The Schroeder-Bernstein Theorem

Suppose

\[ H : Z \to Z \]

is a 1-to-1 function. For each \( a \in Z \), the orbit of \( a \) is the smallest subset of \( Z \) which is closed under \( H \) and which contains the point \( a \). The next three easy lemmas refer to \( H \) and \( Z \). For each \( a \in Z \), we let \( O_a \) denote the orbit given by \( a \).

**Lemma 1.** Suppose \( a, b \in Z \) and let \( O_a \) and \( O_b \) be the orbits given by \( a \) and \( b \) respectively. Then either \( O_a = O_b \) or \( O_a \cap O_b = \emptyset \).

*Proof.* Exercise. \( \square \)

**Definition 2.** Let \( O_a \) be the orbit given by \( a \). A point \( b \in O_a \) is an Eve point if \( b \) is not in the range of \( H \).

An orbit cannot have two different Eve points.

**Lemma 3.** Suppose \( a \in Z \) and \( O_a \) is the orbit given by \( a \). If \( O_a \) has an Eve point then that point is unique.

*Proof.* Exercise. \( \square \)

**Lemma 4.** Suppose \( a \in Z \) and \( O_a \) is the orbit of \( a \) given by \( H \). Let \( H_a \) be the restriction of \( H \) to \( O_a \) so that

\[ H_a : O_a \to O_a \]

Then \( O_a \) has no Eve point if and only if \( H_a \) is a bijection from \( O_a \) to \( O_a \).

*Proof.* Exercise. \( \square \)

**Example**

(1) Let \( Z = \mathbb{R} \) and let \( H \) be the function \( H(x) = x^3 \). Find all the \( a \in \mathbb{R} \) such that \( O_a \) is finite. Does every orbit have an Eve point?

(2) Let \( Z = [0, \infty) \) and let \( H \) be the function \( H(x) = x^2 \). Does every orbit have an Eve point? Find all the finite orbits.

(3) Let \( Z = [0, \infty) \) and let \( H \) be the function \( H(x) = e^x \). Is any orbit finite? Does any orbit have an Eve point? \( \square \)
Theorem 5 (Schroeder-Bernstein). Suppose that $|X| \leq |Y|$ and $|Y| \leq |X|$. Then $|X| = |Y|$. 

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be 1-to-1 functions. We must produce a bijection, $h : X \rightarrow Y$.

Let $F : X \rightarrow X$ be the function $g^{-1} \circ f$ and let $G : Y \rightarrow Y$ be the function $f^{-1} \circ g$. Thus $F$ is a 1-to-1 function from $X$ to $X$ and $G$ is a 1-to-1 function from $Y$ to $Y$.

For each $a \in X$, let $O^X_a$ be the orbit of $a$ given by $F$ and for each $b \in Y$, let $O^Y_b$ be the orbit of $b$ given by $G$.

Note:

(1.1) For each $a \in X$, for each $c \in O^X_a$, $f(c) \in O^Y_{f(a)}$.

(1.2) For each $b \in Y$, for each $d \in O^Y_b$, $g(c) \in O^X_{g(b)}$.

Note:

- Suppose $a \in X$. Then $O^X_a$ has no Eve point if and only if $O^Y_{f(a)}$ has no Eve point.

Similarly,

- Suppose $b \in Y$. Then $O^Y_b$ has no Eve point if and only if $O^X_{g(b)}$ has no Eve point.

Fix $a \in X$. There are essentially three key observations (verify them!).

(I) Suppose $O^X_a$ has no Eve point. Then the restriction of $f$ to the orbit $O^X_a$ is bijection with the orbit $O^Y_{f(a)}$.

The next two key observations concern the situation where $O^X_a$ has an Eve point. Let $c$ be the (unique) Eve point of $O^X_a$.

(II) If $c$ is in the range of $g$ then $g^{-1}$ gives a bijection of $O^X_a$ with $O^Y_{f(a)}$.

(III) If $c$ is not in the range of $g$ then $f$ gives a bijection of $O^X_a$ with $O^Y_{f(a)}$. 

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Now we can easily define the desired bijection
\[ h : X \to Y \]
by defining \( h \) on each orbit \( O^X_a \) where \( a \in X \).

(4.1) If \( O^X_a \) has no Eve point then for each \( d \in O^X_a \), \( h(d) = f(d) \).

(4.2) If \( O^X_a \) has an Eve point and that Eve point is in the range of \( g \) then for each \( d \in O^X_a \), \( h(d) = g^{-1}(d) \).

(4.3) If \( O^X_a \) has an Eve point and that Eve point is not in the range of \( g \) then for each \( d \in O^X_a \), \( h(d) = f(d) \).

It follows that \( h \) is a bijection from \( X \) to \( Y \) (show this!). \( \square \)