5.2. *If P is a measured planar algebra (resp. *-planar algebra) we define the* canonical bilinear form $(,)$ *(resp.* the inner product $\langle , \rangle$*) on each P_n to be:*

$(x, y) = \mu($ 

$)$ *resp.* $\langle x, y \rangle = \mu($ 

$)$.

We would have obtained different bilinear and sesquilinear forms by different placement of the \$'s above. The next condition eliminates that possibility.

Definition 1.5.3. *A measured planar algebra is called* spherical *if the multilinear function $\mu \circ Z_T$ defined for every T with no strings connected to D^T depends only on the isotopy class of T on the 2-sphere compactification of $\mathbb{R}^2$.*

Definition 1.5.4. *A measured planar *-algebra (over $\mathbb{R}$ or $\mathbb{C}$) will be called* positive definite *if the inner product above is positive definite.*

Proposition 1.5.5. *A positive definite measured planar algebra (or a measured planar algebra with non-degenerate canonical bilinear form) is nondegenerate.*

Proof. An element in the kernel of ι_n is necessarily orthogonal to everything for $(,)$ and $\langle , \rangle$. □

A planar algebra may possess a canonical measure.

Definition 1.5.6. *A planar algebra P will be called a* central *planar algebra if* $\dim P_\alpha = 1$ *for each* $\alpha \in \mathfrak{B}_0$.

Proposition 1.5.7. *A unital central planar algebra is a measured planar algebra in a unique way.*

Proof. There is a unique way to identify labelled 0-tangles with the scalars compatible with the gluing. □

1.6 Summary

There have been an unfortunately large number of adjectives to be applied to the term planar algebra. For the convenience of the reader we list them all here.

(1)*Vanilla*	1.2.5
(2)*Oriented*	1.2.7
(3)*Shaded*	1.2.6
(4)*Star*	1.2.11
(5)*Unital*	1.3.2
(6)*Reduced*	1.3.7
(7)*Measured*	1.5
(8)*Nondegenerate*	1.4.1
(9)*Central*	1.5.6
(10)*Positive definite*	1.5.4
(11)*Spherical*	1.5.3
(12)*Finite dimensional*	if P_α is finite dimensional for every α.

Putting most of these together we get the kinds of planar algebras we are most interested in:

Definition 1.6.1. *A* positive planar algebra *is a positive definite unital finite dimensional planar *-algebra. A* subfactor *planar algebra is a central spherical positive shaded planar algebra. A* correpsondence *planar algebra is a positive oriented planar algebra.*

If we drop the sphericality condition from a subfactor planar algebra we will refer to a "non-spherical" planar algebra. Observe that loop parameters are positive in positive planar algebras.

2 Examples

2.1 The trivial examples.

(i) The zero planar algebras.
If one chooses P_n to be arbitrary vector spaces and one sets all the maps Z_T to be zero one obtains a derisory planar algebra.

Perhaps the only thing to say about them is that they can obviously be made unital and any unital planar algebra for which $Z(\Omega) = 0$ is a zero planar algebra.
(ii) The trivial planar algebra.
If F is the ground field and we set $P_n = F$ for all n, and Z_T to be the product map then we get a planar algebra. It is furthermore untial and reduced if we define the images of all the TL tangles to be $1 \in F$, and the loop parameter δ is equal to 1. The oriented and shaded versions are obvious.

This planar algebra is of little interest though it will furnish us with a subfactor planar algebra-1.6.1.

2.2 Tensors-$P^{\otimes}$

We will give unoriented, oriented and shaded versions.

(i) The unoriented case.

Suppose we have a finite dimensional vector space V with a basis $v_1, v_2, ..., v_k$. Then elements of the tensor powers $\otimes^n V$ can be concretely represented by arrays of numbers $R_{i_1,i_2,...,i_n}$ which are the coefficients of the elementary basis tensors $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$.

In order to define a planar algebra we need to give vector spaces P_n and the action of planar tangles. For the tensor planar algebra, $P_n^{\otimes}$ will be $\otimes^n V$.

To define the multilinear map of a planar n-tangle T we may suppose that a tensor has been assigned to every $D \in \mathfrak{D}_T$. Then we have to create an element of $\otimes^n V$. This means assigning a number $R_{i_1,i_2,\cdots,i_n}$ to every n-tuple of integers between 1 and k. To do this, we begin to define a function from the strings of T to $\{1, 2, \cdots, k\}$ by assigning the indices $i_1, i_2, ..., i_n$ to the boundary points of D^T.

If the ath. and bth. boundary points are connected by a string of T and $i_a \neq i_b$ then we set $R_{i_1,i_2,\cdots,i_n} = 0$

So we can suppose the assignment of indices can be extended from the points on the boundary disc to the strings meeting those points. Call a "state" σ of T any extension of this function to *all* the strings of T. Then each σ assigns, for each $D \in \mathfrak{D}_T$, indices to the n_D boundary points of D. Thus for each such disc there is a number R^D_σ given by the tensor that is allotted to D.

We now define

$$R_{i_1,i_2,\cdots,i_n} = \sum_{\sigma} \prod_{D \in \mathfrak{D}_T} R^D_\sigma$$

This R obviously depends multilinearly on the tensors assigned to each $D \in \mathfrak{D}$ and it is a simple matter to check the gluing axiom. Diffeomorphism invariance is obvious. Thus we have a planar algebra $P^{\otimes}$.

$P^{\otimes}$ becomes a planar *-algebra under the operation of complex conjugation and reversing the order of the indices of tensors. It is also a central planar algebra and the canonical sesquilinear form is positive definite.

One might wonder why we are only allowing planar systems of contractions for tensors. It was Penrose ([]) who invented a diagrammatic notation for tensor contractions which allowed for arbitrary pairings of the indices. We contend that the planar restriction is significant as there are important examples of sub planar algebras of $P^{\otimes}$ that are not closed under all contraction systems. Also just the problem of determining the dimensions of a sub planar algebra of $P^{\otimes}$ (given generators of it) is undecidable whereas if one allows arbitrary contractions it is probably algorithmically possible (there is a closely related family of planar algebras where the same problem is algorithmically decidable).

Observe that this planar algebra can immediately be extended to a unital reduced one by using the convention that an empty product is equal to 1. Note also that the loop parameter of this planar algebra is k, the dimension of the auxiliary vector space V. This is because if we are given a closed string then it is not connected to the outside boundary so we must sum over the k possible index values for that string, all other index values being held fixed.

We see immediately a shortcoming of the unoriented tangles-the only obvious symmetry group of the algebra is the permutation group of the basis vectors, and the idea of covariant and contravariant indices is absent.

(ii) The oriented version.

Again V is a finite dimensional vector space of dimension k. In order to give an oriented planar algebra we must assign a vector space to every $\alpha \in \mathfrak{B}_n$ for every n. That is simple enough:

$$P_\alpha = \bigotimes_{i=1}^{n} V^{\alpha(i)}$$

where $V^{\uparrow}$ is V and $V^{\downarrow} =$ the dual $V^{\dagger}$ of V.

A multilinear map from $\underset{D \in \mathfrak{D}_T}{\times} P_{\partial(D)}$ is the same thing as a vector in $P_{\partial(D^T)} \otimes (\bigotimes_{D \in \mathfrak{D}_T} P_{\partial(D)})^{\dagger}$. This may be written as a tensor product of V's and $V^{\dagger}$'s over the set of all boundary points of discs in T. The tangle gives a pairing between all these boundary points with V always paired with $V^{\dagger}$. So we may rearrange the the tensor product as

$$\bigotimes_{\text{non-closed strings of } T} (V \otimes V^{\dagger}).$$

But there is a canonical element of $V \otimes V^{\dagger}$ so taking the tensor product of it over the non-closed strings of T we get a multilinear map from $\bigtimes_{D \in \mathfrak{D}_T} P_{\partial(D)}$ to $P_{\partial(D_T)}$. Z_T is just this map times k^ℓ, ℓ being the number of closed strings in T.

Note how this definition works for tangles without input discs as well so that this planar algebra is a reduced unital one.

If V is a Hiilbert space then there is a conjugate-linear isomorphsims between V and $V^\dagger$ which allows us to make $\overrightarrow{P}^{\otimes}$ into a planar *-algebra in the obvious way. The resulting $\langle , \rangle$ is positive definite so we get a correspondence planar algebra.

Diffeomorphism invariance and naturality are easy, and if one chose a basis of V and the dual basis for $V^\dagger$ one would obtain explicit formulae just like in the unoriented case.

Remark 2.2.1. *Observe that the group $GL(k)$ acts in a canonical way on $\overrightarrow{P}^{\otimes}$. This means that for every subgroup of $GL(k)$ there is a planar algebra for which P_α is the invariant tensors in the tensor power of V and $V^\dagger$ defined by α.*

(iii) The shaded version. As we have observed in 1.2.8, an oriented planar algebra defines a shaded one.If V is a Hilbert space then we get a subfactor planar algebra. There is a far more interesting way to make tensors into a shaded planar algebra defined below in 2.9

2.3 The Temperley-Lieb planar algebra P^{TL}.

(i) Unoriented version.
The vector space P^{TL}_{2n+1} is zero and P^{TL}_{2n+1} is the vector space of formal linear combinations of connected TL diagrams with $2n$ boundary points. The loop parameter δ may be assigned arbitrarily so there is one TL planar algebra for each δ. The action of planar tangles is obvious, just insert the TL diagrams into the internal discs, lining up the distinguished intervals. Then remove any closed loops that are formed one at a time, each time multiplying by δ. This construction would be hard to miss from the operadic standpoint as $\mathfrak{C}_0$ is always an algebra over the operad $\mathfrak{C}$.

P^{TL} also extends to a unital planar algebra (in the obvious way). Moreover the maps defining the unital structure of any planar algebra endow it with a quotient of P^{TL} as a planar subalgebra.

(ii) Oriented version.

The strings of a connected TL diagram D may be oriented to give a diagram $\overrightarrow{D}$. If $\overrightarrow{D}$ has $2n$ boundary points there is an element $\alpha_{\overrightarrow{D}} \in \mathfrak{B}_{2n}$ given by the orientation of the boundary points. The vector space P^{TL}_α is the set of formal linear combinations of such tangles (and is zero for $\mathfrak{B}_{2n+1}$). Oriented planar tangles act in the obvious way, with closed strings being removed with a multiplicative factor of $\delta_\pm$ according to their orientation. It is clear that this oriented planar algebra is reduced and is unital in the obvious way.

Note that for $\alpha \in \mathfrak{B}_{2n}$, $\dim P_\alpha^{TL}$ (i.e. the number of oriented connect TL diagrams) is no longer simply the Catalan number. It is a complicated function of α for which we will soon give an "explicit" formula.

Proposition 2.3.1. $\dim P_\alpha^{TL} = 0 \iff |\alpha^{-1}(\uparrow)| \neq |\alpha^{-1}(\downarrow)|$.

Proof. The only non-obvious thing to prove is that if $|\alpha^{-1}(\uparrow)| = |\alpha^{-1}(\downarrow)|$ then there is an oriented TL diagram having α as its boundary data. This follows by induction- if not all boundary arrows are the same there must be a pair of consecutive boundary points which have different orientations. These two points can be connected by an oriented edge. The remainder of the diagram can be completed induction. □

Now if $\alpha \in \mathfrak{B}_{2n}$ we define a word on the letters X an Y as follows:
Let

$$f_\alpha(i) = \begin{cases} X & \text{if } i \text{ is odd and } \alpha = \uparrow \text{ or } i \text{ is even and } \alpha(i) = \downarrow \\ Y & \text{if } i \text{ is odd and } \alpha = \downarrow \text{ or } i \text{ is even and } \alpha(i) = \uparrow \end{cases}$$

Now let w_α be the word whose ith. letter is $f_\alpha(i)$.

Recall the Voiculescu trace tr_V of [] on the algebra of non-commutative polynomials in X and Y which, on a monomial, is the number of planar pairings between the letters of the word, where X must be paired with X and Y with Y.

Proposition 2.3.2. $\dim P_\alpha^{TL} = tr_V(w_\alpha)$.

Proof. w_α was designed so that a pairing contributing to the Voiculescu trace is the same thing as an oriented TL diagram. □

3) Shaded version.

Do e_i's []

2.4 Van Kampen diagrams and P^Γ.

Let Γ be a (countable discrete) group with a finite generating set Gen. Let V be the vector space having Gen as a basis. We will construct a planar subalgebra of the $\overrightarrow{P}^\otimes$ built on V. Functions from $\{1, 2, \cdots, n\}$ to Gen give tensors in $\overrightarrow{P}_\alpha^\otimes$ by choosing $g \in Gen$ to be a basis element for V for $\uparrow$ and the dual basis element for $\downarrow$. The space P_α^Γ is the vector subspace of $\overrightarrow{P}_\alpha^\otimes$ spanned by all tensors R_f, f being a function from $\{1, 2, \cdots, n\}$ to Gen such that $\prod_{i=1}^{n} f(i)^{\alpha(i)} = 1$ where $g^\uparrow = g$ and $g^\downarrow = g^{-1}$.

We leave it as an exercise to show that the P_α^Γ form a planar subalgebra of $\overrightarrow{P}^\otimes$. As a planar *-subalgebra of $\overrightarrow{P}^\otimes$, P^Γ is a correspondence planar algebra.[]?? Check * property.

The natural subfactor planar algebra defined by the induced shaded planar algebra consists of all words of even length in the generators such that the product of the letters in the word, with alternating exponents ± 1, is equal to the identity.

2.5 Ice

This planar algebra is implicit in Lieb's ice-type model.[] It is an oriented planar algebra. For $\alpha \in \mathfrak{B}_n$ we let P_α^{Ice} be the vector space whose basis is the set of all functions $\iota : \{1, 2, \cdots, n\} \to \{\pm 1\}$ such that $\sum_{i=1}^n (-1)^{\alpha(i)} \iota(i) = 0$. Clearly $P_n^{Ice} = 0$ for n odd and

$$\dim(P_{2r}^{Ice}) = \binom{2r}{r}.$$

(Here $(-1)^\uparrow = 0, \quad (-1)^\downarrow = 1$.)

For every real number λ we now define a structure of a reduced central unital planar algebra $P^{Ice} (= P^{Ice,\lambda})$ with P_α^{Ice} defined above.

So suppose T is an oriented planar n-tangle with n even and we are given a function ι from the boundary points of D^T to $\{\pm 1\}$, and an element $R(D) \in P_{\partial(D)}^{Ice}$ for every $D \in \mathfrak{D}_T$. As for $P^\otimes$, we have to come up with a number R_ι so that $Z_T(R) = \sum_\iota R_\iota \iota$. Define a state σ of T to be any extension of ι to the strings of T (so there are no states if two boundary points of D^T are connected by a string of T and ι is different on those boundary points). A state induces for each $D \in \mathfrak{D}_T$ a function σ_D from its boundary points to $\{\pm 1\}$ so we can talk about $R(D)_{\sigma_D}$. We then let

$$R_\iota = \sum_\sigma \prod_{D \in \mathfrak{D}_T} R(D)_{\sigma_D} f(\sigma)$$

where $f(\sigma)$ is calculated in a similar way to 1.3.9: first isotope T so that all discs are horizontal rectangles with their distinguished intervals to the left, and all strings meet all rectangles at right angles, half at the top and half at the bottom. Then define

$$f(\sigma) = \lambda^{\int_{\mathfrak{S}(T)} \sigma d\theta}$$

where $d\theta$ is the angle 1-form on $\mathbb{R}^2$ normalised so that the integral over a positively oriented circle is equal to 1.

Note that we do not really use the real numbers in the definition since once the tangle is in its standard form the contribution of each string to the integral is at worst a half integer.

Isotopy invariance of Z_T as defined is not quite obvious because of the factors $f(\sigma)$, indeed the formula would not be isotopy invariant without the condition $\sum_{i=1}^n (-1)^{\alpha(i)} \iota(i) = 0$. But, as explained in []burnsthesis, any two planar isotopies of a tangle into the required form can be supposed to produce the same result, up to rotations of the internal rectangles by 2π. For each state σ it is clear that the rotations do not affect $f(\sigma)$. Thus Z_T is isotopy invariant. The naturality of Z_T follows from the obvious additivity of $f(\sigma)$ under gluing of tangles.

The unital structure on P^{Ice} is clear. Once a TL tangle T with boundary function α) is isotoped so that the outside disc is a rectangle and the strings meet the boundary orthogonally one defines

$$Z(T)_\iota = \sum_\sigma f(\sigma)$$

with σ and ι defined exactly as above.

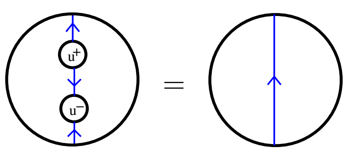
The reduced property for P^{Ice} is obvious with $\delta_+ = \delta_- = \lambda + \lambda^{-1}$.

Note that for $\lambda = 1$ this planar algebra structure is exactly what would be defined by using a basis in the oriented version $\overrightarrow{P}^{\otimes}$ for a two dimensional auxiliary vector space V.

Definition 2.5.1. *Let $\alpha_\pm \in \mathfrak{B}_2$ be defined by $\alpha_+(1) =\uparrow$, $\alpha_+(2) =\downarrow$, $\alpha_1(1) =\downarrow$ and $\alpha_-(2) =\uparrow$. Then define $u^\pm \in P^{Ice}_{\alpha_\pm}$ by*

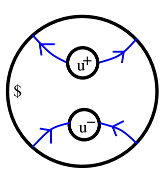
$$u^\pm_{i,j} = \begin{cases} 1 & \text{if } i = -j \\ 0 & \text{otherwise} \end{cases}$$

These elements $u_\pm$ are obviously in P^{Ice} and allow us to change the orientation of a string. Observe the relation:



We can now define Temperley-Lieb like elements in P^{ice}:

Definition 2.5.2. *Let $\mathcal{E} \in P^{Ice}_\alpha$ be the element*



where $\alpha \in \mathfrak{B}_4$ is defined by the picture.

Note that with this choice of α, P^{Ice} is an algebra as in 6.1.1.

Proposition 2.5.3. *For this algebra structure $\mathcal{E}^2 = \delta\mathcal{E}$.*

The reason for insisting on the relation $\sum_{i=1}^n (-1)^{\alpha(i)}\iota(i) = 0$ in the definition of P^{Ice} was to ensure invariance under all planar isotopies. But we could easily define operads based on planar tangles with horizontal rectangles instead of discs and "rigid" planar algebras where we only require invariance of Z under isotopies during which the horizontal rectangles stay horizontal rectangles. Then one could proceed exactly as in the definition of the basis dependent version of $P^{\otimes}$ except that in the definition of Z_T, the contribution of each state would be multiplied by a factor $f(\sigma) = \lambda^{\int_{\mathfrak{S}(T)} \sigma d\theta}$.

Note that the multiplication tangle of 6.1.1 works just as well in rigid planar algebras to give each P_{2n} and algebra structure. Applying this to P^{Ice} we see that each time α is such that P^{Ice}_{α} is an algebra, it is in fact a subalgebra of the $2^n \times 2^n$ matrices. We record here the matrix for $\mathcal{E}$ in the obvious basis:

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 1 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[]Do Kauffman diagrams, Jones braid group rep and polynomial.

Let us agree that for any oriented planar algebra P, P_n will be P_α where $\alpha : \{1, 2 \cdots, 2n\} \to \{\uparrow, \downarrow\}$ is $\alpha(i) = \uparrow$ for $1 \leq i \leq n$ and $\alpha(i) = \downarrow$ for $n+1 \leq i \leq 2n$. Observe that P_n is a unital algebra unitally embedded in P_{n+1} via the appropriately oriented tangles $id \otimes 1$ and $1 \otimes id$ of 5.1.2.

Definition 2.5.4. *For any X in P_2 as above inductively define $X_1 = \mathcal{E}$ and $X_{n+1} = (1 \otimes id)(X)$.*

All the X_n can be considered as elements of the same algebra.

Proposition 2.5.5. *We have*
(i) $\mathcal{E}_n^2 = \delta \mathcal{E}_n$
(ii) $\mathcal{E}_n \mathcal{E}_{n\pm1} \mathcal{E}_n = \mathcal{E}_n$
(iii) $\mathcal{E}_n \mathcal{E}_m = \mathcal{E}_m \mathcal{E}_n \qquad$ *if* $|m - n| > 1$

These are the famous Temperley Lieb relations of [],[],[]. Given any element $X \in P_2^{Ice}$ as above, X_n makes sense using the

Definition 2.5.6. *Let $g = \mathcal{E} - \lambda 1$.*

Lemma 2.5.7. *We have the braid relations*
(i) $g_n g_{n+1} g_n = g_{n+1} g_n g_{n+1}$
(ii) $g_n g_m = g_m g_n \qquad$ *if* $|m - n| > 1$
together with the (Hecke) relation
(iii) $g_n - g_n^{-1} = (\lambda^{-1} - \lambda)1$.

Lemma 2.5.8. *If we define $R(x) = e^x g - e^{-x} g^{-1}$ then*

$$R_n(x) R_{n+1}(x+y) R_n(y) = R_{n+1}(y) R_n(x+y) R_{n+1}(x)$$

Proof. This is an exercise, using only the relations of the previous lemma. There are 8 terms on each side of the relation. 6 of these are equal using just the braid relations. Using (iii) and the braid relation, the others reduce to $g_n^2 - g_n^{-2}$ on one side and $g_{n+1}^2 - g_{n+1}^{-2}$ on the other. But squaring (iii) shows that both of these are the same multiple of the identity. $\square$

Let us now write out the matrices for g and $R(\theta)$ explicitly. From the definition of g and the matrix for $\mathcal{E}$ we have

$$g = \begin{pmatrix} -\lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} - \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} \text{ and } g^{-1} = \begin{pmatrix} -\lambda^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & -\lambda^{-1} \end{pmatrix}$$

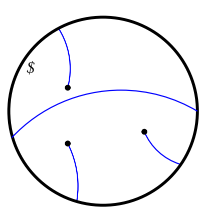
From which we get immediately up to a global factor of 2, with $e^{-\phi} = \lambda$,

$$R(\theta) = \begin{pmatrix} \sinh(\phi - \theta) & 0 & 0 & 0 \\ 0 & e^{\theta} \sinh\phi & \sinh\theta & 0 \\ 0 & \sinh\theta & e^{\theta}\sinh\phi & 0 \\ 0 & 0 & 0 & \sinh(\phi - \theta) \end{pmatrix}$$

In section 8 we will see that the entries of $R(\theta)$ supply the Boltzmann weights for a statistical mechanical model called the "Ice-type" model. We see that provided $\phi \geq \theta \geq 0$ these Boltzmann weights are positive and so make physical sense.

2.6 The Motzkin planar algebra

We describe only the unoriented version, the modifications necessary for the other versions are now obvious. By definition P_n^{Motz} is the vector space spanned by isotopy classes of connected planar n-tangles with no closed strings and and all input discs having exactly one boundary point. By connectedness any input disc must be connected to the boundary disc by a string so P_n^{Motz} is finite dimensional. Here is a picture of an element in P_5^{Motz} where we have shrunk the input discs down to dots:



We will call such a tangle a "Motzkin diagram". Counting the Motzkin diagrams is similar to counting TL diagrams. Note that the 1-discs and their strings could be shrunk to the boundary points and one obtains the standard objects counted by the Motzkin numbers-see []. The first few Motzkin numbers (and therefore the dimensions of the P_n^{Motz}) are

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835.$$

If we write a_n for the nth. Motzkin number (with $a_0 = 1$) then it is obvious that

$$a_{n+2} = a_{n+1} + \sum_{j=0}^{n} a_j a_{n-j}$$

so that the generating function $\sum_{n=0}^{\infty} a_n z^n$ satisfies

$$z^2 A^2 + (z - 1)A + 1 = 0.$$

Solving the quadratic gives explicit expressions for a_n as sums of products of binomial coefficients.

The planar algebra structure on the Motzkin algebra is defined exactly as for TL. Besides closed strings one inevitably encouters strings ended by dots. These are handled like closed strings by removal with another multiplicative constant. But in fact we may as well assume that this constant is 1, by multiplying each basis element by a constant depending on its number of dots.

Exercise: In the oriented version of Motzkin, interpret the dimensions of the P_α as a Voiculescu-type trace.

2.7 Knots and links.

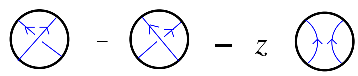
The planar algebra we are about to define was implicitly present in Conway's paper []. We do the oriented and unoriented definitions together.

For each even n let P_n^{Conway} (resp. $\overrightarrow{P}_\alpha^{Conway}$) be the vector space of formal linear combinations of (3-dimensional) isotopy classes of link diagrams (resp. oriented link diagrams) with the $2n$th. roots of unity as boundary points and the interval on the unit circle preceding 1 in clockwise order as the distinguished boundary interval. By three dimensional isotopy class we mean that two link diagrams are identified if they can be obtained one from another by the three Reidemeister moves and planar isotopy.

The action of planar tangles on the vector spaces P_n^{Conway} is just as in TL, without removal of closed strings, by gluing in tangles using an appropriate orientation preserving diffeomorphism of the unit disc to the relevant disc in the planar tangle. []MAKE THIS MORE EXPLICIT FOR TL. The unital structure is obvious.

Proposition 2.7.1. *The algebra P_0^{Conway} (resp. $\overrightarrow{P}_0^{Conway}$) is the polynomial algebra with one generator for each non-split link (resp. oriented link) in $\mathbb{R}^3$.*

Note that this planar algebra is non-degenerate but not reduced. Each P_n is infinite dimensional even as a module over P_0. Conway's "linear skein theory" for $\overrightarrow{P}_\alpha^{Conway}$ was to take the quotient of this planar algebra by the ideal generated by the element:



(where z is either an indeterminate or a fixed element of the field according to context).

Conway showed that the quotient $\overrightarrow{P}^{Alexander}$ of this planar algebra satisfies $\dim \overrightarrow{P}_0^{Alexander} = 2$ and that if the oriented link L is considered as an element of $\overrightarrow{P}_0^{Alexander}$ then it is equal to $\Delta_L(\sqrt{t} - \dfrac{1}{\sqrt{t}})$ times the the tangle O (with either orientation) where Δ_L is the Alexander polynomial of L and $z = \sqrt{t} - \frac{1}{\sqrt{t}}$.

It was observed in [] that the Jones polynomial can be defined by changing the coefficients slightly in the Alexander polynomial skein relation above and in [],[] it was shown that arbitrary coefficents may be used to obtain what is now called the HOMFLYPT polynomial. it is worth observing that the Alexander skein relation implies that the Alexander polynomial of a split link is zero so that although $\overrightarrow{P}^{Alexander}$ is not strictly speaking reduced it does have the property that it is almost so, with loop parameter zero, since $O^2 = 0$ so that any closed string may be removed and the tangle multiplied by zero, provided there is something else to the tangle. For the Jones and many other such invariants one my further quotient by a relation to make the planar algebra reduced.

Kauffman observed in [] that if one leaves out the first Reidemeister move one obtains a theory which works also in the unoriented case, obtaining a version of the Jones polynomial called the Kauffman bracket ([]) and a two variable polynomial invariant of oriented links called the Kauffman polynomial.

2.8 The BMW algebra

2.9 Spin models

Spin models only exist for shaded planar algebras.

As for vertex models we take an auxiliary Q-dimensional vector space V with basis $S = \{s\}$. The vector spaces for P^{spin} are:
$P_{0,+}$= the ground field, $P_{0,-} = V$ and $P_{n,\pm} = \otimes^n V$ (recall that for shaded planar algebras "n'' means half the number of boundary points for a disc). The action of the operad is defined as follows:
Observe first the the shaded intervals of an n-disc can be numbered $1, 2, \cdots, n$ so that we can identity basis elements of $\otimes^n V$ and functions ψ from the shaded boundary intervals of an n-disc to S. As for vertex models, given a shaded planar tangle n-

tangle T and a function f from $\mathfrak{D}_T$ to the appropriate tensor powers of V, we will give the coefficients $R_{s_1,s_2,\cdots,s_n}$ of $Z_T(f)$ in this basis of functions.

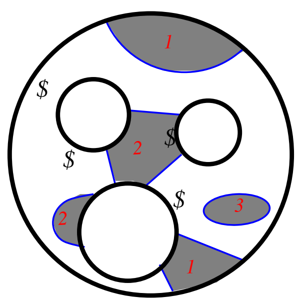
If two shaded intervals i and j of D^T are part of the boundary of some shaded region of T, and $s_i \neq s_j$, put $R_{s_1,s_2,\cdots,s_n} = 0$.

Otherwise we may extend the function $i \mapsto s_i$ from the shaded intervals of D^T to all the shaded regions meeting the shaded intervals of D^T. Call a "state" σ of T any extension of this function to *all* the shaded regions of T. Then each σ assigns, for each $D \in \mathfrak{D}$, indices to the shaded boundary intervals of D. Thus for each such disc there is a number R^D_σ given by the tensor that f assigns to D.

We now define

$$R_{s_1,s_2,\cdots,s_n} = \sum_\sigma \prod_{D \in \mathfrak{D}_T} R^D_\sigma$$

This R obviously depends multilinearly on the tensors assigned to each $D \in \mathfrak{D}$ and it is a simple matter to check the gluing axiom. Diffeomorphism invariance is obvious. Thus we have a planar algebra P^{spin}. Below is a picture of a state on a shaded planar tangle T where $S = \{1, 2, 3\}$.



We see that P^{spin} is *reduced* with loop parameters $\delta_+ = Q$ (closed string around a shaded region) and $\delta_- = 1$ (closed string around an unshaded region).

Remark 2.9.1. Sometimes it is advantageous to change the values of δ so that P^{spin} becomes spherical. This is possible by 1.3.9. We will call the resulting plana algebra P^{Spin}. The only difference between P^{spin} and P^{Spin} is in the action of the tangles which differ by the combinatorial multiplicative factor of 1.3.9

The spin planar algebra is *not* a central planar algebra. It is however a measured planar algebra with μ assigning 1/3 to each of the minimal projections in $P^{spin}_{0,-}$

Also P^{spin} is clearly acted upon by any group of permutations of the set S of spins. If this action is transitive the fixed points are a central planar algebra.

Definition 2.9.2. *If G acts transitively on S as above with point stabiliser H, we call $P^{G,H}$ the planar algebra of fixed points for the action on P^{spin}. The special case when G is finite and $|H| = 1$ will be called the group planar algebra P^G.*

This gives interesting examples. It was shown by Izumi ([]) that under favourable circumstances, for instance if the action is primitive, that G and H can be recovered from $P^{G,H}$.

A central planar *-subalgebra of P^{spin} (such as those coming from transitive group actions) defines an association scheme. To see this in detail we will use the following:

Proposition 2.9.3. *If P is a central planar subalgebra of P^{spin} then $dim P_{1,+} = 1 = dim P_{1,-}$.*

Proof. There is a unique connected annular tangle which maps $P^{spin}_{1,+}$ to $P_{0,-}$ which is the identity when both these spaces are identified with V. So if $P_{0,-}$ is one dimensional, so is $P_{1,+}$. □

Exercise 2.9.4. *If P is a central planar subalgebra of P^{spin} then the identity of $P_{1,+}$ is a minimal projection for comultiplication.*

Now to see how to get an association scheme, observe that $P_{2,-}$ is an abelian C*-algebra which is thus spanned by its minimal projections. Each such projection corresponds to a subset of $\{1, 2, \cdots, Q\}$. From the above exercise the identity of $P_{2,+}$ is such a minimal projection. This, and the closure of $P_{2,+}$ under multiplication, comultiplication and * are precisely the conditions of an association scheme ([]). The algebra $P_{2,+}$ is called the Bose-Mesner algebra of the association scheme. It would be interesting to find obstructions that prevent an association scheme from coming thus from a spin model planar algebra.

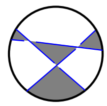
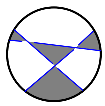
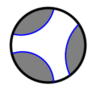
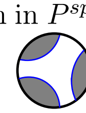
Exercise 2.9.5. *Show that if $P_{i,j}$ is a minimal projection in $P_{2,-}$ then $|\{j : P_{i,j} = 1\}|$ is independent of j.*

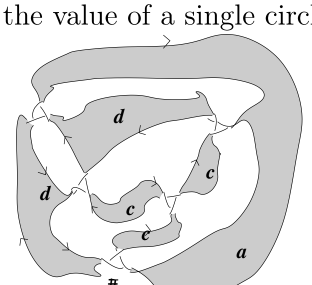
This fact is true for an association scheme. Note that it implies $dim P_{2,+} \leq Q$.

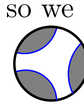
Given a group action as above one may consider another planar algebra which is the one generated by the association scheme (i.e. generated by $P_{2,+}$). ln general this is different from the fixed points under G. A case where they are the same is for the dihedral group on a set with five elements (see []). They are different for Jaeger's Higman-Sims model ([],[]) — although the dimensions of the two planar algebras agree for a while, they have different asymptotic growth rates, one being that of the commutant of Sp(4) on $(\mathbb{C}^4)^{\otimes k}$ and the other being 100^k.

Here is an interesting example for a doubly transitive group. It connects with Example [] and gives a new kind of "spin model" for link invariants from links projected with only triple point singularities.

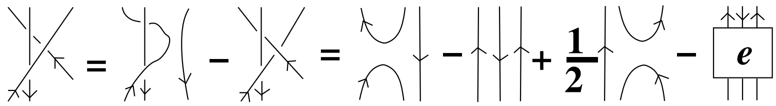
The alternating group $G = A_4$ is doubly transitive on the set $\{1, 2, 3, 4\}$ with point stabiliser H but there are two orbits on the set of ordered triples (a, b, c) of distinct elements according to whether $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$, $4 \mapsto d$ (with $\{a, b, c, d\} = \{1, 2, 3, 4\}$) is an even or odd permutation. Let $e \in {P^{G,H}}_{3,+}$ be the

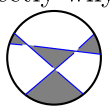
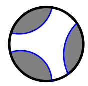
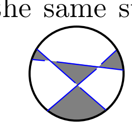
characteristic function of the even orbit. Define a mapping from the free shaded planar algebra on the generator  (the position of the \$ is immaterial) to $P^{G,H}$ by sending  to $e - \frac{1}{2}$  . To prove that this map passes to the quotient P^H (the planar algebra of []) with parameters $t = i = x($ 1,–1 in $\ell - m$ variables) it would suffice to show that twice the value of the HOMFLY polynomial of the link obtained from the free planar algebra above is the partition function in P^{spin} (with $Q = 4$) given by filling the discs in the free planar algebra with $e - \frac{1}{2}$  .
We give a sample calculation below which illustrates all the considerations. Note that, for $t = i = x$, the value of a single circle in the HOMFLY skein is 2.



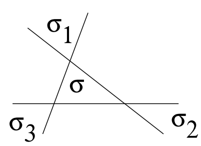
Smoothing all the 3-boxes leads to a single negatively oriented circle so we must divide the final partition function by 2. Replacing the 3-boxes by $e - \frac{1}{2}$  we look for spin states, i.e. functions from the shaded regions to $\{1, 2, 3, 4\}$ for which each 3-box yields a non-zero contribution to the partition function. Around each 3-box this means that either the three spin values are in the even orbit under A_4, or they are all the same. The first case contributes $+1$ to the product over boxes, the second case contributes –1 (**not** $-\frac{1}{2}$ because of the maxima and minima in the box). If the box labeled (†) is surrounded by the same spin value, all the spin states must be the same for a nonzero contribution to Z. This gives a factor $4 \times (-1)^5$. On the other hand, if the spins at (†) are as in Figure 2.8.3 with (a, b, c) in the even orbit, the other spin choices are forced (where $\{a, b, c, d\} = \{1, 2, 3, 4\}$), for a contribution of –1. The orbit is of size 12 so the partition function is $\frac{1}{2}(-12 - 4) = -8$. For this link the value of the HOMFLY polynomial $P_L(1, -1)$ is –4. The factor of 2 is accounted for by the fact that our partition function is 2 on the unknot. Thus our answer is correct. Note how few spin patterns actually contributed to Z!

If we wanted to use non-alternating 3-boxes we could simply use the HOMFLY skein relation to modify the 3-box. For instance



In general by [LM], $P_L(1,-1)$ is $(-1)^{c-1}(-2)^{\frac{1}{2}d}$ where c is the number of components of L and d is the dimension of the first homology group (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) of the triple branched cover of S^3, branched over L. It would be reassuring to be able to see directly why our formula gives this value. This would also prove directly that the map  $\mapsto e - \frac{1}{2}$  passes to the HOMFLY quotient. Our derivation of this is a little indirect — one may show that the planar subalgebras [] and $P^{G,H}$ are the same by showing they arise as centralizer towers from the same subfactor (constructed in []). Thus there must be a 3-box corresponding to:  and we obtained the explicit expression for it by solving an obvious set of equations.

As far as we know, this is the first genuine "3-spin interaction" statistical mechanical model for a link invariant. Of course one may produce 3-spin interaction models by taking a 2-spin one and summing over the internal spin σ in the picture

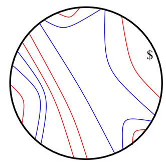


but that is of little interest. One may check quite easily that the above model does not factorize in this way.

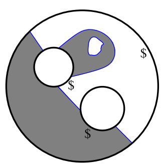
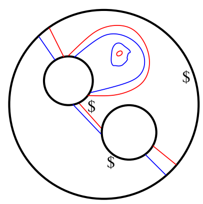
2.10 Fuss Catalan.

This planar algebra was discovered by Bisch and the author in their explorations of intermediate subfactors. We will first give the original definition as a shaded planar algebra then show it can be extended to a coloured planar algebra (with three colours).

Definition 2.10.1. *A* (positive) Fuss Catalan basis tangle *will be the planar isotopy class of a planar 4n TL tangle whose boundary points are labelled by a and b, in clockwise order starting from the first one after \$ in the sequence abbaabbaa....bba in such a way that strings only connect boundary points with the same label (so that the labelling extends to the strings themselves). A negative basis tangle is the same except that the \$ lies between two b's.*

A positive Fuss-Catalan basis tangle  (red=a, blue=b)

To define the operad action, take a shaded tangle T and add to each string a red one which is a close parallel in the unshaded region to obtain $FC(T)$ thus:

$T =$  $FC(T) =$ 

The action of T is now clear - take appropriately isotoped Fuss Catalan basis tangles and glue them into the input discs of $FC(T)$. Any closed strings are removed counting a multiplicative factor of δ_a for an a loop and δ_b for a b string.

Thus we obtain a shaded planar algebra P^{FC} which is unital nondegenerate, reduced, central with loop parameter $\delta_a\delta_b$, spherical and may be given *-structure in the obvious way. The dimension of $P^{FC}_{n,\pm}$ is the second Fuss-Catalan number $\frac{1}{2n+1}\binom{3n}{n}$.

It is a subfactor planar algebra for $\delta_a, \delta_b \geq 2$. This follows from []. If $\delta_a = 2\cos\pi/m$ or ≥ 2 and $\delta_b = 2\cos\pi/n$ or ≥ 2 the kernel of the canonical inner product 1.5.2 is an ideal and the quotient is a subfactor planar algebra. A subfactor $N \subseteq M$ has P^{FC} as a sub planar algebra of its canonical planar algebra 9.0.1 iff it has an intermediate subfactor $N \subseteq P \subseteq M$. The shadings of a shaded planar algebra are naturally by N and M so in this case it is natural to consider the coloured planar algebra over tangles whose regions are coloured N, P and M, the restrictions on the colouring (corresponding to the shading conditions) being that N and M can only be adjacent to P and P cannot be adjacent to itself.

Exercise 2.10.2. *There is a bijection between Fuss-Catalan basis tangles and connected planar tangles with no input discs, whose boundary interval colouring pattern is $NPMPMPNP....MP$, coloured by N, P and M with the above adjacency rules.*

We will see the virtue of this picture when we analyse the algebra structure of P^{FC}.

The above idea has been noticed by many people, it was Dylan Thurston who first explained it to the author.

2.11 Quantum groups.

2.12 The planar algebra of a graph.

3 Presentations of planar algebras

3.1 The free planar algebra on a set of generators

3.2 Planar skein theory

3.3 Knot skein theory

3.4 The exchange relation

3.5 Yang Baxter skein relations

3.6 Jellyfish

4 Operations on planar algebras.

4.1 Cabling

4.2 Direct sum

4.3 Tensor product

4.4 Stitching

4.5 Free product

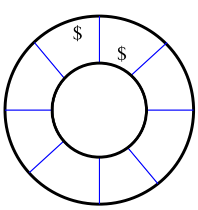
4.6 Free stitching

4.7 duality

5 Linear tangles

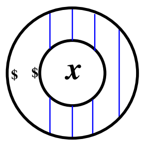
5.1 Annular categories.

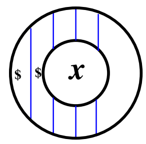
Definition 5.1.1. *The* rotation *will be the element of* $Ann^{TL}_{n,n}$ *for* $n > 0$ *defined by the following linear tangle:*



The meaning of ρ in the oriented case is obvious. For shaded planar algebras it is a pair of maps from $P_{n,\pm}$ to $P_{n,\mp}$ given by the figure 5.1.1 with the two possible shadings.

Definition 5.1.2. *The $id \otimes 1$ and $1 \otimes id$ from P_n to P_{n+2} will be the maps defined as follows:*

$(id \otimes 1)(x) =$ 

$(1 \otimes id)(x) =$ 

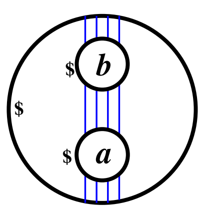
In the oriented case there are two versions of both $id \otimes 1$ and $1 \otimes id$ according to the orientation of the string going from the outside boundary to itself. In the shaded case $id \otimes 1$ goes from $P_{n,\pm}$ to $P_{n+1,\pm}$ and $1 \otimes id$ goes from $P_{n,\pm}$ to $P_{n+1,\mp}$.

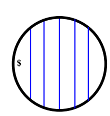
6 Algebra structures.

Certain tangles define associative multiplications on planar algebras. We have not yet been able to list all such structures and the problem is a bit open-ended as the multiplications may not be defined on all elements etc. But in this section we record the main algebras we have investigated.

6.1 Inductive limit algebra.

Proposition 6.1.1. *If P is a planar algebra then any labelled tangle isotopic to the one drawn below defines an associative algebra structure on P_{2n} for each $n \geq 0$.*

$ab =$ 

If P is unital, so is P_{2n} for every n, the identity being given by  .

Proof. This is the same as exercise 1.2.2 □

Remark 6.1.2. *Whenever we refer to P_{2n} as an algebra this is the structure we mean.*

If P is a shaded planar algebra the above tangle defines algebra structures on both $P_{n,+}$ and $P_{n,-}$. When we write P_n for a shaded planar algebra we will mean $P_{n,+}$.

The oriented case is more intersting. If the above tangle is to define an algebra structure on P_α then α, thought of as a word on $\uparrow$ and $\downarrow$, must be of the form ww^* where w^* is w read backwards and with the arrows reversed. If we refer to P_α as an algebra it is this structure we will mean.

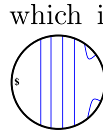
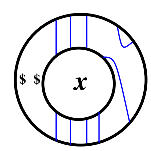
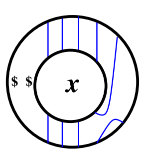
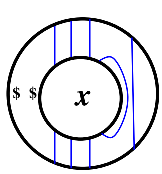
Proposition 6.1.3. *Both $id \otimes 1$ and $1 \otimes id$ (of 5.1.2) define unital algebra homomorphisms from P_{2n} to P_{2n+2} and $(id \otimes 1)(1 \otimes id) = (1 \otimes id)(id \otimes 1)$.*

Proof. Simple pictures. □

Definition 6.1.4. *We call P_∞ the inductive limit algebra for the maps $id \otimes 1$ and $P_{r,\infty}$ for the subalgebra which is the image of $(1 \otimes id)^r$.*

Theorem 6.1.5. *Let P be a unital nondegenerate reduced planar algebra with δ a non-zero scalar. Then the centraliser $Z_{P_\infty}(P_{r,\infty})$ of $P_{r,\infty}$ in P_∞ is P_{2r}.*

Proof. A simple diagram shows that $P_{2r} \subseteq Z_{P_\infty}(P_{r,\infty})$.

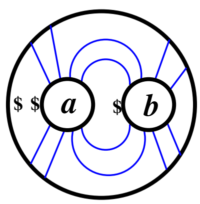
Now suppose that $x \in P_\infty$ commutes with $P_{r,\infty}$. Then x must be in P_m for some m. If $m \leq r$ then $x \in P_r$ and there is nothing to prove. So suppose $x \in P_m$ with $m > r$. We will show that this implies $x \in P_{m-1}$ so that iterating, x in fact belongs to P_r. To see this consider $id \otimes 1(x)$ which is of course the same element in P_∞ as x. Since $m > r$, the element $E =$  with $m-1$ vertical strings (illustrated with $m = 5$), is in $P_{r,\infty}$ so we must have $xE =$  $=$  $= Ex$. Surrounding both these pictures by an obvious annular tangle we get $x = \delta$  . Thus $x \in P_{m-1}$ as claimed. □

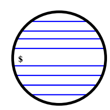
Corollary 6.1.6. *If P is as in the previous theorem then P_∞ is a central algebra iff P is a central planar algebra.*

It is clear what $P_{\infty,\pm}$ means in the shaded case-note that $1 \otimes id$ goes from $P_{n,\pm}$ to $P_{n+1,\mp}$ and defines an algebra embedding. The above theorem is true for both $P_{\infty,\pm}$ but care is needed with the corollary. The oriented case is more exciting. There will be inductive limit algebras as above for every infinite sequence of $\uparrow$'s and $\downarrow$'s.

6.2 Comultiplication.

Proposition 6.2.1. *If P is a planar algebra then any labelled tangle isotopic to the one drawn below (for $n = 4$)defines an associative algebra structure on P_{4n} for each $n \geq 0$.*

$a \circ b =$ 

If P is unital, so is $(P_{4n}, \circ)$ for every n, the identity being given by  .

The corresponding definitions for the shaded and oriented cases are clear though note that in the oriented case comultiplication, just like multiplication, can only be defined on certain P_α's.

Proposition 6.2.2. *The map ρ^n gives an isomorphism between P_{4n} and $(P_{4n}, \circ)$ in the unoriented case. In the shaded case it gives an isomorphism between $P_{2n,+}$ and $(P_{2n,-}, \circ)$.*

Exercise 6.2.3. *What does ρ^n do in the oriented case? When do multiplication and comultiplication coexist?*

Proof. In fact we could have defined $a \circ b$ as $\rho^{-n}(\rho^n(a)\rho^n(b))$, the pull back of multiplication □

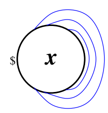
Remark 6.2.4. Note that this does *not* mean that $P_{n,+}$ and $P_{n,-}$ are isomorphic! For a counterexample one may take the P^G of 2.9.2 when G is not abelian:

Exercise 6.2.5. *Show that $P^G_{2,+} \cong \mathbb{C}G$ and $P^G_{2,-} \cong \ell^\infty(G)$.*

Exercise 6.2.6. *For a subfactor planar algebra (or more generally for a central planar algebra for which the canonical bilinear form is non-degenerate) there is a canonical isomorphism between P_2 and its dual. Show that comultiplication as defined above can be dualised to obtain what is normally called a comultiplication on P_2, i.e. a map from P_2 to $P_2 \otimes P_2$ satisfing the dual of associativity. Show that this comultiplication is an algebra homomorphism for P^G. Find an example where it is not.*

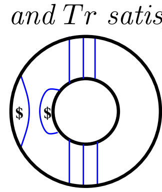
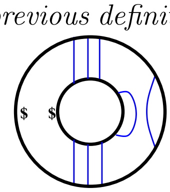
6.3 The Markov trace

If P is a measured planar algebra we may define a map $Tr : P_{2n} \to \mathbb{F}$ by the following diagram:

$Tr(x) = \mu($  $)$ where we have as usual suppressed the outsideboundary disc when there are no strings meeting it.

Proposition 6.3.1. *$Tr(ab) = Tr(ba) = (a, b)$ so if P is unital, $Tr(x) = (x, 1)$, and in the planar *-algebra case $\langle x, y\rangle = tr(xy^*)$.*

Definition 6.3.2. *If $A \subseteq B$ are finite dimensional algebras and $\phi : B \to \mathbb{F}$ is a linear function for which the bilinear forms $[a, b] \mapsto \phi(ab)$ on B and A are nondegenerate, then the unique map $E_A : B \to A$ for which $\phi(E_A(b)a) = \phi(ba)$ for all $b \in B$, $a \in A$ is called the* conditional expectation *from B to A. (Similarly for sesquilinear forms and *-algebras.)*

Proposition 6.3.3. *If P is a central spherical planar algebra with nondegenerate canonical bilinear form and $\delta \neq 0$, then if $B = P_{2n}$ and $A = (1 \otimes id)(P_{2n-2})$ or $A = (id \otimes 1)(P_{2n-2})$ then A, B and Tr satisfy the hypothesis of the previous definition and E_A is the linear tangle $\frac{1}{\delta}$*  *in the first case and $\frac{1}{\delta}$*  *in the second.*

Proof. Just draw the pictures for $Tr(E_a(b)a)$ etc. □

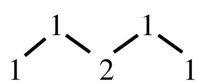
Corollary 6.3.4. *If we write $P_{1,n-1,1}$ for $(id \otimes 1)(1 \otimes id)(P_{2n+2})$ then the following algebras form a commuting square in the sense of []:*

$$\begin{array}{ccc} (id \otimes 1)(P_{2n}) & \subset & P_{2n+2} \\ \cup & & \cup \\ P_{1,n-1,1} & \subset & (1 \otimes id)(P_{2n}) \end{array}$$

Remark 6.3.5. For a positive planar algebra the Markov trace may be renormalised as $tr = \delta^{-n} Tr$ on $P_{n,\pm}$. Then we get a positive definite trace on P_∞. One may perform the GNS constuction [] since left multiplication operators are bounded by finite dimensionality. The resulting von Neumann algebra will be called M_P. M_P may or may not be a factor even when P is central. Since $1 \otimes id$ preserves tr it extends to M_P and we will call the resulting von Neumann subalgebra N_P. The commuting square property above shows that the inclusion $N_P \subseteq M_P$ is proper unless $dim P_\alpha = 1 \forall \alpha$. M_P will always be hyperfinite in the sense of [].

6.4 The inductive limit structure of the P_{2n}, Bratteli diagrams.

We are most interested in cases where P_{2n} is semisimple, and to simplify the presentation we will assume it is a direct sum of matrix algebras over $\mathbb{F}$. Any unital inclusion $A \subseteq B$ of such algebras is completely given by a simple matrix Λ_A^B which describes the inclusion map on K_0 or alternatively how the minimal idempotents of P_{2n} decompose as sums of minimal idempotents of P_{2n+2}. Thus for instance if we were considering the inclusion of group algebras $\mathbb{C}S_2 \subseteq \mathbb{C}S_3$ (for symmetric groups) we would find the matrix: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. This is most conveniently represented by a bipartite graph whose vertices are the matrix algebra summands, remembered by their size, and the number of edges between vertices is the entry of the matrix for the inculsion. Thus in the above example we would get:



For towers of inclusions of algebras one just stacks graphs as above to obtain what is called the Bratteli diagram ([]) for the inductive limit algebra. The unital property of the inclusions is reflected in the propery of the Bratteli diagram that the number at each vertex is the sum of the numbers connected to it on the next row.

In general it is quite difficult to calculate the algebra structure of P_{2n} but it is known in most cases simply because they occurred first as inductive limit algebras and then the algebra structure was observed to extend to that of a planar algebra. Planar algebra structure is most useful when the identity structure is richest, i.e. for shaded and vanilla planar algebras. When this is not available the algebra structure can be known from outside the theory.

Example 6.4.1. *The Hecke algebra.*

The term Hecke algebra here refers to the algebras one would get as the Hecke algebras of double cosets for group/subroup pairs when the group is $GL(n)$ over a finite field and the subgroup is that of upper triangular matrices. For fixed n, if q is the order of the field, this algebra has presentation on generators $g_i, i = 1, 2, \cdots n-1$ (see [])
$g_i^2 = (q-1)g_i + q$
$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$
$g_i g_j = g_j g_i$ if $|i-j| > 1$.

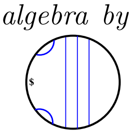
This algebra can be obtained from knot theory. Indeed $P^{Homflypt}$ is up to a change of variables the quotient of $\overrightarrow{P}^{Conway}$ by the first relation above where g is a positive crossing in $\overrightarrow{P}^{Conway}_{\uparrow,\uparrow,\downarrow,\downarrow}$. So $P^{Homflypt}_\alpha$ (with $\alpha(i) = \uparrow$ for $i = 1, 2, \cdots n$ and $\downarrow$ for $i = n+1, \cdots 2n$) is the same algebra as the Hecke algebra with n as above. A deformation argument allows one to obtain the structure of the Hecke algebra for generic values of q, since for $q = 1$ the algebra is semisimple for each n the structure is that of the of the group algebra $\mathbb{C}S_n$. The Bratteli diagram for this algebra is well known to be the Young lattice whose vertices are Young diagrams connected by the induction-restriction rule for representations of the symmetric group, see []. For special values of q degenerations occur and quantum invariant theory sees the Young lattice as the large N limit of the algebra tower $P_\alpha^{U_q(sl_N)}$. For different choices of α the Bratteli diagrams will be different but may always be calculated using the tensor powers of the N dimensional representation of sl_N and its dual, and a deformation argument. This will not work as easily for knot invariants obtained from higher dimensional representations and I know of no general recipe for calculating these algebra structures even for the adjoint representation of sl_n where the corresponding knot theory planar algebra does not generated the invariants for the tensor powers.

6.5 The Temperley Lieb algebra

A unital planar algebra P contains by definition a planar subalgebra spanned by TL diagrams. We will call it the TL subalgebra of P. If P is nondegenerate, its TL subalgebra depends only on δ. So the term "TL planar algebra" has two meanings, one the planar algebra defined in 2.3 and the other its quotient by the kernel of the canonical bilinear form which is the same as the one we have just defined.

In this section we will always be referring to the nondegenerate version of TL.

We will determine the inductive limit algebra structure in the shaded version (which also determines the vanilla version and the oriented version with alternating boundary orientations). We will write TL_n for the algebra P_n^{TL} (shaded).

Exercise 6.5.1. *TL_n is generated as an algebra by $\{E_i, i = 1, 2, .., n-1\}$ where $E_i = (1 \otimes id)^{i-1}(E_1)$, E_1 being the tangle*  *where the shading is implicit.*

Show that the E_i satisfy the relations
*(i) $E_i^2 = \delta E_i$ and $E_i^* = E_i$ in the planar *-algebra case.*
(ii) $E_i E_{i\pm1} E_i = E_i$
(iii) $E_i E_j = E_j E_i$ if $|i - j| \geq 2$.

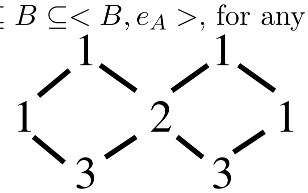
Suppose δ is such that P^{TL} is a positive definite planar algebra, we will determine the Bratteli diagram for TL_n and the possible values of δ. This argument is well known so we limit ourselves to a sketch. We will rely heavily on the "basic construction" where for simplicity we limit ourselves to multimatrix algebras.

Definition 6.5.2. *If $A \subseteq B$ is a unital inclusion of multimatrix algebras, $A = \oplus_{i=1}^n M_{k_i}, B = \oplus_{i=1}^m M_{l_i}$ with $m \times n$ inculsion matrix Λ_A^B and a trace tr on B which is nondegenerate on both A and B then the* basic construction *$< B, e_A >$ is the algebra of linear transformations on B generated by B (acting by left multiplication) and the conditional expectation e_A (see 6.3.2).*

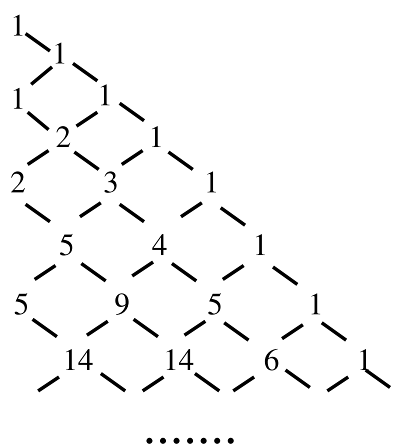
Theorem 6.5.3. *The basic construction* $< B, e_A >$ *is equal to the algebra* $End_{-A}(B)$ *of all right* A*-linear maps on* B *and*
(i) $< B, e_A >$ *is a multimatrix algebra and if* p *is a minimal idempotent in* A *then* pe_A *is a minimal idempotent of* $< B, e_A >$*, canonically defining a bijection between the set of simple summands of* A *and those of* $< B, e_A >$*.*
(ii) $\Lambda_B^{<B,e_A>}$ *is the transpose of* Λ_A^B*.*
(iii) $e_A b e_A = e_A(b) e_A$ *for* $b \in B$
(iv) For $b \in B$*,* $b \in A \iff be_A = e_A b$
(v) The map $B \otimes_A B \to < B, e_A >$ *defined by* $x \otimes y \mapsto x e_A y$ *is a* $B - B$ *bimodule isomorphism.*
(vi) If C *is any unital algebra generated by* B *and an idempotent* e *with* $ebe = e_A(b)e$ *for* $b \in B$ *then* $xe_A y \mapsto xey$ *is an algebra homomorphism onto a 2-sided ideal of* C*.*

Proof. See [],[]. □

Thus for instance in the example $A = \mathbb{C}S_2 \subseteq \mathbb{C}S_3 = B$ as above, the Bratteli diagram for $A \subseteq B \subseteq < B, e_A >$, for any suitably nondegenerate trace, is

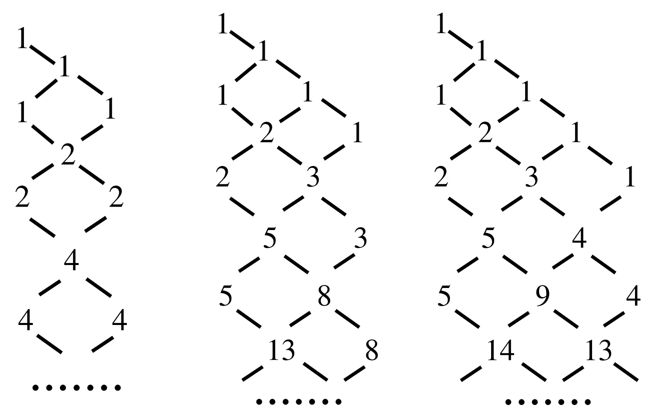


Theorem 6.5.4. *Suppose* P *is a positive definite planar algebra. Then either* $\delta = 2\cos\pi/n$ *for some integer* $n \geq 3$ *or* $\delta > 2$*. Moreover if* $\delta > 2$ *the tower of algebras defined by the inductive limit TL planar subalgebra of* P *has the following Bratteli diagram:*



where the kth row (starting with $k = 0$*) gives the sizes of the matrix summands of the algebra* TL_k*. There are* $[\frac{n+1}{2}]$ *such summands and if we number them with* i *starting from* 0 *from right to left, they are then numbers* $\binom{k}{i} - \binom{k}{i-1}$*, and the trace of* *a minimal projection in the corresponding summand is* $T_{k-2i+1}(\delta)\delta^{-k}$*,* T_k *being the Tchebychev polynomial defined by* $T_k = \frac{q^k - q^{-k}}{q - q^{-1}}$ *with* $\delta = q - q^{-1}$*.*

If $\delta = 2\cos\pi/n$ *the Bratteli diagram is obtained from the one above by eliminating the last matrix summand of* TL_{n-2} *and all those vertically below it, and adjusting the sizes of the matrix summands to account for unitality of the inclusions. Thus for* $\delta = 4, 5$ *and* 6 *we obtain the following Bratteli diagrams:*



Proof. (sketch). One proceeds indcutively. First suppose $\delta > 2$. Verifying the relationship between TL_0 and TL_1 is trivial. Now suppose we know the inclusion matrix for $TL_n \subseteq TL_{n+1}$, together with the weights giving the normalised Markov trace tr as a weighted sum of traces. On TL_n this may be thought of as a vector $\overrightarrow{t}_n$ whose indices are the central summands of TL_n, thus ranging from 0 to $[\frac{n+1}{2}]$ and if $\overrightarrow{dim_n}$ is the vector whose entries are the sizes of matrix algebras, the condition

$$\langle \overrightarrow{t}_n, \overrightarrow{dim_n} \rangle = 1$$

expresses precisely the normalisation $tr(1) = 1$.And of course $\dim TL_n = ||\overrightarrow{dim_n}||^2$. By (vi) of 6.5.3 we know there is an algebra homomorphism ϕ from $< TL_{n+1}, e_{TL_n} >$ to a 2-sided ideal of TL_{n+2} where as e we take e_{n+1} of 2.3 and use a diagram to obtain the condition of (vi) from 6.3.3. We claim that ϕ is injective. Since $< TL_{n+1}, e_{TL_n} >$ is multimatrix, it suffices to check that $\phi(q) \neq 0$ for some element q in each matrix summand of $< TL_{n+1}, e_{TL_n} >$. Thus by 6.5.3, it suffices to show that $\phi(pe_{TL_n}) \neq 0$ for any minimal projection $p \in TL_n$. But the Markov trace of such an element is equal to $\delta^{-2}tr(p)$ which is non-zero by induction (the Tchebychev polynomials are never zero for $\delta \geq 2$).

But now binomial identities show that the sizes of the matrix algebras in $\phi(< TL_{n+1}, e_{TL_n} >)$ are as required and the sum of their squares is equal to $\frac{1}{n+3}\binom{2n+4}{n+2} - 1$. Thus the ideal $\phi(< TL_{n+1}, e_{TL_n} >)$ is of codimension at most 1 in TL_{n+2}. A further binomial identity shows that the sum of the traces of the central projections in $\phi(< TL_{n+1}, e_{TL_n} >)$ is equal to $1 - T_{n+3}(\delta)\delta^{-n-2}$. But $T_{n+3}(\delta) > 0$ for all n for $\delta \geq 2$. This exhibits TL_{n+2} with its subalgebra TL_{n+1} and trace $\overrightarrow{t}_{n+2}$ as required.

Now suppose $\delta < 2$. The method so far shows that the Bratteli diagram for $TL_0 \subseteq TL_1 \subseteq \cdots \subseteq TL_{n+2}$ is identical to that for $\delta \geq 2$ as long as $T_k(\delta) > 0$ for $k \leq n+3$. Clearly if $T_k(\delta) > 0$ for $k \leq n+2$ but $T_{n+3}(\delta) < 0$ then the planar algebra cannot be positive definite. It is easy to check that this rules out any value of δ lying between the values $2\cos\pi/r$ and $2\cos\pi/(r+1)$, for $r = 3, 4, 5, \cdots$.

But if $\delta = 2\cos\pi/r$ then $T_k(\delta) > 0$ for $k < r$ and $T_r(\delta) = 0$. So there is no contradiction but we conclude that the map ϕ is actually onto TL_{r-1}! The relations between the trace vector and the inclusion matrices show that the same is true for all subsequent basic constructions in the TL tower and we are done.

Explicit formulae for the dimensions of the simple summands for $\delta = 2\cos\pi/r$ are available-[].

□

Remark 6.5.5. Note that the existence of a positive definite planar algebra, and hence a positive definite TL quotient, for $\delta = 2\cos\pi/n$ follows from 2.12 using the A_{n-1} Coxeter graph.

Remark 6.5.6. The above method of proof actually proves a lot more. For any field, provided δ is not a zero of any of the Tchebychev polynomials $T_2, \cdots, T_{n+1}$ it shows that TL_n is in fact multimatrix and has a Bratteli diagram equal to the first one in 6.5.4 up to the nth row.

Remark 6.5.7. The whole planar algebra structure was not used in the proof. In fact we determined the structure of the $C*^*$-algebra generated by $e_1, e_2, e_3, \cdots e_n$ satisfying the familiar relations $e_i^2 = e_i = e_i^*, e_i e_{i\pm1} e_i = \delta^{-2} e_i, e_i e_j = e_j e_i$ for $|i-j| \geq 2$ possessing a positive normalised trace tr such that

$$tr(we_{n+1}) = \delta^{-2} trw \text{ where } w \text{ is a word on } e_1, e_2, \cdots, e_n.$$

Remark 6.5.8. It is obvious that the ideal $\mathcal{I}$ in TL_{n+2} given by the basic construction is in fact the linear span of all non-empty words on the $e_i's$. In the C^*-algebra above this defines a canonical minimal, central projection f_{n+1} for which $f_{n+1}\mathcal{I} = 0$. This is known as the Jones-Wenzl idempotent-see [],[].

6.6 Principal graphs.

If P is a positive definite shaded planar algebra, the method of proof of theorem 6.5.4 gives a particular structure to the Bratteli diagram of the P_n. Indeed if $\Lambda_{P_n}^{P_{n+1}}$ is known together with their trace vectors $\overrightarrow{t}_n$ and $\overrightarrow{t}_{n+1}$ defined by the Markov trace, the basic construction embeds via ϕ as an ideal in P_{n+2} exactly as for TL except of course that it is not necessarily of codimension 1.

Let us write e_n for $e_{P_{n-1}} \in P_{n+1}$. Two cases arise:

Case (i) ϕ is surjective.

This happens if the traces of the minimal central projections in $\phi(< P_{n+1}, e_{n+1} >)$ add up to 1. But the traces of minimal projections in $\phi(< P_{n+1}, e_{n+1} >)$ are just δ^{-2} times their values on the corresponding minimal projections in P_n so $\overrightarrow{t}_n$ is a positive eigenvector of $\Lambda_{P_n}^{P_{n+1}}(\Lambda_{P_n}^{P_{n+1}})^t$. This implies that the trace vector of P_{n+1} is a positive eigenvector of $\Lambda_{P_{n+1}}^{P_{n+2}}(\Lambda_{P_{n+1}}^{P_{n+2}})^t$ so that the homomorphism ϕ is surjective at all subsequent levels in the tower.

When this case arises we say that P is "finite depth" and the depth is defined to be smallest value of $n+1$ for which the homomorphism ϕ is surjective.

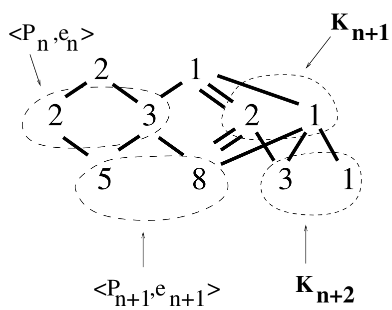
Case (ii) $\phi(< P_{n+1}, e_{n+1} >)$ is a proper ideal in P_{n+2}.

Then the Bratteli diagram for $P_{n+1} \subseteq P_{n+2}$ contains two parts, the first being the basic construction part and the second being "new stuff". By semisimplicity the new stuff is canonically a direct summand of P_{n+2}. We will denote this summand by K_{n+2} (for $n = 0, 1, 2, \cdots$). Thus $P_{n+2} = \phi(< P_{n+1}, e_{n+1} >) \oplus K_{n+2}$.

Lemma 6.6.1. $\phi(< P_n, e_n >)K_{n+2} = 0$.

Proof. Since $\mathcal{P}_{n+2}$ is an ideal It suffices to show that $K_{n+2}e_n = 0$. But e_n is a multiple of $e_n e_{n+1} e_n$ and certainly $K e_{n+1} = 0$. □

We thus see that the Bratteli diagram for $P_n \subseteq P_{n+1} \subseteq P_{n+2}$ must look as below:



Definition 6.6.2. *The bipartite graph underlying that part of the Bratelli diagram connecting* $< P_n, e_n >$ *to* $< P_{n+1}, e_{n+1} >$ *is unaltered by* K_{n+1} *and* K_{n+2} *and so converges as* $n \to \infty$ *to a bipartite graph called the* <u>*Principal graph*</u> Γ_P *of* P*.The principal graph of the dual planar algebra to* P *is called the* dual principal graph $\hat{\Gamma}_P$ *of* P*.*

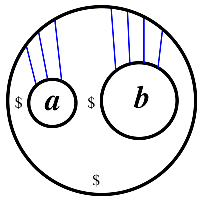
Obviously to say that P is of finite depth is the same as saying that Γ_P is finite.

Exercise 6.6.3. *Show that* P *is of finite depth iff* $\hat{\Gamma}_P$ *is finite.*

6.7 Graded algebras.

There are many graded algebras defined by a planar algebra. We begin with the simplest.

Definition 6.7.1. *If P is a planar algebras we call $Gr(P)$ the $\mathbb{N}\cup\{0\}$-graded algebra whose degree n graded component is P_n and with multiplication $\wedge : P_m \times P_n \to P_{m+n}$ defined by:*

$a \wedge b =$  *(here $m = 3$ and $n = 4$)*

*$x \mapsto x^*$ makes $Gr(P)$ into a *-algbera if P is a planar *-algebra.*

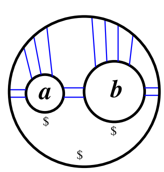
Thus $\overset{\infty}{\underset{n=0}{\oplus}} P_n$ becomes an associative algebra $Gr(P)$ which is unital if P is, the identity being given by Ω.

Remark 6.7.2. *This algebra has a curious commutativity property- a picture shows that $a \wedge b = \rho^{deg(a)}(b \wedge a)$*

Example 6.7.3. *$Gr(P^{\otimes})$ is the algebra of non-commuting polynomials in as many variables as the dimension of the auxiliary vector space V, or in other words the tensor algebra of V.*

If P is shaded, there are two graded algebras according to the shading, to be unambiguous we call $Gr(P)$ the one for which the regions containing the $'s are unshaded. The oriented case is more interesting as one must grade the algebra by the semigroup of all words $\uparrow$ and $\downarrow$. In the case of $\overrightarrow{P}^{\otimes}$, and the subsemigroup of words on just $\uparrow$, one obtains the tensor algebra of V and if one takes the fixed point planar algebra for some group in $GL(V)$ one obtains thus the algebra of (non-commutative) invariants of G.

For each k we can make $\{P_{n+2k} | n = 0, 1, 2, \cdots\}$ into a graded algebra ($Gr(P) = Gr_0(P)$) with the multiplication $\wedge_k : P_{m+2k} \times P_{n+2k} \to P_{m+n+2k}$ defined by (illustrated with $m = 3, n = 4, k = 2$):

$a \wedge_k b =$ 

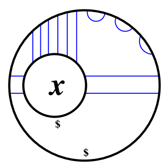
Thus for each k we have an associative algebra $Gr_k(P) = \overset{\infty}{\underset{n=0}{\oplus}} P_{n+2k}$ which is unital if P is, the identity being the same as that of the algebra P_{2k} (see 6.1.2)

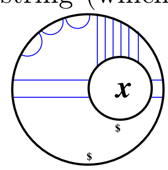
Proposition 6.7.4. *The map $1 \otimes id$ defines an algebra homomorphism of $Gr_k(P)$ into $Gr_{k+1}(P)$.*

Proof. Just draw the picture. □

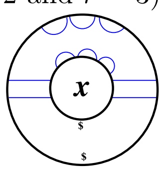
Observe that the degree zero component of $Gr_k(P)$ is nothing but the algebra P_{2k}. Observe also that $Gr_k(P)$ is a *-algebra if P is a planar *-algebra.

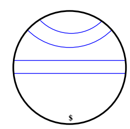
Theorem 6.7.5. *Let P be a unital nondegenerate reduced planar algebra with $\delta \neq 1, 0$. Then the centraliser $Z_{Gr_k(P)}(Gr(P))$ is P_{2k}.*

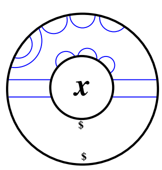
Proof. Suppose $x \in Gr_k(P)$ commutes with $Gr(P)$. We may suppose $x \in Gr_k(P)_n$ for some n. There are two cases:
(i) $n = 2r > 0$. Call $\cup$ the TL basis element with one string (which is in $Gr(P)$ hence $Gr_k(P)$). Then $x\cup^r =$ 

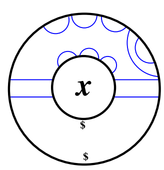
$= \cup^r x =$ 

(illustrated with $k = 2$ and $r = 3$). Capping off the cups on one of these figures one gets

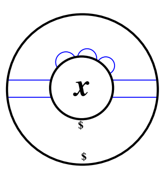
$x = \delta^{-r}$ 

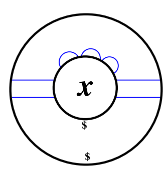
. Now taking the commutator with $d =$  $\in Gr(P)$

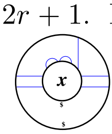
one gets 

$=$ 

. Now cap off with $d\cup^r$ to obtain

$(\delta^{r+2} - \delta^r)$ 

$= 0$. So either 

$= 0$ in which case $x = 0$ or $\delta = 1$, a contradiction.
(ii) $n = 2r + 1$. Proceeding as in (i) with $\cup^r$ one obtains (illustrated with $r = 2$) $x = \delta^{-r}$ 

. Taking the commutator with $\cup$ once again and capping off the cup we get $x = \delta x$ and argue as before.

□

Corollary 6.7.6. *Let P be as in the previous theorem, then $Gr(P)$ is a central algebra iff P is a central planar algebra.*

Remark 6.7.7. For a subfactor planar algebra the Voiculescu trace on $Gr(P)$ is defined by the augmentation. In [] it is shown to be positive definite and one may perform the GNS constuction [] since left multiplication operators are bounded ([]) The resulting von Neumann algebra will be called $\mathfrak{N}_P$. One may define traces on each $Gr_k(P)$ by augmenting a partial Markov trace as in []. Since $1 \otimes id$ preserves these traces these traces one obtains and embedding of $\mathfrak{N}_P$ inside $\mathfrak{M}_P$, the GNS closure of $Gr_1(P)$. By orthogonality one can show that the inclusion $\mathfrak{N}_P \subseteq \mathfrak{M}_P$ is proper iff $\delta > 1$. These considerations apply without centrality of P but of course $\mathfrak{M}_P$ will not be a factor if P is not central-its centre will be $P_{0,+}$. Unlike $\mathfrak{M}_P$ will only be hyperfinite if $\delta = 1$ ([]).

6.8 Fusion algebras.

7 Connections.

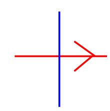
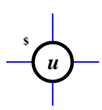
7.1 Bi-invertibles and Bi-unitaries

A bi-invertitble is an element of P_4 in a given (unital) planar algebra P that satisfies relations akin to the type II Reidemeister moves.

Definition 7.1.1. *Let P be a unital vanilla finite dimensional planar algebra. An element $u \in P_4$ will be called* bi-invertible *if*

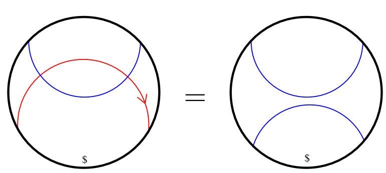
$$u\rho(u) = 1 \quad \text{in the algebra } P_4$$

*(6.1.2) In a planar *-algebra the bi-invertible u will be called bi-unitary if it is unitary.*

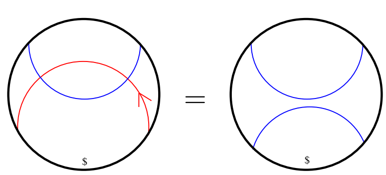
Remark 7.1.2. We adopt the diagrammatic convention that a double point  should be replaced by 

and the rest of the red string should become blue as well.

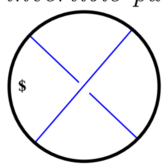
Bi-invertibility is thus equivalent to the following identity in P_4:



and since $u\rho(u) = 1$ is equivalent to $\rho(u)u = 1$ this in turn is equivalent to:



(The position of the \$'s is immaterial as long as it is the same on both sides of the equations.)

Example 7.1.3. *The bi-invertible par excellence is the crossing in knot theory. If we consider the element* 

in P_2^{Conway}*. A single picture shows that this element is biinvertible.*

Example 7.1.4. *In general we may choose an invertible (unitary in the * case) element of* P_1

Example 7.1.5. *In* $P^{\otimes}$ *the simplest example is the permutation tensor:*

$$R_{i,j,k,l} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise} \end{cases}$$

This can be elaborated by choosing permuations π_i *for each* i *and setting*

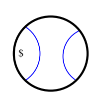
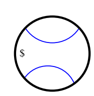
$$R_{i,j,k,l} = \begin{cases} 1 & \text{if } j = l \text{ and } i = \pi_l(k) \\ 0 & \text{otherwise} \end{cases}$$

(If the tensor indices are a finite group, ρ_g *is conjugation by* g *and we map* P^{Conway} *to* $P^{\otimes}$ *by sending a positive crossing to this* R *and a negative one to its inverse, the element of* P_0 *defined by a link is the number of homomorphisms of the fundamental group of the link complement to the group.)*

And further if we can find permuations π_i *and* ρ_j *with* $\rho_k^2 = 1$ *and* $\pi_{\rho_k(l)} = \pi_l$

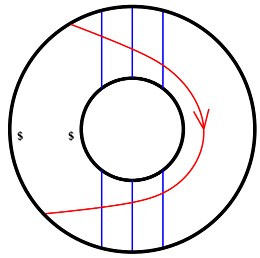
$$R_{i,j,k,l} = \begin{cases} 1 & \text{if } i = \pi_l(k) \text{ and } j = \rho_k(l) \\ 0 & \text{otherwise} \end{cases}$$

(see [],[])

Exercise 7.1.6. *Show that the only bi-invertibles in* P^{TL} *are* A  $+A^{-1}$  *where* $A^2 + A^{-2} = -\delta$*. This is biunitary iff* $|A| = 1$ *and* P^{TL} *is a subfactor planar algebra. Thus planar algebras may admit no bi-unitaries but any unital planar algebra has bi-invertibles.*

One use of a bi-invertible is to define an *endomorphism* of the algebra P_∞ into itself.

Definition 7.1.7. *If* $u \in P_2$ *is bi-invertible, define the map* $\psi_u : P_{2n} \to P_{2n+2}$ *by the following annular tangle:*



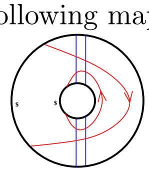
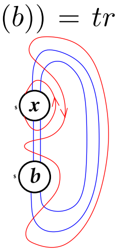
Proposition 7.1.8. *ψ defines a unital algebra embedding of P_{2n} into P_{2n+2} which is compatible with the inclusions $P_k \subseteq P_{k+1}$ and so defines an endomorphism of P_∞. If u is biunitary then ψ is a *-algebra embedding/endomorphism.*

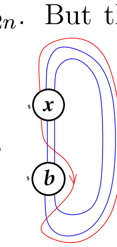
Proof. Just draw the pictures and use the bi-invertible property. □

Biinvertibles and biunitaries are intimately related with "commuting squares" or "orthogonal pairs" ([]) of algebras.

Proposition 7.1.9. *Suppose P is central, finite dimensional, with nondegenerate canonical bilinear form, and $\delta \neq 0$ Fix n and put $C = P_{2n+2}, A = (id \otimes 1)(P_{2n})$ and $B = \psi_u(P_{2n})$. Then the bilinear form defined by tr is nondegenerate on B and $A \cap B$ and $E_A E_B = E_B E_A = E_{A \cap B}$.*

Proof. Non-degeneracy of tr on B follows from the fact that ψ preserves the Markov trace. Consider the following map from P_{2n+2} to itself:

$\mathcal{E} = \frac{1}{\delta}$  To show that $\mathcal{E}$ is the conditional expectation onto B one needs to show that $tr(\mathcal{E}(x)\psi(b)) = tr(x\psi(b)) \forall x \in C$ and $b \in P_{2n}$. But the left hand side of this equation is $\frac{1}{\delta}$ 

and the right hand side is 

.

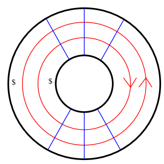
After a little isotopy and moves as in 7.1.2 we see that these two numbers are equal. To complete the proof is a matter of composing the annular tangles for E_A and E_B in both orders. Non-degeneracy on $A \cap B$ is immediate. □

So A, B, C and $A \cap B$ forming a commuting square.

Corollary 7.1.10. *The map on P_∞ defined by $\mathcal{E}$ above gives a conditional expectation from P_∞ onto $\psi(P_\infty)$.*

We see that 7.1.9 Allows us to control the inclusion $\psi(P_\infty) \subseteq P_\infty$. In particular it is proper. In the von Neumann algebra case this will be particularly useful. But of more interest is the situation concerning $\psi(P_{2n})$ $(= B)$ and $(1 \otimes id)(P_{2n})(= D)$.

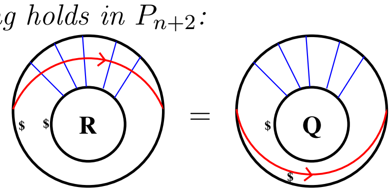
Proposition 7.1.11. *Identifiying P_{2n} with $\psi(P_{2n})$ using ψ, the "angle operator" $E_B E_D E_B$ is given by the following diagram:*

$\frac{1}{\delta^2}$ 

(And of course if we identify D with P_{2n} using $1 \otimes id$ we get the same picture for $E_D E_B E_D$ with the orientations reversed.)

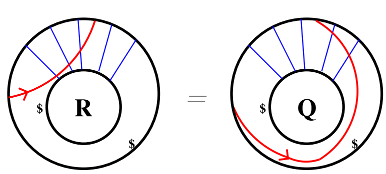
7.2 Flatness.

Definition 7.2.1. *Suppose we are given a planar algebra P and a bi-invertible element u in P_4. An element $R \in P_n$ is said to be* flat *for u if there is a $Q \in P_n$ so that the following holds in P_{n+2}:*



Remark 7.2.2. *Note that if n is even this is equivalent to saying that $\psi_u(R) \in (1 \otimes id)(R)$*

Proposition 7.2.3. *$R \in P_n$ is flat iff for any integer $p, 0 \leq p \leq n$ there is a $Q \in P_n$ with*



where the red string crosses p strings in the picture containing R.

Proof. Just surround the flatness picture by an appropriate annular tangle and use 7.1.2 to obtain the pictures above. □

Lemma 7.2.4. *If P is a positive planar algebra, an element is flat iff it is a fixed vector for the angle operator (7.1.11).*

Proof. The angle operator is indeed the angle operator for the two subspaces D and B. The eigenspace with eigenvalue 1 is precisely the intersection of the two subspaces.

□

We see that the union of the eigenvalues of the angle operator as n varies forms the pure point spectrum of the angle operator ([]) between the two von Neumann subalgebras $\psi(M^P)$ and N^P of 6.3.5. This is very interesting in light of [] where it is proved that the intersection of finite index subfactors is of finite index iff the spectrum of the angle operator is finite. But the intersection of $\psi(M^P)$ and N^P is just the closure of the flat elements so the multiplicity of the eigenvalue 1 of the angle operator actually influences the number of its eigenvalues.

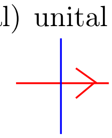
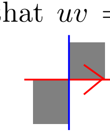
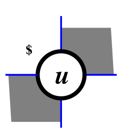
Perhaps the main interest in biinvertibles and their flat elements is the following:

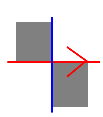
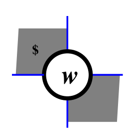
Theorem 7.2.5. *If P is a unital vanilla finite dimensional planar algebra and $u \in P_4$ is bi-invertible, then the flat elements for u form a planar subalgebra of P which is unital if P is and a planar *-subalgebra if P is a planar *-algebra and u is biunitary.*

Proof. We have to show that any planar tangle labelled with flat elements is again flat. This is not hard-draw the diagram of the left hand side of flatness. The red string can then be moved through the labelled tangle to the bottom, passing through any blue strings by 7.1.2 and any labelled discs by 7.2.3. Unitality follows from 7.1.2 and the *-algebra property follows from applying an orientation reversing diffeomorphism to the picture in 7.2.1 and using $\rho(u) = u^*$. □

Definition 7.2.6. *If u is a biinvertible or biunitary in P we call P^u the planar subalgebra (sub *-algebra) of flat elements for u.*

7.3 The shaded case, Hadamard matrices.

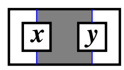
Consider now the case of shaded (not necessarily spherical) unital reduced planar algebras. There are two possible shadings for the picture  . So we define in this case a biinvertible to be a pair u, v of elements in $P_{2,+}$ such that $uv = 1$ and $\rho(u)\rho^{-1}(v) = \frac{\delta_+}{\delta_-}1$ (in $P_{2,-}$). Now we adopt the convention that  is to be replaced by 

and  is to be replaced by 

(with $w = \rho^{-1}(v)$). For u to be biunitary (when P is a planar *-algebra) means that u itself is unitary. This is equivalent to $\rho(u)$ being a multiple of a unitary in $P_{2,-}$.

With these conventions all the definitions and results of this section apply in the shaded case.

Proposition 7.3.1. *Choose invertibles (resp. unitaries) $x, y \in P_{1,+}$ and define*

$w(x, y) =$  .

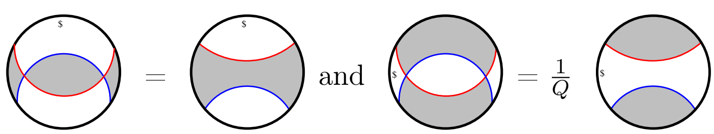
Then if u is biinvertible (resp. biunitary), so are $uw(x, y)$ and $wu(x, y)$.

Proof. This follows immediately from pictures. □

We see that the set of biinvertibles (resp. biunitaries) is a union of double cosets for the subgroup of invertibles (unitatries) of the form $w(x, y)$ above. Changing u by w's is called a *gauge transformation.*

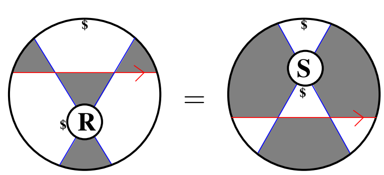
Of considerable interest is the case of P^{spin}. In this case a biunitary element is precisely the same as a unitary matrix $u_{i,j}$ with $|u_{i,j}| = \frac{1}{\sqrt{Q}} \quad \forall i, j = 1, 2, \cdots Q$. If the field is $\mathbb{R}$ this is precisely the same notion as a Hadamard matrix -[] (after multiplication by $\sqrt{Q}$). In the complex case such matrices are called complex Hadamard matrices.

Note that the biunitarity condition translated into diagrams is just:



with both orientations allowed on the red line.

An element R of $P_{2,+}$ is flat in this case if there is a $S \in P_{2,-}$ such that



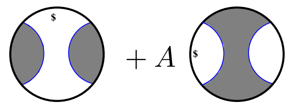
This is a version of what is known as the “star-triangle” equation. Written out in algebraic notation it becomes, according to our conventions,

$$\sum_a \overline{u}_{a,i} u_{a,j} R_{k,a} = u_{k,j} \overline{u}_{k,i} S_{i,j} \qquad \forall i, j, k = 1, 2, \cdots Q$$

These equations apply no matter how the action of tangles in P^{spin} is normalised. See 1.3.9. The only effect of changing the values of δ on closed loops will be to change the S corresponding to a given R by a scalar.

Elements of TL are always flat. Let us record in detail the flatness relation for TL elements in $P^{Spin}_{2,+}$. It also applies to any biinvertible in a *spherical* shaded planar algebra. (see 2.9.1).

Definition 7.3.2. *For an invertible scalar A let*

$R(A) =$  $\in P^{Spin}$

Proposition 7.3.3. *The star triangle equation is satisfied for*

$$R = R(A) \text{ and } S = AR(\frac{1}{A})$$

.

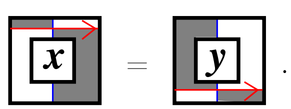
Proof. This is easy to check from the star triangle equation and biinvertibility. □

Hadamard matrices have been intensively studied. Apart from an obvious example when $Q = 2$, they only exist for Q a multiple of 4. For information on Hadamard matrices see [].

Gauge transformations alter a complex Hadamard matrix by multiplying rows and columns by scalars of modulus one (± 1 in the real case). This, together with permutations of the rows and columns, gives what is called *Hadamard equivalence* of (generalized) Hadamard matrices. Row and column permutations produce equivalent P^u's so any information about u obtained from P^u alone will be *invariant under Hadamard equivalence.* (The endomorphism ψ_u of 7.1.7 itself contains more information than just P^u.)

Proposition 7.3.4. *If u is a complex Hadamard matrix, P^u is central. Moreover, $dim P^u_1 = 1$, and $P^u_{2,+}$ and $P^u_{2,-}$ are abelian.*

Proof. We have $\dim P_{0,+} = 1$. Here is the equation for flatness of $x \in P_{1,+}$.



If the bottom shaded region is assigned a spin a, and the top region a spin b, the left-hand side gives $u_{b,a}x_a$ and the right-hand side gives $u_{b,a}y_b$, so x_a is independent of a, and $\dim P^u_{1,+} = 1$. It follows that $\dim P_{0,+} = 1$.

$P^u_{2,+}$ is abelian because $P_{2,-}$ obviously is and $\rho^3\psi$ defines an antiisomorphism between $P^u_{2,+}$ and $P^u_{2,-}$. □

So a complex Hadamard matrix u yields a subfactor planar algebra. In fact the corresponding subfactor was the starting point of the theory of planar algebras, as the equations for P^u are those for the relative commutants of a spin model commuting square given in []. We now determine $P^u_{2,\pm}$ for a complex Hadamard matrix u.

Definition 7.3.5. *Given a $Q \times Q$ complex Hadamard matrix $u_{a,b}$ we define the $Q^2 \times Q^2$ profile matrix $Prof(u)$ by*

$$Prof(u)_{a,b}^{c,d} = \sum_x u_{x,a}\overline{u}_{x,b}\overline{u}_{x,c}u_{x,d}.$$

The profile matrix is used in the theory of Hadamard matrices. We will see that it determines P^u.

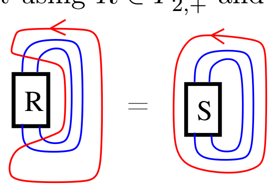
Definition 7.3.6. *Given the $Q^2 \times Q^2$ matrix $Prof(u)$, define the directed graph $\mathfrak{G}_u$ on Q^2 vertices by $(a,b) \to (c,d)$ iff $Prof(u)_{a,b}^{c,d} \neq 0$.*

The isomorphism class of $\mathfrak{G}_u$ is an invariant of Hadamard equivalence.

Theorem 7.3.7. *If u is a $Q \times Q$ generalized Hadamard matrix thought of as a biunitary for the spin planar algebra $P = P^{spin}$, then the minimal projections of the abelian C^*-algebra $P_{2,+}^u$ are in bijection with the connected components of the graph $\mathfrak{G}_u$. Moreover the (normalized) trace of such a projection is $\dfrac{n}{Q^2}$ where n is the size of the connected component, which is necessarily a multiple of Q.*

Proof. For matrices $R_{a,b}, S_{a,b}$, the flatness equations of the star-triangle equations above amount to saying that, for each (i,j), the vector $v_{(i,j)}$ whose x^{th} component is $u_{x,j}\overline{u}_{x,i}$ is an eigenvector of the matrix R with eigenvalue $S_{i,j}$. The profile matrix is just the matrix of inner products of these eigenvectors, $\langle v_{(d,c)}, v_{(a,b)}\rangle$. The $v_{(i,j)}$ span the space since, for fixed i, the biunitary equations show that the $v_{(i,j)}$ are orthogonal. So let p be a nonzero proection in $P_{2,+}^u$. For each (i,j) either $pv_{(i,j)} = 0$ or $pv_{(i,j)} = v_{(i,j)}$. If $pv_{(i,j)} = v_{(i,j)}$ and there is an edge between (i,j) and (k,ℓ) on $\mathfrak{G}_U$ then $pv_{(k,\ell)} = v_{(k,\ell)}$. Hence the image of p is spanned by the $v_{(i,j)}$'s with (i,j) in a union of connected components of $\mathfrak{G}_u$. The orthogonal projection p_C onto the linear span of $v_{(i,j)}$'s with (i,j) in a connected component C is in $P_{2,+}^u$ since all the $v_{(i,j)}$'s are eigenvectors for this projection. Such a p_C is clearly minimal.

If the matrix R is a minimal projection, $S_{a,b}$ is either 1 or 0 depending on whether (a,b) is in the connected component or not. Consider the picture below where the shadings are implicit using $R \in P_{2,+}^u$ and $S \in P_{2,-}^u$.



Applying Reidemeister type II moves and summing we obtain the assertion about the trace. (It is a multiple of $1/Q$ since x is a $Q \times Q$ matrix.) □

If G is a finite abelian group and $g \mapsto \hat{g}$ is an isomorphism of G with its dual $\hat{G}$ (=Hom$(G, \mathbb{C}^*)$), we obtain a generalized Hadamard matrix u, with $Q = |G|$, by setting $uh, g = \frac{1}{\sqrt{Q}}\ \hat{h}(g)$. We call this a *standard* generalized Hadamard matrix. It is Hadamard if $G = (\mathbb{Z}/2\mathbb{Z})^n$ for some n.

Exercise 7.3.8. *Show that if u is standard P^u is exactly the planar algebra P^G of 2.9.2. In particular $dim(P^u_{k\pm}) = Q^k$.*

Exercise 7.3.9. *Show that if $dim P^u_{2,\pm} = 2$ then u is gauge equivalent to a standard complex Hadamard matrix.*

(Hint. Being Abelian, $P^u_{2,+}$ is $\ell^\infty(X)$ with $|X| = Q$. Use comultipilcation to define a group structure on X.)

We have, together with R. Bacher, P. de la Harpe, and M.G.V. Bogle performed many computer calculations. So far we have not found a generalized Hadamard matrix u for which $\dim(P^u_{2,\pm}) = 2$ but $\dim(P^u_{3,\pm}) > 5$. The five 16×16 Hadamard matrices have $\dim P^u_{2,\pm} = 16$,8,5,3 and 3, and are completely distinguished by the trace. There are group-like symmetries in all cases corresponding to the presence of normalizer in the subfactor picture. Burstein in [] has completely determined the structure of P^u in the case $dim P^u_{2,\pm} = 8$. The Hadamard matrix itself decomposes as a twisted tensor product []

Haagerup has shown how to construct many interesting examples and given a complete classification for $Q = 5$. In the circulant case he has shown there are only finitely many examples for fixed prime Q (see [ ]).

Perhaps somewhat surprisingly, the presence of a lot of symmetry in u can cause $P^u_{2,\pm}$ to be small! The kind of biunitary described in the following result is quite common — the Paley type Hadamard matrices give an example.

Proposition 7.3.10. *Suppose $Q-1$ is prime and let u be a $Q \times Q$ complex Hadamard matrix with the following two properties (the first of which is always true up to gauge equivalence):*

(i) There is an index $*$ with $u_{a,*} = u_{*,a} = 1$ for all a.

(ii) The group $\mathbb{Z}/(Q-1)\mathbb{Z}$ acts transitively on the spins other than $*$, and $u_{ga,gb} = u_{a,b}$ for all $g \in \mathbb{Z}/(Q-1)\mathbb{Z}$.

Then $dim(P^u_{2,\pm}) = 2$ or u is gauge equivalent to a standard matrix.

Proof. The nature of the star triangle equations makes it clear that $\mathbb{Z}/(Q-1)\mathbb{Z}$ acts by automorphisms on $P^u_{2,+}$, obviously fixing the projection e which is the matrix $R_{a,b} = 1/Q$. Thus the action preserves $(1-e)P^u_{2,+}(1-e)$. Since $(Q-1)$ is prime there are only two possibilities: either the action is non-trivial and $\dim(P^u_{2,+}) = Q$ so P^u is standard, or every solution of the star triangle equations is fixed by $\mathbb{Z}/(Q-1)\mathbb{Z}$. In the latter case let $R_{a,b}, S_{a,b}$ be a solution of the star triangle equations. Then putting $c = *$ we obtain $\sum_d u_{d,a}\overline{u}_{d,b}R_{*,d} = S_{b,a}$, so $S_{b,a}$ is determined by the two numbers $R_{*,*}$ and $R_{*,d}$, $d \neq *$. So by 2.11.7 we are done. □

We would like to make the following two open problems about matrices quite explicit. Both concern a generalized Hadamard matrix u.

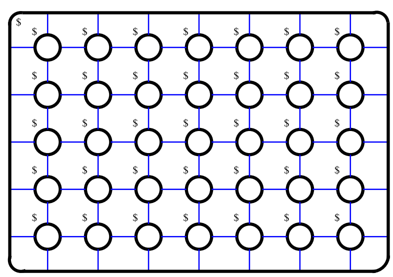
(i) Is the calculation of $dimP^u_{k,\pm}$ feasible in the polynomial time as a function of k?

(ii) Is there a u for which $dimP^u_{k,\pm} = \frac{1}{k+1}\binom{2k}{k}$? (i.e., P^u is just the shaded Temperley-Lieb algebra).

8 2D Statistitical mechanical models.

8.1 Generalities

If we consider the following tangle:



Given $R \in P_4^{\otimes}$ we consider the value of this tangle T with all the inputs being R. We have

$$Z_T = \sum_{\sigma} \prod_{D \in \mathfrak{D}_T} R_{i,j,k,l}$$

Where σ runs over all functions from the strings of the tangle to the set $\{1, 2, \cdots k\}$ and i, j, k, l are the values of σ on the four strings surrounding D. Obviously something needs to be done about the boundary but let us ignore that for the moment.

If all the $R_{i,j,k,l}$ are non-negative they can be written $exp(-\frac{E(i,j,k,l)}{kT})$ and we recognise the partition function for what is called a "vertex model" on a square lattice in two dimensional equilibrium classical statistical mechanics. The discs in $\mathfrak{D}_T$ are to be thought of as "atoms" interacting with their neighbours on the lattice with the possible states of each atom being given by the quadruple (i, j, k, l). Then $E(i, j, k, l)$ is the energy of the atom in that state. What we do with the boundary will change the answer but since all the terms are positive, not by much and certainly not enough to affect the growth rate of Z_T as the lattice gets larger and larger in size. It is this growth rate, $\frac{1}{|\mathfrak{D}T|}\log(Z_T)$ that is one of the main objects of study, called the free energy-see [].

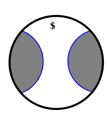
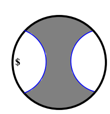
But we see that our planar algebra formalism allows us to consider Z_T when R is any element of any planar algebra in the space corresponding to 4 boundary points. If we consider the graph planar algebras of [] we get the so-called IRF models (interaction round a face). We will not in general expect convergence of the growth rate of Z_T without some kind of positivity assumption on R.

Of particular interest are the spin models. We suppose 8.1 is shaded so that the top left hand region is unshaded. Then the shaded regions form a square lattice that is rotated by $\pi/4$ with the shaded regions being the vertices of this ("semidual") lattice and the internal discs of 8.1 being the edges. The partition function would then be

$$Z_T = \sum_{\sigma} \prod_{D \in \mathfrak{D}_T} R_{i,j}$$

where now σ is a way of assigning a spin in $\{1, 2, \cdots, Q\}$ to the shaded regions which are now the "atoms". And i and j are the two spin values assigned by the state to the two atoms in the shaded regions around D. The R matrices in the disc give the Boltzmann weights of the interaction between two neighbouring atoms in their spin states given by σ (so we need the $R_{i,j}$ to be positive, or at least non-negative, for all values of (i, j)). This was the setup for the original Ising model which is the case $Q = 2$. (We consider here the case without an external magnetic field. One can be applied by inserting 2-discs diagonally between the discs of 8.1.

Any choice of positive numbers $R_{i,j}$ defines a model and many choices have been looked at. [] [] If the set of spin values has no structure then one can only allow $R_{i,j}$ to depend on whether $i = j$ or not. It is easy to see that any such R is realised by a TL element of $P^{spin}_{2,+}$ so we are led to consider

R =  $+ A$  and

This is the most general form since just multiplying R by a consant will have a simple effect on the partition function which will only change the free energy by an additive scalar.

This choice of R is known as the (isotropic) *Potts model* which we will investigate below.

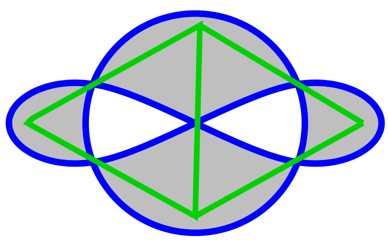
8.2 Kramers-Wannier duality.

Before Onsager solved the square lattice Ising model in [], the critical temperature was located by a simple argument which supposed the existence and uniqueness of the critical temperature. We give a planar algebra version of the argument which applies in greater generality and reposes on our concept of biinvertible element. The argument will have two components-first using a biinvertible element to identify the partition function on a planar graph with that of the dual graph, and the second identifying the graph and the dual graph for certain lattices.

As a preliminary we explain the medial (four-valent) graph for planar graphs. If T is a shaded planar 0-tangle with input discs all having four boundary points, one may form the planar graph G_T whose vertices are the shaded regions and whose edges are the input discs of T (each of which is connects a pair of (not necessarily distinct) regions. We sat that the four-valent planar graph obtained from T by shrinking all the internal discs to points is the *medial graph* for G_T.

Exercise 8.2.1. *Show that for any (finite) planar graph G there is a 0-tangle T with $G_T = G$. Show that the unshaded regions of G_T define the planar dual of G.*

We include a picture showing a planar graph together with its medial graph.



Now suppose we are given a measured shaded planar algebra (P, μ) and an element $R \in P_{2,+}$. Then we may define labelled planar 0-tangle given any planar graph G by inserting R into the input discs of the tangle T with $G = G_T$. We say that R defines a statistical mechanical model on G whose partition function$Z_{G,R}$ is $\mu(Z_T)$.

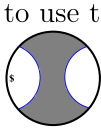
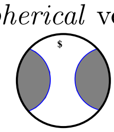
Proposition 8.2.2. *If (P, μ) is spherical and reduced with invertible loop parameter. Suppose R is flat with respect to some biinvertible, satisfying the star triangle equation with S and $\hat{G}$ is the planar dual of G then $Z_{G,R} = Z_{\hat{G},S}$.*

Proof. Form the planar tangle T giving the medial graph. $Z_{G,R}$ is μ of the element of $P_{0,+}$ obtained by putting R in all the internal discs of T. Now introduce a small closed string outside all the strings o T. This simply multiplies $Z_{G,R}$ by the loop parameter. Now the string can be passed right through the labelled tangle, producing the same tangle with the shading reversed, with all internal discs labelled by S, by flatness. By sphericality the closed string may be removed and we see precisely the picture for $Z_{\hat{G},S}$. □

We can now undertake the discussion that will give the critical temperature for the Ising model. Let G be a large square lattice with N^2 vertices and some way of closing the lattice on the boundary. Then apart from the boundary of G and $\hat{G}$, $\hat{G}$ is also a large square lattice, of the same size. If P is realised as a concrete planar algebra (e.g. P^{spin}) and all the elements of R and S are positive in some basis, it is to be expected that what happens at the boundary will have a negligible effect on the partition function. It is also expected that $\lim_{N\to\infty} \frac{1}{N^2} \log Z_{G,R}$ will exist and define a function called the *free energy per site* of the system, $F(R)$. It should not depend on boundary conditions so that we expect, by 8.2.2, $F(R) = F(S)$. But if we are in the isotropic Potts model with $R = R(A)$ then $S(A) = AR(\frac{1}{A})$ by 7.3.3 so that $Z_{G,R(A)} = (A)^{N^2} Z_{\hat{G},R(\frac{1}{A})}$ and

8.2.3. $$F(R(A)) = \log(A) + F(R(\tfrac{1}{A})).$$

The assertions about the existence of F and the independence on boundary conditions are quite reasonable and no doubt proved in considerable generality in the mathematical physics literature. But now comes the interesting part. Criticality is supposed to correspond to some kind of singularity in $F(R(A))$ but by the equation above, if a singularity occurs at some value of A it also occurs at $\frac{1}{A}$. So if we make the (big) assumption that there is a unique critical point (phase transition), it must occur when $A^2 = 1$.

We now only have to connect the parameter A with the physical parameters in the Potts model and we have the critical temperature, assuming it exists and is unique.. In P^{spin} the entries of the matrix R are the Boltzmann weights for the interactions between neighbouring spins and of course in the Potts model they only depend on whether the two spins are the same or not. A global change in the base energy will affect the free energy simply by a constant so we may assume $E(\sigma, \sigma')$ to be any fixed quantity if if $\sigma \neq \sigma'$. Now we have to be careful since in order to apply 8.2.2 we need to use the *spherical* version P^{Spin}. This has no effect on the meaning of the tangle  but  acts by $\sqrt{Q}$ times it's action in the asymmetric version of P^{spin}. This means that the tangle $R(A)$ represents Boltmann weights of $A + \sqrt{Q}$ on the diagonal and $\sqrt{Q}$ off the diagonal. These are all positive provided $A > -\frac{1}{\sqrt{Q}}$. In []Baxter it is supposed that the two Boltzmann weights are inverse to one another with the case $\sigma = \sigma'$ being e^K. This means that $\frac{A}{\sqrt{Q}} = e^{2K} - 1$ and the equation for cricicality is

$$e^{2K} = 1 + \sqrt{Q}.$$

In the Ising case, $Q = 2$ and we obtain $K = \frac{\log(1+\sqrt{2})}{2}$ in accordance with [].

The functional equation 8.2.3 relates $F(R(A))$ to $F(R(A^{-1}))$. Since A is related to K by $\frac{A}{\sqrt{Q}} = e^{2K} - 1$, if K is positive, $K \to 0$ is the same as $A \to 0$ and $K \to \infty$ is the same as $A \to \infty$. But $K = \frac{-E}{k_B T}$ so that the functional equation relates high temperature behaviour in the ferromagnetic case $E < 0$ to high temperature behaviour.

One may extend this example in many ways. First of all, structure on the graph may allow a natural assignment of different R matrices to different interactions. For a rectangular lattice In the above this could mean a value of A for horizontal interactions ($\frac{A}{\sqrt{Q}} = e^{2K} - 1$) and a value B for vertical ($\frac{B}{\sqrt{Q}} = e^{2L} - 1$). If $F(A, B)$

is the resulting rectangular lattice free energy, the extension of 8.2.2 and the above argument give immediately the functional equation

$$F(A,B) = \log(AB) + F(\frac{1}{B}, \frac{1}{A})$$

from which we see that if singularities occur on one side of the line $AB = 1$ then they occur on the other side as well, which gives the "self-dual" or "critical" Potts model equation

$$(e^{2K} - 1)(e^{2L-1}) = Q$$

first obtained by Potts [].

Another extension of the argument is to non-TL solutions of the star-triangle equation. For instance one could take the solutions we know for complex Hadamard matrices. We would like to use elements of $P^u_{2,+}$ as Boltzmann weights for a statistical mechanical model and isolate a critical or at least "self-dual" variety. But the Boltzmann weights must be positive for the model to make physical sense so we require both the R and S entries in the star triangle equation to be positive. We know all solutions for S from the proof of 7.3.7. They are simply functions that are constant on the connected components of the graph $\mathfrak{G}_u$, and all the entries in the matrix will be positive is the same as saying S is a positive function. So to determine the variety of all positive solutions we need to find all such $S(a,b)$ for which

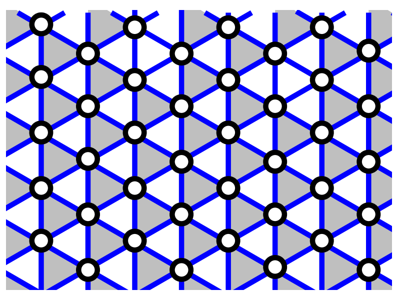
$$\sum_{i,j} S_{i,j} \bar{u}_{a,i} u_{a,j} u_{b,i} \bar{u}_{b,j} > 0 \quad \forall a, b$$

The solution space is at least two dimensional since it contains the Temperley Lieb solutions which give the Potts model but we can easily see that there is more. If we rewrite the star-triangle relation so that the side of the equation with a summation involves S, we see that the entries $R(a,b)$ are in fact the eigenvalues of $S(a,b)$ so as soon as $S(a,b)$ is positive definite as a matrix, the numbers $R(a,b)$ will be positive. On the other hand the diagonal $\{(a,a)\}$ is a connected component of $\mathfrak{G}_u$ so we may ensure positivity by diagonal dominance. As soon as the diagonal entries are large enough compared to the other entries the matrix is positive. To determine the dimension of the space of physically relevant value one thus only needs to know the action of ρ^2 on the set of mininal projections in $P_{2,-}$ or in other words on the vertices of the graph $\mathfrak{G}_u$. The exact nature of the space might be difficult to determine. One case is easy to complete and that is the case where u is the Fourier transform matrix for a finite abelian group. If G is such a group (of order Q) and we choose an isomorphism $g \mapsto \hat{g}$ from G to $\hat{G}$ then we can define $u_{g,h} = \frac{1}{\sqrt{Q}}\hat{g}(h)$. Flat elements are then given by any matrix $R_{g,h} = R(g-h)$ and the corresponding S matrix is obtained from the Fourier transform of $R(g)$. The rotation acts by sending g to g^{-1} so the dimension of the space of flat matrices with positive Boltzmann weights is easy in terms of the number of involutions in G. For instance if G is $\mathbb{Z}/5\mathbb{Z}$ the dimension is three-R must be of the form

$$R(n) = \begin{cases} r_0 & \text{if } n = 0 \\ r_1 & \text{if } n = 1 \text{ or } 4 \\ r_2 & \text{if } n = 2 \text{ or } 3 \end{cases}$$

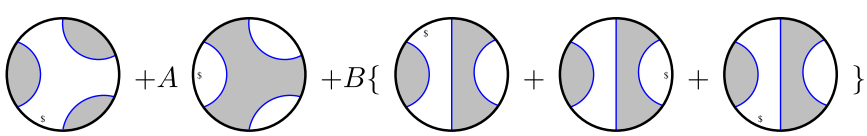
And in fact the space of all R with all Boltzmann weights positive is the intersection of the positive cone in $\mathbb{R}^3$ with $\{(r_0, r_1, r_2) | r_0 + r_1 \cos 2\pi/5 + r_2 \cos 4\pi/5 > 0\}$. CONTINUE

Another way to generalise this duality result is to use R-matrices in n-box spaces for $n > 2$. The most obvious candidate is to use TL elements and get a 3-spin interaction Potts model on a triangular lattice. Here is a picture of the lattice, ignoring boundary conditions:

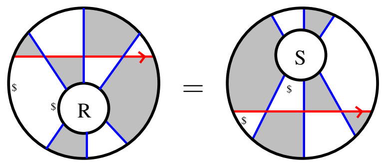


This was analysed in detail in [](Baxter Temperley Ashley, Proc. R. Soc. Lond. A 16 January 1978 vol. 358 no. 1695 535-559). We illustrate in the simple isotropic case where R is invariant under ρ^2 so there is no need to specify the position of the $'s. The geometry of the lattice could be used to accomodate an arbitrary TL element.

The internal discs are all to be filled with the same element R of TL.After normalising the energy as in the previous case we can assume

$R(A, B) =$ 

Choosing any biinvertible in P^{Spin} it is clear that $R(A, B)$ satisfies the flatness condition



with $S(A,B) = AR(\frac{1}{A}, \frac{B}{A})$. So repeating the argument of 8.2.2 and the discussion of Kramers-Wannier duality we expect the following functional equation:

$$F(A,B) = \log A + F(\frac{1}{A}, \frac{B}{A})$$

where $F(A,B)$ is the free energy per site of the large limit of the triangular lattice above where all the circles are labelled with $R = R(A,B)$. So that the self-dual and supposedly "crtitical" situation is just $A = 1$.

Relating the values of A and B to the Boltzmann weights is a bit more interesting. There are 3 Potts configurations:
(i) All three σ (in the shaded regions) distinct: call the Boltzmann weight w_3.
(ii) All three σ equal: call the Boltzmann weight w_1.
(iii) Two of the σ's equal and the other one different: call the Boltzmann weight w_2.

These give the equations:

$$\begin{aligned} w_3 =& \sqrt{Q} \\ w_2 =& \sqrt{Q} + B \\ w_1 =& \sqrt{Q} + 3B + \frac{A}{\sqrt{Q}} \end{aligned}$$

(The factors or $\sqrt{Q}$ are because of the action of tangles in P^{Spin} as opposed to P^{spin}.)

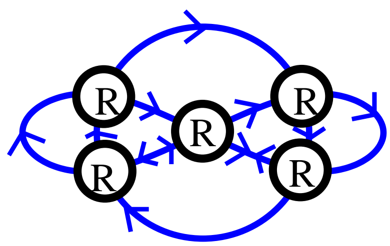
We can zero out the energy as before by dividing by w_3 and setting $\frac{w_2}{w_3} = e^K$ and $\frac{w_1}{w_3} = e^L$ to obtain the equation for self-duality:

$$e^L = 3e^K + \frac{1}{Q} - 2.$$

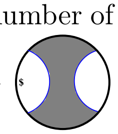
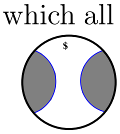
Note that in the ferromagnetic case $L > 0, K > 0$ there is a physical value of L for every K and in the antiferromagnetic case there is a physical pair (K, L)provided $|K|$ is small enough.

8.3 Temperley Lieb equivalence

If we consider a planar graph G with its medial graph as in 8.2.1 we have shown how to define a Potts model on it. We can also define an "Ice-type" model on it as follows. Orient the strings (edges of the medial graph) at each vertex of the medial graph so that two edges bounding the same shaded region are ingoing and the other two are outgoing. This orientation may or may not extend to the strings. If it does not, simply insert u^+ or u^- somewhere on that string. Blow up the crossings so that they are discs into which elements of P_2^{Ice} may be inserted. Putting $R(\theta)$ into each disc we obtain a labelled 0-tangle in P_0^{Ice}. Here is what would be obtained for the tangle of 8.2.1:



where we have suppressed the $u^{\pm}$ between the two arrows on two of the strings for clarity.

If we replace each R by $x\mathcal{E} + y1$ then we get a sum over 2^k terms, where k is the number of vertices of the medial graph. Each term contributes a power of x and y and a factor δ^r where r is the number of closed loops formed. (A closed loop must contain an even number of $u^{\pm}$ which all cancel.) But if we used P^{Spin} on the original graph and used x  $+y$  we would get exactly the same sum. This is an abstract version of Temperley-Lieb equivalence as in [],[].

8.4 The transfer matrix.

8.5 The Yang-Baxter equation.

8.6 Commuting transfer matrices.

9 Subfactor planar algebras.

Definition 9.0.1. *The canonical planar algebra of a subfactor $N \subseteq M$ is the planar algebra with $P_{n,+} = H^0_{N-N}(\otimes^n_N M)$, $P_{n,-} = H^0_{M-M}(\otimes^{n+1}_N M)$ endowed with the action of shaded tangles defined above.*