Second homework assignment.

1) Let $m$ be a non-zero natural number and define the relation $x \equiv y$ on $\mathbb{Z}$ by $x \equiv y$ iff $x - y$ is a multiple of $m$. Show that $\equiv$ is an equivalence relation and identify the equivalence classes. Show that one may define addition and multiplication on the set $\mathbb{Z}/\equiv$ of equivalence classes by

$$\bar{a} + \bar{b} = \bar{a + b}$$

and

$$\bar{ab} = \bar{ab}.$$ 

Show that the resulting structure is not a field if $m$ is not prime.

2) It was shown in the last homework that every positive real $x$ has a unique positive $n$th root $x^{\frac{1}{n}}$ for every $n \in \mathbb{N}$ Show that

$$(x^{\frac{1}{m}})^n = (x^n)^{\frac{1}{m}}$$

so we may talk unambiguously about $x^q$ for all $q \in \mathbb{Q}$.

3) If $A$ is a bounded set of real numbers and if $|a - b| < 1$ for all $a, b \in A$, show that $\sup A - \inf A \leq 1$. Does it also follow that $\sup A - \inf A < 1$?

4) If a sequence $(q_n)$ has the property that $|q_n - q_{n+1}| \leq r|q_{n-1} - q_n|$, for some $r, 0 < r < 1$, show that $q_n$ is Cauchy.

5) Show any (algebraic) automorphism of a complete ordered field is the identity, i.e. if $\alpha : \mathbb{R} \to \mathbb{R}$ is a bijection with $\alpha(x + y) = \alpha(x) + \alpha(y)$ and $\alpha(xy) = \alpha(x)\alpha(y)$ then $\alpha(x) = x$ for all $x \in \mathbb{R}$. (Hint show it first for $\mathbb{Q}$ then use the fact that positive numbers are squares.)