Runge Kutta methods are ways of finding numerical solutions to the ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial value \((x_0, y_0)\). The simplest such method is Euler’s method where to approximate the solution \(y\) at some value \(x\) we divide the interval \([x_0, x]\) up into intervals of equal size \(h\) (called the step size) and we define the sequence \((x_n, y_n)\), by

\[
x_{n+1} = x_n + h \\
y_{n+1} = y_n + hf(x_n, y_n).
\]

We hope that as \(h \to 0\), the calculated approximation to \(f(x)\) will tend to the actual value of \(f(x)\).

Thus in between \(x_n\) and \(x_{n+1}\) we are approximating the solution by a straight line whose slope is \(f(x_n, y_n)\). The corresponding change in \(y\) is \(hf(x_n, y_n)\).

In the special case where \(f(x, y)\) is independent of \(y\), call it \(f(x)\), the solution to the differential equation is \(y = \int f(x)dx\). Thus Euler’s method would be a numerical integration of the function \(f(x)\). The value \(hf(x_n)\) is the area of a rectangle which would occur if we were using the “left end-point” integration. Thus Euler’s method generalises the left end-point rule, a very crude method.

Runge Kutta methods use different values of \(f(x, y)\) for \(x\) in the interval \([x_n, x_{n+1}]\) to come up with an approximation of \(y_{n+1}\). The most well known of these appears to be RK4, also known as the Runge-Kutta method.

To define RK4 all we have to do is specify how \(y_{n+1}\) is obtained from \(y_n\). For this purpose one defines (with \(x_{n+1} = x_n + h\))

\[
k_1 = f(x_n, y_n) \\
k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\
k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \\
k_4 = f(x_{n+1}, y_n + hk_3)
\]

EXERCISE 1: Draw a diagram of a direction field for \(x\) between \(x_n\) and \(x_{n+1} = x_n + h\) to illustrate the meaning of the slopes \(k_1, k_2, k_3, k_4\).

Then \(y_{n+1}\) is defined to be

\[
y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\]

In the case where \(f(x, y)\) is independent of \(y\) we see that \(k_2 = k_3\) and that RK4 is nothing but Simpson’s rule!

EXERCISE 2: Use RK4 with step size 0.1 to approximate the solution to \(\frac{dy}{dx} = x + y\) with \(x_0 = 0, y_0 = 1\) for \(x = 0, 0.1, 0.2, 0.3\). Check that \(y = 2e^x - x - 1\) is the actual solution to the ODE and compare the values with those calculated by RK4 and Euler’s method (values calculated in the book, page 576).