Coherent-constructible correspondences and log-perfectoid mirror symmetry for the torus

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Abstract

Suppose k is an arbitrary field (of any characteristic) and X is a smooth complete toric variety, which is a compactification of the torus $T := (\mathbb{G}_m)^n$. Let $S := (S^1)^n$ be a topological torus on the Langlands dual lattice to T. The N-power map $P^N : T \to T$ extends to a ramified power map $P^N : X \to X$, giving a tower of spaces $X^{(N)}$ over X all isomorhpic to X. The "perfectoid completion", $X^{\mathrm{perf}} := \varprojlim X^{(N)}$ (originally defined in a local context, but for toric varieties still defined over k) has non-finite type scheme structure, first studied (in a local context) by Scholze in the context of perfectoid spaces. The space X^{perf} "compactifies" the completed universal cover T^{perf} of the torus T/k, and in particular has action by the geometric Galois group, Γ_T of T, with quotient stack $X^{\mathrm{perf}}/\Gamma_T$ "compactifying" $T(= T^{\mathrm{perf}}/\Gamma_T)$ in a Galois-compatible way. The function sheaf $\mathbb{O}(X^{\mathrm{perf}})$ is equipped with a unipotent boundary ideal, a_∂, and using the theory of "almost mathematics" of Faltings and Gabber-Ramero we define a category of "almost quasicoherent sheaves" $\mathrm{Qcoh}^a(X^{\mathrm{perf}})$, and its Γ_T-equivariant version, $\mathrm{Qcoh}^a(X^{\mathrm{perf}})^{\Gamma_T}$. We show that, regardless of the compactification X of the torus, the equivariant almost quasicoherent category $\mathrm{Qcoh}^a(X^{\mathrm{perf}})^{\Gamma_T}$ is derived equivalent to the category $\mathrm{Shv}(S)$ of all topological sheaves on S (no constructibility restrictions). Via results of Nadler-Zaslow and others, this should be thought of as a version of the Fukaya category of the mirror of the algebraic variety T, which is the symplectic torus $T^\vee := T^*(S)$. Based on this result we argue that the derived category $D^b\,\mathrm{Qcoh}^a(X^{\mathrm{perf}})^{\Gamma_T}$ (which is independent of choice of compactification X) should be thought of as the correct compromise between (quasi)coherent sheaves on T and compactly supported sheaves on T, both of which are studied but have some complementary deficiencies in the context of mirror symmetry. We make some conjectures which generalize this mirror symmetry statement in the wider SYZ context. By incorporating singular support conditions on X into the equivalence, we deduce a strong generalization of the coherent-constructible correspondence conjecture of [FLTZ]. Along the way, we show that the category of quasicoherent sheaves on X is equivalent (as an Abelian category) to the category of sheaves on a certain Grothendieck topology $\overline{\Theta}$ related to S, a fact first observed without proof by James Dolan.

1 Introduction

1.1 Overview and motivation

Homological Mirror Symmetry studies equivalences between derived categories associated to algebraic and differential-geometric objects. The original formulation (due to Kontsevich, [Kontsevich]) conjectures that to any (suitably nice) algebraic variety X one can associated a "mirror" symplectic variety $X^\vee$ together with an equivalence of dg categories $D^b\operatorname{Qcoh}(X) \cong \operatorname{Fuk}(X^\vee)$: here $D^b\operatorname{Qcoh}(X)$ is the derived ategory of quasicoherent sheaves on X and $\operatorname{Fuk}(X^\vee)$ is the (dg enhancement) of the A_∞ (idempotent completed) Fukaya category associated to $X^\vee$. In practice, known instances of mirror symmetry are variations on this theme, with $D^b\operatorname{Qcoh}(X)$ and $\operatorname{Fuk}(X^\vee)$ modified in some way.

1.1.1 Mirror symmetry for the cylinder

Let $\mathbb{C}^*_{sym}$ be the cylinder, viewed as a symplectic manifold with Kähler form $\omega := d|z| \wedge d\theta$, for $|z|, \theta$ the absolute value and angle coordinates. The mirror of $\mathbb{C}^*_{sym}$ is the algebraic variety $\mathbb{C}^*_{alg}$ (or, over more general fields, $\mathbb{G}_m$), with functoriality with respect to scalar multiplication making $\mathbb{C}^*_{sym}, \mathbb{C}^*_{alg}$ naturally into dual tori. Mirror symmetry results take the form of the statement that a Fukaya category on $\mathbb{C}^*_{sym}$ is derived equivalent to a coherent category on $\mathbb{C}^*_{alg}$. However, since $\mathbb{C}^*_{sym}, \mathbb{C}^*_{alg}$ are open, there is some ambiguity to which category should be taken on either side. Namely, on $\mathbb{C}^*_{sym}$ there are two standard natural Fukaya categories. First, the *compact* Fukaya category $\operatorname{Fuk}\mathbb{C}^*_{sym}$ has objects exact compact Lagrangians $L \subset \mathbb{C}^*_{sym}$ together with a local system. The *wrapped* Fukaya category, $\operatorname{Fuk}_{ot}\mathbb{C}^*_{sym}$ has objects all Lagrangians which are allowed to go to ∞, and must be deformed to wrapp around at ∞ in a certain asymptotically periodic way when composed. The known mirror symmetry statements for the cylinder take the following form:

Theorem 1. *1. The (idempotent-completed dg enhancement of) the compact Fukaya category of the cylinder $\mathbb{C}^*_{sym}$ is equivalent to the derived category of coherent sheaves on $\mathbb{C}^*$ with bounded and compactly-supported cohomology.*

2. The (idempotent-completed dg enhancement of) the wrapped Fukaya category of the cylinder $\mathbb{C}^*_{sym}$ is equivalent to the derived category of all coherent sheaves on $\mathbb{C}^*$.

There are some problems with both categories. Namely, the compact Fukaya category does not remember as much geometric information about the symplectic variety $\mathbb{C}^*_{sym}$ as one would like, and the wrapped Fukaya category loses some of the finite-dimensionality and symmetry that the ordinary Fukaya category has. Now since $\mathbb{C}^*_{sym}$ is naturally a cotangent bundle, $\mathbb{C}^*_{sym} \cong T^*S^1$, there is a third category, defined by Nadler and Zaslow in [**?**], which interpolates between the compact and wrapped versions. We call this category the *microlocal Fukaya category*, $\operatorname{Fuk}_\mu(\mathbb{C}^*_{sym})$; its objects are Lagrangians with some conicity conditions at ∞. Conjecturally, such a category can be defined more generally for any symplectic manifold which admits a Lagrangian skeleton: for an informal definition for Weinstein manifolds, see [**?**]. It is reasonable to ask now whether there is a new *coherent* category of sheaves on $\mathbb{C}^*_{alg}$ which is mirror symmetric to $\mathrm{Fuk}_\mu(\mathbb{C}^*_{sym})$. In this paper, we define such a category, which we call the *log-perfectoid* category of coherent sheaves.

1.1.2 Coherent-constructible correspondence: the Nadler-Zaslow approach to the n-dimensional cylinder

One way to interpolate between the two different coherent categories above (compactly supported sheaves on $\mathbb{G}_m$ and all sheaves on $\mathbb{G}_m$) is to study sheaves on the (unique) compactification $\mathbb{P}^1$ of $\mathbb{G}_m$. It turns out that the derived category of sheaves on $\mathbb{P}^1$ is indeed related to a certain modified Fukaya category for the cylinder. This approach becomes more interesting at higher dimensions, where the cylinder $\mathbb{C}^*_{sym} = T^*S^1$ is replaced by $(T^*S^1)^n$ and the algebraic mirror is replaced by $\mathbb{G}_m^n$. Then the compactification is no longer unique, and the most well-studied compactifications are toric varieties. This gives us a whole family of flavors of mirror symmetry statements for $(T^*S^1)^n$ parametrized by toric varieties. As it turns out, for each (smooth and compact) toric variety X compactifying $\mathbb{G}_m^n$ there is a fully faithful functor from the derived category $D^b(\mathrm{coh}(X))$ to the microlocal Fukaya category, $\mathrm{Fuk}_\mu(T^*S^1)^n$. The image of this functor should then be cut out by a combinatorial boundary condition on Lagrangians in $(T^*S^1)^n$. One very fruitful point of view on the resulting theory originates from a very old observation of Bondal, [**?**], that the derived category of coherent sheaves on an n-dimensional toric variety X can be faithfully embedded in the category of constructible sheaves on the topological torus $S := (S^1)^n$ (proved by [**?**]). Now it follows from [**?**], the category of constructible sheaves on S is equivalent to a certain Fukaya category on $T^*S \cong (T^*S^1)^n$, and boundary conditions on this Fukaya category translate into *singular support conditions* (a notion imported from microlocal analysis) on constructible sheaves on $(S^1)^n$. This point of view results in a constellation of results and conjectures known as the *coherent-constructible correspondence conjectures* (ccc conjectures for short), announced with sketch of proof in an early version of this work, [**?**] and proved by Kuwagaki, [**?**].

1.1.3 Some motivation: coherent-constructible correspondence at the highly ramified limit

From now on we assume access to (possibly a variant of) the philosophy of [**?**] to identify the Fukaya category of the n-dimensional symplectic torus T^*S with some category of topological sheaves on S itself. The fact that toric *compactifications* of the torus $\mathbb{G}_m^n$ give different full derived subcategories of $\mathrm{Constr}(S)$ is suggestive: perhaps there is some universal, or "limit compactification" of $\mathbb{G}_m^n$ which sees the entire category of constructible, or even all topological, sheaves on S. One candidate is to take the limit of all toric varieties (which naturally form a projective system by fan refinement), but this is not quite good enough: for example, in the one-dimensional context, there is a unique compactification ($\mathbb{P}^1$) and it only "sees" constructible sheaves on $S = S^1$ which are locally constant outside the basepoint $1 \in S^1$. We can do better by extending our world from toric varieties to toric stacks. Namely, suppose (for now) that the basefield k is algebraically closed of characteristic 0. Then for any positive integer N, the group $\mathbb{Z}/N$ acts on both $\mathbb{G}_m$ and on $\mathbb{P}^1$ by multiplication by roots of unity. Consider the unique étale N-fold cover $\mathbb{G}_m^{(N)}$ of $\mathbb{G}_m$. It is isomorphic to $\mathbb{G}_m$, but it will be convenient for us to differentiate it notationally to think of it as a cover of the original $\mathbb{G}_m$ on which we want a mirror symmetry result. Now the map $\mathbb{G}_m^{(N)} \to \mathbb{G}_m$ is the projection map induced by factoring out the action of $\mathbb{Z}/N$ on $\mathbb{G}_m^{(N)}$. Let $(\mathbb{P}^1)^{(N)}$ (also isomorphic to $\mathbb{P}^1$ but thought of as fibering over it) be the unique smooth compactification of $\mathbb{G}_m^{(N)}$. It maps to $\mathbb{P}^1$ with n-fold ramification at the boundary points, $0, \infty$. Now we can study the quotient stack $(\mathbb{P}^1)^{(N)}/(\mathbb{Z}/N)$, which compactifies the free quotient stack $\mathbb{G}_m^{(N)}/(\mathbb{Z}/N) \cong \mathbb{G}_m$. The category of coherent sheaves on this stack will simply be the category of $\mathbb{Z}/N$-equivariant sheaves on the ramified cover $(\mathbb{P}^1)^{(N)}$ (isomorphic to $\mathbb{P}^1$). The generalization of the coherent-constructible correspondence in this context says that the derived category of sheaves on $(\mathbb{P}^1)^{(N)}/(\mathbb{Z}/N)$ is derived equivalent to the category of constructible sheaves on S^1 which are locally constant outside the N roots of unity in S^1. As $N \to \infty$, the tower of orbifolds $(\mathbb{P}^1)^{(N)}/(\mathbb{Z}/N)$ forms a projective system. One would naively expect the category of coherent sheaves on this limit (a priori, a notion that can be defined in many diffent ways) to be derived equivalent to the category of sheaves "with exceptional fibers at $\mathbb{Q}/\mathbb{Z} \subset S^1$", which from the point of view of constructible sheaves is a nonsense object. Miraculously, there is a way to make sense of such a construction to get a much more complete result.

1.1.4 Inverse systems, almost quasicoherent categories and the main theorem

Note that given a family of surjective maps of affine varieties $X_0 \leftarrow X_1 \leftarrow \dots$, the projective limit can be understood as an honest affine variety, albeit not of finite type. Namely, the rings $\mathbb{O}(X_0) \to \mathbb{O}(X_1) \to \dots$ are a sequence of injective maps of commutative rings, and we can formally define $\underleftarrow{\lim} X_i := \operatorname{Spec}(\underrightarrow{\lim} \mathbb{O}(X_i))$ to be the spectrum of the direct limit of these rings. A similar result holds for group schemes, and the inverse limit of the group schemes $\mathbb{Z}/N = \operatorname{Spec}(k[\mathbb{Z}/n])$ is the group scheme

$$\mathbb{Z}/\infty := \operatorname{Spec}(k[\mathbb{Q}/\mathbb{Z}]).$$

This is the geometric version of the algebraic fundamental group of $\mathbb{G}_m$, and acts on the universal cover of $\mathbb{G}_m$, which is $\underleftarrow{\lim} \mathbb{G}_m^{(N)} := \operatorname{Spec}(k[\mathbb{Q}])$ (note that we are taking finite polynomials with exponents in $\mathbb{Q}$, not series). Similarly, the inverse limit of ramified covers $(\mathbb{A}^1)^{(N)}$ has an affine limit,

$$(\mathbb{A}^1)^{(\infty)} := \operatorname{Spec} k[\mathbb{Q}^{\geq 0}].$$

Applying this construction locally (on the upper and lower hemispheres), we get a ramified limit $(\mathbb{P}^1)^{(\infty)}$. Now the affine object $(\mathbb{A}^1)^{(\infty)}$ (or, more often, a completion of its pro-p piece) comes up in number theory, where one associates to it a new category of sheaves using an algebraic theory (originally studied by Faltings, [Faltings]) called "almost mathematics". The map of rings $k[\mathbb{Q}^{\geq 0}] \to k$ taking a polynomial to its constant coefficient gives a point $0 \in (\mathbb{A}^1)^{(\infty)}$. Given a vector space V, it gives a skyscraper sheaf δ_V over 0 on $(\mathbb{A}^1)^{\infty}$ (with $k[\mathbb{Q}^{\geq 0}]$ action factoring through the unit coefficient). This is a Serre subcategory, Qcoh_∂ (something which is not true at any finite level), and in particular we have a well-defined quotient category, which is known as the "almost quasicoherent category" $\mathrm{Qcoh}^a((\mathbb{A}^1)^{(\infty)}) := \mathrm{Qcoh}((\mathbb{A}^1)^{(\infty)}) / \mathrm{Qcoh}_\partial$. It was first observed by Faltings that, in the context of number theory, the category of almost quasicoherent sheaves $\mathrm{Qcoh}^a((\mathbb{A}^1)^{(\infty)})$ is in many ways better behaved than the category of all quasicoherent sheaves (and, for some applications, even better than the category of coherent sheaves at any finite stage, because the variety $(\mathbb{A}^1)^{(\infty)}$ over a local field will be *perfectoid*). A similar category makes sense for $(\mathbb{P}^1)^{(\infty)}$ (quotient out the Serre subcategory of skyscraper sheaves at 0 and ∞). This category has action by the geometric fundamental group $\mathbb{Z}/\infty$, and it makes sense to consider the equivariant category, which we think of as the category of almost quasicoherent sheaves on the orbifold,

$$\mathrm{Qcoh}^a(\mathbb{P}^1_{\text{orb-}\infty}) := \mathrm{Qcoh}^a\left((\mathbb{P}^1)^{(\infty)}/(\mathbb{Z}/\infty)\right):$$

the "correct" inverse limit of the quasicoherent categories of $(\mathbb{P}^1)^{(n)}/(\mathbb{Z}/n)$.

Our main theorem in the one-dimensional case is now, in terms of this notation, as follows.

Theorem 2. *We have an equivalence of derived categories*

$$D^b\,\mathrm{Qcoh}^a(\mathbb{P}^1_{orb\text{-}\infty}) \cong D^b\,\mathrm{Shv}(S^1),$$

where $\mathrm{Shv}(S^1)$ *is the category of all topological sheaves on* S^1.

It turns out that taking the inverse limit over all N is overkill, and we can write down a similar statement for the tower of ramified covers $(\mathbb{P}^1)^{(p^n)}$ for powers of p, or any other infinite tower of this sort. In the higher-dimensional case, the result will turn out to be independent not only on the tower of ramified covers, but also (on the level of derived categories) on the choice of toric compactification X of $\mathbb{G}_m^n$. Namely, for any toric variety X and any integer N there is a canonical variety $X^{(N)}$ over X, which is isomorphic to X and extends the map $t \mapsto t^n : T \to T$ on the open orbit. This map has action by the N-torsion group $(\mathbb{Z}/N)^n$ in the torus. In the limit, we get a variety $X^{(\infty)}$ which has an action by the geometric fundamental group $(\mathbb{Z}/\infty)^n$ of $T = \mathbb{G}_m^n$. It also has a well-defined category $\mathrm{Qcoh}^a(X^{(\infty)})$ of almost-quasicoherent sheaves, which is the Serre quotient by sheaves pushed forward from the boundary variety. We once again write down the quotient $X^{(\infty)}/(\mathbb{Z}/\infty)^n := X_{\mathrm{orb}-\infty}$ which compactifies T, and define the category $D^b\,\mathrm{Qcoh}^a(X_{\text{orb-}\infty})$ as the category of equivariant objects in $D^b\,\mathrm{Qcoh}^a(X^{(\infty)})$. Now in the general case, we have the following theorem.

Theorem 3. *We have an equivalence of derived categories*

$$D^b\,\mathrm{Qcoh}^a(X_{orb\text{-}\infty}) \cong D^b\,\mathrm{Shv}(S),$$

where $\mathrm{Shv}(S)$ *is the category of all topological sheaves on* S.

(See Theorem 5 for a more precise statement).

In particular, the derived category $D^b\,\mathrm{Qcoh}^a(X_{\text{orb-}\infty})$ does not depend on the choice of compactification X of the torus. (Note however that the *Abelian* category $\mathrm{Qcoh}^a(X_{\text{orb-}\infty})$ does depend on the specific compactification X.)

Our proof of the coherent-constructible-correspondence in this context uses the toric structure on X. But the set-up for the category $\mathrm{Qcoh}^a(X_{\text{orb-}\infty})$ does not. Indeed, say $U \subset X$ is any subvariety with normal crossings (or, more generally, locally toric) boundary and with geometric fundamental group surjecting onto the local (monodromy) fundamental group near any point of the boundary. Then given a cofiltration $\Gamma_1 \leftarrow \Gamma_2 \leftarrow \dots$ of the geometric fundamental group $\pi_1^g(U)$ by finite quotients, we get a sequence of covers $U^{(1)} \leftarrow U^{(2)} \leftarrow \dots$. It is not hard to see that there is a unique extension of $U^{(i)}$ to a maximally ramified cover $X^{(i)}$ locally of toric type. We now obtain a quasicoherent category by the perfectoid-inspired procedure:

1. Take an inverse limit $X^{(\infty)}$ of the $X^{(N)}$, viewed as a (non-finite type) scheme and similarly view the inverse limit Γ_∞ of the Γ_N as a group scheme.
2. Consider the category of "almost coherent" sheaves, defined as a quotient of $\mathrm{Qcoh}(X^{(\infty)})$ by sheaves pushed forward from the boundary.
3. Consider the category of Γ_∞-equivariant objects in $\mathrm{Qcoh}^a(X^{(\infty)})$.

The output of this procedure is a category $\mathrm{Qcoh}_{\text{log-perf}}(U, X)$ which we call the "log-perfectoid" category. We conjecture that this is the right object to consider on the algebraic side in more general microlocal mirror symmetry conjectures (when the B side involves an open variety and the A side has a symplectic skeleton).

1.2 Coherent-constructible correspondence: history and context

A consequence of the main result of this paper is a new proof of the coherent-constructible correspondence conjecture of [FLTZ], which should be attributed to Tatsuki Kuwagaki, [K] (and to [FLTZ] in the equivariant coherent case). In fact, the present paper grew out as an extension of the draft [V] on the author's website, which announced and sketched an earlier incomplete proof of the coherent-constructible correspondence. The author hopes to apologize for the inconvenient state of the literature this caused. Here we sketch a brief history of the problem outside the context of mirror symmetry.

The original paper [FLTZ] proves a general coherent-constructible correspondence (from now on, "ccc") statement in a very slightly modified version. Namely, the action of T on X lets us introduce a new category related to $\mathrm{Qcoh}(X)$: the category of torus-equivariant quasicoherent sheaves, $\mathrm{Qcoh}^T(X)$. The corresponding modification on the constructible side turns out to be replacing the topological torus $S = (S^1)^n$ with its universal cover $\widetilde{S}$ (a real affine space) and the relevant singular support submanifold $\Lambda \subset T^*S$ by its pullback $\widetilde{\Lambda}$, and Bondal's coherent-constructible functor from [B] can be adapted to the equivariant case. The analogue of the coherent-constructible correspondence becomes the "equivariant coherent-constructible correspondence", an equivalence

$$D^b \operatorname{Qcoh}^T(X) \cong D^b \operatorname{Constr}_{\widetilde{\Lambda}} \widetilde{S}. \tag{1}$$

The paper [FLTZ] shows that when X is smooth and projective, the equivariant ccc correspondence holds after imposing some finiteness conditions on both sides. Namely, they show that

$$D^b \operatorname{coh}^T(X) \cong D^b \operatorname{Constr}^{\mathrm{fin}}_{\widetilde{\Lambda}} \widetilde{S}. \tag{2}$$

Here the left hand side is the category of complexes of (equivariant quasicoherent) sheaves on X with coherent cohomology and the superscript "fin" on the right means that we study complexes of constructible sheaves which have cohomology with finite-dimensional fibers which are 0 outside a compact set. Remarkably, the full coherent-constructible correspondence can be deduced easily (see e.g. section 11 of the present paper) from the equivariant coherent-constructible correspondence if one were able to drop the finiteness conditions above, i.e. if one were able ot extend the known equivalence (2) to the conjectural (1) (this is in particular the point of view adopted in the present paper). The paper [FLTZ] introduced the basic language of affine polyhedral cones and polytopes which have been subsequently used in all work on toric varieties including the present one. It would not be inaccurate to say that the proof given here for the coherent-constructible correspondence can be boiled down to "[FLTZ] + some considerations of derived limits at infinity" (see also [V]).

Special cases of the (nonequivariant) ccc conjecture have been worked out in [FLTZ, Treumann, 2-dim case, SS]. A general proof (also deducing the non-equivariant ccc conjecture from the equivariant (1) above) was first announced and sketched in the preprint [V]. This preprint used a combination of two methods: a notion of derived descent datum and a geometric deformation argument for deforming the fiber functor to a polytope. The first complete proof of the result was written down in the Kuwagaki's paper [K], which proved a very general statement (for possibly singular and stacky toric varieties) using a slightly different method. Namely, for smooth varieties, Kuwagaki used induction from the affine smooth case using derived glueing and desingularization, and used a convolution argument to control singular support (in place of the deformation argument of [V]). A more recent paper, [Z], extended the deformation argument from [V] into another proof. The present paper grew out of a revision of the paper [V] with the goal of incorporating the two arguments of [V] into a more elegant Grothendieck-topological picture. The derived descent method is replaced by an equivalence of (Abelian, not derived) categories between coherent sheaves on the toric variety X and topological sheaves on a certain Grothendieck topology Θ. The deformation method is replaced by a fully faithful embedding of the category of quasicoherent sheaves on the toric variety X into a new category of quasicoherent sheaves on a certain "large" non-Noetherian scheme (much larger than $X^{(\infty)}$). Namely, the category $\mathrm{Qcoh}^a(X_\Sigma^{Nov})^{S^\sim}$ of "almost local" (once again in the sense of [Faltings]'s "almost mathematics") sheaves with a certain equivariance condition on the *toric Novikov variety* X_Σ^{Nov} associated to a toric fan Σ (what in [V] was the deformation procedure translates in this context to applying the adjoint functor to this embedding). The coherent-constructible correspondence statement for the toric variety X now follows from a statement for this new category with no singular support condition: namely, we prove that it is equivalent to the category of *all* topological sheaves on the torus S, i.e.

$$\mathrm{Qcoh}^a(X_\Sigma^{Nov})^{S^\sim} \cong \mathrm{Shv}(S).$$

1.3 Topos-theoretic interpretation of coherent categories

The guiding principle of the present paper is to replace algebraic categories and functors with topos-theoretic ones using (Abelian) categories of sheaves on Grothendieck sites. (The fact that constructible sheaves on a toric variety is a Grothendieck topos was first observed without proof in the note [**?**], and interestingly, implies exotic tensor product structures on the categories we consider.) While many of the arguments can be interpreted using modern derived theory with derived glueing of categories replacing topoi (similarly to [**?**]), the point of view of sheaves on sites seems essential (or at least very convenient) for certain "analysis-style" limit arguments: especially, for comparing categories in the "Novikov" and "perfectoid" contexts where the relevant equivalences of Grothendieck topologies simply falls out of the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ (see Lemma 39).

Recall that a Grothendieck topology (also known as a "site") is a category-theoretic extension of the notion of topological space with just the right axioms to be able to define a sheaf condition. Grothendieck initially introduced Grothendieck topologies (and the associated theory of topoi) to work with étale sheaves. Recall that étale opens on a topological manifold M are maps $U \to M$ which interpolate between the notions of open embedding $U \hookrightarrow M$ and covering space $\widetilde{M} \twoheadrightarrow M$. Grothendieck observed that sections of a sheaf on different étale U (defined as global sections of the pullback) satisfy a sheaf condition with respect to "étale covers" (surjective étale maps $\bigsqcup U_i \to X$). He formalized the data of étale opens and étale covers into the category-theoretic notion of "site" (here called Grothendieck topology). All functorial assignments

$$(\text{étale open}) \to \text{vector space}$$

satisfying the sheaf condition with respect to étale covers are called *sheaves* in the étale topology. For an algebraic variety X, one can similarly study sheaves, either in the Zariski topology or in the étale topology (where the étale maps $U \to X$ are also required to be algebraic). Grothendieck observed that for smooth manifolds, sheaves in the étale topology coincide with sheaves in the ordinary topology (with respect to open subsets and open covers). But in the context of algebraic geometry, sheaves in the étale topology are in fact very different from sheaves in the Zariski topology, and (in the case where the base field is $\mathbb{C}$) are a much better approximation (especially for doing homological algebra) to the category of topological sheaves on the underlying space.

In section 5, we introduce a new pair of Grothendieck topologies Θ, $\overline{\Theta}$ and a pair of equivalences of (Abelian) categories

$$\mathrm{Qcoh}^T(X) \cong \mathrm{Shv}(\Theta)$$

$$\mathrm{Qcoh}(X) \cong \mathrm{Shv}(\overline{\Theta}).$$

Since coherent sheaves are fundamentally (at least locally) an algebraic object, it is remarkable that they admit an interpretation in this topological language, and this algebro-topological relatinoship should be understood as a mirror symmetric phenomenon at the Abelian cataegory level: the more so because the Grothendieck topology $\overline{\Theta}$ is closely related to the étale topology on the topological manifold S involved in the coherent-constructible correspondence (and Θ, correspondingly, is related to the ordinary topology on $\widetilde{S}$).

We briefly sketch how the topologies $\Theta, \overline{\Theta}$ are defined, and how they produce a mirror symmetry functor. If X is an n-dimensional toric variety and L is an ample line bundle on X, X inherits a Kähler structure from the projective embedding associated to the full pencil for L. Given the Kähler structure together with the toric action gives a polytope $\Delta_L \subset \mathbb{R}^n$: the image of the moment map on X (viewed as a symplectic variety) associated with the (real) Lie algebra action. These polytopes can be interpreted in a more algebraic way, as the convex hulls in $\mathbb{R}^n$ of the weights of (homogeneous) sections of L. The key observation which relates $\mathrm{Qcoh}^T(X)$ with topos theory is the fact that, for fixed $\mathcal{F}$, the assignment $\mathcal{F} \mapsto \mathrm{Hom}_{\mathrm{Qcoh}^T}(L, \mathcal{F})$ behaves in a very sheaf-like way with respect to Δ_L as we vary L. Indeed, if a collection of polytopes $\Delta_{L_1}, \ldots, \Delta_{L_k}$ happen to cover another moment polytope Δ_L , then so long as for each collection of indices $i_1, \ldots, i_k$

$$\text{each intersection } \bigcap_j \Delta(L_{i_j}) \text{ is itself a moment polytope } \Delta_{L_{i_1,\ldots,i_d}} \tag{P1}$$

(something that happens in a positive density of cases), then for any equivariant sheaf $\mathcal{F} \in \mathrm{Qcoh}^T(X)$ we have

$$\text{the diagram } \mathrm{Hom}_{eq}(L, \mathcal{F}) \to \bigoplus_i \mathrm{Hom}(L_i, \mathcal{F}) \rightrightarrows \mathrm{Hom}(L_{ij}, \mathcal{F}) \text{ is an equalizer.} \tag{P2}$$

Implicit in this statement is that the arrows above can be canonically defined: this follows from the fact that equivariant line bundles on X (up to isomorphism) are in bijection with equivariant Cartier divisors, which form a partially ordered set.

The property (P1) is too restrictive to turn the poset of moment polytopes into a Grothendieck topology. Instead what we do is weaken the condition (P1) and consider *all* equivariant line bundles on X. In this category diagrams of the form (P2) have a natural generalization, and this turns out to be a *bona fide* Grothendieck topology. For our purposes, it is convenient to extend the poset of equivariant Cartier divisors to a larger poset, where we allow some components of multiplicity of $+\infty$ (but not $-\infty$). We denote these (Cartier) "quasidivisors," QDiv. Such "divisors" are still associated with line-bundle-like sheaves (though now possibly quasicoherent rather than coherent), and when the resulting sheaf is ample its moment polytope is a convex *unbounded* polytope in $\mathbb{R}^n$ (thought of as having some faces at ∞). We write the resulting Grothendieck topology (QDiv, Θ) or Θ for short. A very useful property of the topology Θ is "existence of (enough) points", which says that this category possesses a collection of stalk functors which are conservative (i.e. which distinguish isomorphisms of modules). The "mirror" to the stalk functors are functors from the category $\mathrm{Qcoh}^T(X)$ to Vect, which turn out to be functors of (graded) sections on standard affine subsets. This means that the "stalkwise local" point of view on sheaves on Θ corresponds under our equivalence of Abelian categories to the "affine local" point of view on $\mathrm{Qcoh}(X)$! The interplay between these two points of view (especially after generalizing this picture to Novikov varieties) will be one of the most important technical tools in this paper.

The Grothendieck topology (QDiv, Θ) on equivariant quasidivisors has action by the character lattice $X^*(T) \cong \mathbb{Z}^n$, which can be thought of either as shifts by equivariant characters or equivalently as shifts by homogeneous principal divisors. We can "quotient out" (in a category-theoretic sense) by this equivariance to get a new Grothendieck topology $(\overline{\mathrm{QDiv}}, \overline{\Theta})$. In the same way that objects of QDiv behave as a generalization of polytopes in $\mathbb{R}^n$, objects of $\overline{\mathrm{QDiv}}$ behave as the same polytopes, but now mapping as étale opens to the quotient $\mathbb{R}^n/\mathbb{Z}^n = S$. Using the duality result in section 2 (together with some formal nonsense from Appendix [topoi and equivariance]) we deduce that sheaves on the Grothendieck topology $(\overline{\mathrm{QDiv}}, \overline{\Theta})$ are equivalent as an Abelian category with the category of non-equivariant sheaves, $\mathrm{Qcoh}(X)$.

Note (in the equivariant case) that if it were the case that all objects of QDiv correspond to convex polyhedra in a functorial way compatible with open covers, then we would have a map of Grothendieck topologies $B : \mathrm{Opens}(\mathbb{R}^n) \to \Theta$ with the "pullback" of a divisor being the interior associated polytope (an object of the category underlying the topology, $\mathrm{Open}(\mathbb{R}^n)$). In fact, this is true if and only if X is an affine toric variety, in which case the derived functors of the pullback and pushforward with respect to B coincide with the [**?**] ccc functor $B^* = ccc : D^b\, \mathrm{Qcoh}^T(X) \to D^b\, \mathrm{Constr}(\mathbb{R}^n)$ and its adjoint. When X is not affine, it is no longer true that all line bundles are ample, and so the convex polytope Δ_L cannot be defined in general. Instead, what becomes associated to a general divisor is a certain derived sheaf of *sets*. In the language of ∞-categories these give a (finite) derived correspondence $B_{der} : \mathrm{Opens}(\mathbb{R}^n) \to \Theta$. It is still possible to define pullback and pushforward along this correspondence, so long as one is dealing with derived categories. The pullback and pushforward functors $(B_{der})^*, (B_{der})*$ with respect to this correspondence will then be dg functors, which provide us with the ccc functor and its adjoint in the general case. In cases where there is no ambiguity, we still write B^*, B_* for the derived functors $(B_{der})^*, (B_{der})_*$.

1.4 Novikov toric varieties and almost local sheaves

Recall that a toric variety is a compactification of the torus $T := \operatorname{Spec} k(\mathbb{Z}^n)$. A Novikov toric variety is a compactification (in a non-Noetherian sense) of the *Novikov torus*, $T_{Nov} := \operatorname{Spec} k(\mathbb{R}^n)$. Here $k(\mathbb{R}^n)$ is the group algebra. We denote elements $p \in k(\mathbb{R}^n)$ "Novikov polynomials" and write $p = \sum_{\lambda \in \mathbb{R}^n} p_\lambda t^\lambda$ with almost all $p_\lambda = 0$. The notation is in analogy with "Novikov series", which are certain infinite sums $\sum p_r t^r$ over $r \in \mathbb{R}$, which appear on the Fukaya side of most mirror symmetry statements in a related, but not entirely analogous, context.

Given a full-dimensional cone $\Lambda \subset \mathbb{R}^n$ (closed, convex with nonempty interior, and invariant with respect to multiplication by $\mathbb{R}^+$,) we define $k(\Lambda)$ to be the (unital, commutative) semigroup algebra of Λ. We observe that in this case the ring of functions $k(\mathbb{R}^n)$ on the Novikov torus is a localization of $k(\Lambda)$: indeed, in order to get $k(\mathbb{R}^n)$ from $k(\Lambda)$ it is sufficient to invert any t^λ for λ in the interior of Λ. If $\Sigma \subset \mathbb{R}^n$ is a polyhedral fan then the $k[\sigma^\vee]$ are all partial completions of T_{Nov} which glue to form a variety, X_Σ^{Nov}. If Σ is a *rational* polyhedral fan with (ordinary) toric variety X associated to Σ then we write $X_{Nov} = X_\Sigma^{Nov}$ the "Novikov variety associated to X". A general Novikov fan will not be rational and in particular will not be associated to an ordinary toric variety.

Now the short exact sequence of groups $\mathbb{Z}^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ gives a short exact sequence of group schemes $\operatorname{Spec} k(\mathbb{R}^n/\mathbb{Z}^n) \to \operatorname{Spec} k(\mathbb{R}^n) \to \operatorname{Spec} k(\mathbb{Z}^n)$. We introduce the notation $\operatorname{Spec} k(\mathbb{R}^n/\mathbb{Z}^n) =: S^\sim$. Now T_{Nov} acts on X_Σ^{Nov} for any fan Σ in a similar way to how T acts on a toric variety. This means that $(S^n)^\sim \subset T_{Nov}$ also acts on X_Σ^{Nov}. For Σ a rational fan with associated toric variety X, the quotient $X_{Nov}/S^\sim$ is a stack which compactifies (in a suitable sense) $T_{Nov}/S^\sim \cong T$. In fact, this stack should be thought of as having underlying variety X and stabilizer isomorphic to $\operatorname{Spec} k(\mathbb{R}^k/\mathbb{Z}^k)$ on each $n-k$-dimensional open orbit in X. Thus one possible point of view on toric varieties (and the point of view we essentially take here) is that in fact the variety X_{Nov} is a "more fundamental" object than X and X is simply the space

$$\text{"}X_{Nov}//S^\sim\text{"}$$

underlying the stack $X_{Nov}/S^\sim$.

We will be interested not in the entire category $\operatorname{Qcoh}(X_{Nov})$ but rather a slightly smaller subcategory, $\operatorname{Qcoh}^a(X_{Nov})$ of "almost local modules". The qualifier "almost" comes from Faltings's theory of almost geometry (see [**?**]) transposed to a global context. In the terminology of [**?**], if R is a valuation ring with non-discrete valuation group, with residue field k and fraction field K, we say that a module V over R is almost local if the restriction map $V(= \operatorname{Hom}(R, V)) \to \operatorname{Hom}(\mathbf{a}, V)$ is an isomorphism. When the valuation is not discrete, this property is strictly weaker than being a module over K: for example, the maximal ideal $\mathbf{a}$ is itself almost local, but never a module over K. In fact, in the non-discretely valued case, the category $\operatorname{Mod}^a(R)$ of almost local modules is much closer to the category of R-modules than to K-modules, as there is a Serre exact sequence of categories $\operatorname{Mod}(k) \to \operatorname{Mod}(R) \to \operatorname{Mod}^a(R)$ (whereas in the discretely valued case, the Serre kernel of the functor $\operatorname{Mod}(R) \to \operatorname{Mod}(K)$ would be the much larger category of all modules with support some thickening of the special fiber). In this paper, a coherent sheaf over a Novikov variety will be called *almost local* if its restriction (as well as the derived functors of the restriction) to any proper closed equivariant Novikov subvariety is trivial. Define the ring of Novikov taylor series to be the completion $\widehat{R} := \widehat{k(\mathbb{R}_{\geq 0})}$ of the semigroup ring $R := k(\mathbb{R}_{\geq 0})$. One can then show that almost locality of a sheaf $\mathcal{F}$ is equivalent to almost locality in the sense of [**?**] of certain completions of $\mathcal{F}$ over the non-discretely valued ring $\widehat{R}$. Similarly to the case of perfectoids, we have a Serre sequence of categories $\mathrm{Qcoh}(\partial X_{Nov}) \to \mathrm{Qcoh}(X) \to \mathrm{Qcoh}^a(X)$, where $\mathrm{Qcoh}(\partial X_{Nov})$ is the category of sheaves over the (closed) boundary $X \setminus T_{Nov}$. Despite the fact that in the definition of almost locality we "puncture" the closed fiber of the complement $X_\Sigma^{Nov} \setminus T_{Nov}$, the category of almost local sheaves on X_Σ^{Nov} depends in a nontrivial way on the fan Σ and is in general quite far from the category of sheaves on T_{Nov}. When Σ is a regular rational fan with toric variety $X = X_\Sigma$ the (usual) category of quasicoherent sheaves on X will be naturally a full subcategory of $\mathrm{Qcoh}^a(X_\Sigma^{Nov})$.

We work with three equivariant flavors of $\mathrm{Qcoh}^a(X_\Sigma^{Nov})$: other than $\mathrm{Qcoh}^a(X_\Sigma^{Nov})$ itself, we also consider equivariance under the Novikov torus $\mathrm{Qcoh}^a(X_\Sigma^{Nov})^{T_{Nov}}$ and under the subgroup $S^\sim$ $(= \mathrm{Spec}\, k(\mathbb{R}^n/\mathbb{Z}^n)) \subset T_{Nov}$. With these three equivariant flavors, we prove the following results. Let Σ be a fan in $\mathbb{R}^n$ and let $\Lambda := \cup_{\sigma \in \Sigma} \sigma \subset \mathbb{R}^n$ be its support. View $T^*(\mathbb{R}^n)$ as $\mathbb{R}^n \times \mathbb{R}^n$ with the second coordinate the normal coordinate, and similarly view $T^*S \cong S \times \mathbb{R}^n$, where $S := \frac{\mathbb{R}^n}{\mathbb{Z}^n}$. Then the manifold $\mathbb{R}^n \times \Lambda \subset T^*\mathbb{R}^n$ gives a valid singular support condition for sheaves on $\mathbb{R}^n$, and similarly $S \times \Lambda \subset T^*S$ is a valid singular support condition for S. (Here we should think of the coordinate of the ambient space of the fan as being the second, "momentum" coordinate.)

Theorem 4. *Then*

$$D^b\,\mathrm{Qcoh}^a(X)^{T_{Nov}} \cong D^b\,\mathrm{Shv}_{\mathbb{R}^n \times \Lambda}(\mathbb{R}^n), \tag{3}$$
$$D^b\,\mathrm{Qcoh}^a(X)^{S^\sim} \cong D^b\,\mathrm{Shv}_{S \times \Lambda}(S), \text{ and} \tag{4}$$
$$D^b\,\mathrm{Qcoh}^a(X) \cong D^b\,\mathrm{Shv}_{\mathbb{R}^n \times \Lambda}(\mathbb{R}^n)^{\mathbb{R}^n}. \tag{5}$$

The notation Shv here means sheaves in the topological sense (which might not be constructible with respect to any stratification). The category $\mathrm{Shv}_{\mathbb{R}^n \times \Lambda}(\mathbb{R}^n)^{\mathbb{R}^n}$ is the category of $\mathbb{R}^n$-equivariant sheaves on $\mathbb{R}^n$, which includes much more than just constant sheaves: for example, the direct sum of skyscraper sheaves at all points of $\mathbb{R}^n$ is naturally $\mathbb{R}^n$-equivariant. Lines (4) and (5) above follow via some formal nonsense (see section 2) from (3), which can be considered the main result of this paper.

1.5 Statement of main results

Here we collect all of our results together. Let Σ be a toric fan (with cones that are polyhedral but are allowed to not be rational) in the space $N_\mathbb{R}$ with dual space $M_\mathbb{R}$, with "support" $\Lambda_\sigma := \cup_\sigma \sigma \subset N_\mathbb{R}$. Let $A \subset N_\mathbb{R}$ (for most of the paper, we will work with $A = N_\mathbb{R}$) be a dense subgroup which contains $M \subset M_{\mathbb{R}}$ and such that each cone $\sigma \in \Sigma$ is spanned by its intersection with A. The group scheme T_{Nov-A} is defined to be the affine group scheme with ring of functions $k[A]$. Write $S_A = A/M$ (when $A = M_{\mathbb{R}}$ this is the mirror torus); it is a group, and (ignoring any topology on S), we define $S_A^{\sim}$ to be the affine group scheme $\mathrm{Spec}(k[S_A])$. We have an inclusion $S_A^{\sim} \subset T_{Nov-A}$ (with quotient $T = \mathrm{Spec}(k[M])$, as shown in Lemma 12). Associated to the fan Σ we define a T_{Nov}-equivariant scheme X_Σ^{Nov-A} which compactifies T_{Nov}. The scheme X_Σ^{Nov-A} supports a quasicoherent category $\mathrm{Qcoh}(X_\Sigma^{Nov})$ and inside an Abelian subcategory $\mathrm{Qcoh}^a(X_\Sigma^{Nov})$ of "almost local" quasicoherent sheaves, which is equivalent to the almost category (in the sense of [Faltings]) associated to the idempotent sheaf of defining ideals for the boundary $\partial \subset X_\sigma$. There is a sense in which an additive category over k can be equivariant with respect to a group scheme $\mathbb{G}$, and the category $\mathrm{Qcoh}^a(X_\Sigma^{Nov})$ is equivariant with respect to T_{Nov-A}, and hence also with respect to $S_A^{\sim} \subset T_{Nov-A}$. In particular, we can define categories of equivariant objects $\mathrm{Qcoh}^a(X_\Sigma^{Nov-A})^{T_{Nov-A}}$ and $\mathrm{Qcoh}^a(X_\Sigma^{Nov-A})^{S_A^{\sim}}$ with respect to these actions. On the other hand, let $\mathrm{Shv}(M_{\mathbb{R}})$ be the category of all topological sheaves on $M_{\mathbb{R}}$. This category has equivariance with respect to $A \subset M_{\mathbb{R}}$ (viewed as a discrete group and acting by transations) and also with respect to $M \subset A$. We can also define categories of equivariant objects with respect to these actions, and observe that since the action of M on $M_{\mathbb{R}}$ is discrete and free, the category of equivariant objects $\mathrm{Shv}(M_{\mathbb{R}})^M \cong \mathrm{Shv}(S)$ for $S := M_{\mathbb{R}}/M$.

With this set-up we prove the following mirror symmetry results. (An analogue for the dense subgroup A of theorem 4):

Theorem 5. *Suppose the fan Σ is compete, i.e. $\Lambda_\Sigma = N_{\mathbb{R}}$. Then we have the following three equivalences of derived categories.*

$$D^b\,\mathrm{Qcoh}^a(X_\Sigma^{Nov-A})^{T_{Nov-A}} \cong D^b\,\mathrm{Shv}(M_{\mathbb{R}}). \tag{6}$$

$$D^b\,\mathrm{Qcoh}^a(X_\Sigma^{Nov-A})^{S_A^{\sim}} \cong D^b\,\mathrm{Shv}(M_{\mathbb{R}})^M \big(\cong D^b\,\mathrm{Shv}(S)\big). \tag{7}$$

$$D^b\,\mathrm{Qcoh}^a(X_\Sigma^{Nov-A}) \cong D^b\,\mathrm{Shv}(M_{\mathbb{R}})^A \tag{8}$$

It is from part (7) above, applied to the dense subgroup $A = \mathbb{Q} \otimes M \subset M$, that we deduce the log-perfectoid mirror symmetry interpretation (Theorem 3).

For a fan Σ which is not complete (and has support Λ_Σ), we deduce an analogous triad of results with equivariant versions of $\mathrm{Shv}(M_{\mathbb{R}})$ on the right replaced by the category $\mathrm{Shv}_\Lambda(M_{\mathbb{R}})$ of "topological sheaves with singular support $M_{\mathbb{R}} \times \Lambda \subset T^*M_{\mathbb{R}}$" (which we define).

2 Equivariant Categories and Spectral Pontrjagin Duality

Recall that if G is a topological commutative group, its Pontrjaging dual is $G^\vee := \mathrm{Hom}(G, U(1))$. This is equivalent to the unitary spectrum of G, with product structure corresponding to tensor product of unitary representations. A closely related concept in algebraic geometry: if $\mathbb{G}$ is a finite group scheme over a field k, then the group of its characters $\mathrm{Hom}(\mathbb{G}, \mathbb{G}_m)$

is also a finite affine commutative group scheme, called the "Cartier dual" $\mathbb{G}^\vee$ to $\mathbb{G}$. If $\mathbb{G} = \operatorname{Spec} H$ for H a finite dimensional commutative, cocommutative Hopf algebra then $G^\vee \cong \operatorname{Spec}(H^*)$. In this section, we will study categorical actions associated with an intermediate kind of duality we call spectral Pontrjagin duality, which is between a discrete commutative group G and a dual group scheme $G^\sim$ over k.

Definition 1. *If G is a discrete commutative group, write $G^\sim := \operatorname{Spec} k(G)$ for the affine group scheme with function ring the group algebra of G, viewed as a Hopf algebra.*

Remark 1. *When G is a finite group, the duality $G \to G^\sim$ is a special case of Cartier duality. Indeed, the discrete group scheme $G \times \mathrm{pt}_k$ and $\operatorname{Spec} k(G)$ are Cartier dual finite group schemes. However, when G is an infinite group, the dual Hopf algebra $k(G)^*$ is too large an algebra, and $\operatorname{Spec}(k(G)^*)$ does not behave, e.g. from the point of view of representation theory, in the same way as G. It might be possible to express the duality $G \leftrightarrows G^\sim$ as a special case of a generalization of Cartier duality if we pass to a larger category than affine group schemes: e.g., by thinking of $G \times \mathrm{pt}_k$ as a group object in Ind-schemes. However, in order to avoid introducing unnecessary formalism in this section we will restrict ourselves to studying the representation theoretic behavior of the duality functor $G \mapsto G^\sim$ from discrete groups to affine groups schemes.*

2.1 Categorical equivariance

Definition 2. *Suppose $\mathcal{C}$ is a category and G is a group. A (strict) G-equivariant structure on $\mathcal{C}$ is a strict representation of G on endofunctors of $\mathcal{C}$. I.e., it is a collection of functors $X \mapsto {}^gX$ ($g \in G$) with isomorphisms ${}^h({}^gX) \cong {}^{(hg)}X$ and a cocycle condition that says that the two different natural isomorphisms ${}^i({}^h({}^gX)) \cong^{(ihg)} X$ coincide.*

Definition 3. *Suppose $\mathcal{C}$ is a k-linear category and $\mathbb{G}$ is an affine group scheme over k. Suppose further that $\mathcal{C}$ is Abelian and has all colimits (this is not strictly necessary, but simplifies exposition). Write $\mathcal{C}_\mathbb{G}$ for the category of objects of $\mathcal{C}$ fibered over the scheme $\mathbb{G}$, i.e. with $\mathbb{O}(\mathbb{G})$ action. A $\mathbb{G}$-action on $\mathcal{C}$ is then an Abelian functor $A : \mathcal{C} \to \mathcal{C}_\mathbb{G}$ together with an isomorphism $A \circ A \cong \mu^* A$ of functors $\mathcal{C} \to \mathcal{C}_{\mathbb{G}\times\mathbb{G}}$. Here $\mu^* : \mathcal{C}_\mathbb{G} \to \mathcal{C}_{\mathbb{G}\times\mathbb{G}}$ is the pullback functor along the multiplication map of schemes $\mu : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$. It is well defined because pullback is a colimit (actually, a tensor product) and $\mathcal{C}$ has all colimits.*

Writing $\mu_3 : \mathbb{G} \times \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ for the (associative) multiplication map, we have two natural isomorphisms $A \circ A \circ AX \cong \mu_3^* AX$ for $X \in \mathcal{C}$ an object. We require these to be equal.

A consequence of these axioms is that the composition $\mathcal{C} \xrightarrow{A} \mathcal{C}_\mathbb{G} \xrightarrow{\mathrm{fib}} \mathcal{C}$ is the identity, where $\mathcal{C}_\mathbb{G} \xrightarrow{\mathrm{fib}} \mathcal{C}$ is the fiber at $1 \in \mathbb{G}$ functor (well-defined because of the existence of colimits).

Definition 4. *For G a discrete group, let $\mathcal{E}_G$ be the category with objects $X_g \mid g \in G$ and a unique invertible map $X_g \to X_h$ for any g, h. This category has an obvious (and unique) G-equivariant structure with ${}^hX_g = {}^{hg}X$. If $\mathcal{C}$ is an arbitrary G-equivariant category, define the category $\mathcal{C}^G$*

of G-equivariant objects in $\mathcal{C}$ to be the category $Fun_G(\mathcal{E}_G, \mathcal{C})$ of functors strictly compatible with G-equivariance. More concretely, an object of $\mathcal{C}^G$ is an object $X \in \mathcal{C}$ with isomorphisms $\rho_g : X \cong {}^gX$ satisfying $\rho_1 = 1$ and such that the diagram

$$\begin{tikzcd} X \arrow[r, "\rho_g", "\sim"'] \arrow[d, "\rho_{gh}"', "\sim"] & {}^gX \arrow[d, "{}^g\rho_h"', "\sim"] \\ {}^{gh}X \arrow[r, "\sim"] & {}^g({}^hX) \end{tikzcd} \tag{9}$$

commutes. Here ${}^g\rho_h$ is obtained by applying the g-action functor to the arrow $X \xrightarrow[\rho_h]{\sim} {}^hX$.

Now we make a similar definition for a group scheme $\mathbb{G}$.

Definition 5. *Suppose $\mathbb{G}$ is a group scheme and $\mathcal{C}$ is an Abelian $\mathbb{G}$-equivariant category with all colimits. We say that a $\mathbb{G}$-equivariant object of $\mathcal{C}$ is an object X of $\mathcal{C}$ together with a functorial isomorphism of $\mathbb{G}$-fibered objects (objects in $\mathcal{C}_{\mathbb{G}}$): $A(X) \cong \underline{X}_{\mathbb{G}}$. Here $\underline{X}_{\mathbb{G}}$ is the pullback of X along the map $\mathbb{G} \to \mathrm{pt}_k$.*

The isomorphism is required to satisfy an analogue of the commutativity restriction of the diagram (9) written in a fibered way: namely, the diagram

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\rho"'] & \underline{X}_{\mathbb{G}} \arrow[d, "A(\rho)"'] \\ \underline{X}_{\mathbb{G}} \arrow[r, "\Delta"] & \underline{X}_{\mathbb{G}\times\mathbb{G}} \end{tikzcd} \tag{10}$$

must be commutative.

It is easy to check that these definitions of G-invariants as well as $\mathbb{G}$-invariants are functorial. Further, the following lemma is standard:

Lemma 6. *If $\mathcal{C}, \mathcal{C}'$ are G-equivariant categories as above and $\mathcal{C} \to \mathcal{C}'$ is a G-equivariant functor which is an equivalence of categories, then the functor $\mathcal{C}^G \to (\mathcal{C}')^G$ is an equivalence of categories. Similarly, if $\mathcal{C}, \mathcal{C}'$ are $\mathbb{G}$-equivariant categories and $\mathcal{C} \to \mathcal{C}'$ is a $\mathbb{G}$-equivariant functor which is nonequivariantly an equivalence, then $\mathcal{C}^{\mathbb{G}} \to (\mathcal{C}')^{\mathbb{G}}$ is an equivalence of categories.*

We will extensively use the following lemma, which essentially says that if we consider a category $\mathcal{C}$ with trivial $G^{\sim}$-action then equivariant objects are the same as G-graded objects.

Definition 6. *1. Given a category $\mathcal{C}$, write $\mathcal{C}_{gr\ G}$ for the category of G-graded objects of $\mathcal{C}$ (i.e. collections of G-tuples objects of $\mathcal{C}$ with $\mathrm{Hom}(\{X_g\}, \{Y_g\}) := \prod_{g\in G} \mathrm{Hom}(X_g, Y_g)$.)*

2. *If $\mathcal{C}$ has symmetric monoidal structure and all colimits, the category $\mathcal{C}_{gr\ G}$ has graded (unsigned) symmetric monoidal structure, with*

$$(\{X_g\} \otimes \{Y_g\})_h := \oplus_{g_1+g_2=h} X_{g_1} \otimes Y_{g_2}.$$

3. *Similarly, if $\mathcal{C}$ is a module category over the symmetric monoidal category $\mathbf{a}$ then $\mathbf{a}_{gr\ G}$ acts on $\mathcal{C}_{gr\ G}$ in a similar way.*

Definition 7. 1. *Write $\underline{\mathcal{C}}_{\mathbb{G}}$ for the category of "sheaves of $\mathcal{C}$-objects over the scheme $\mathbb{G}$", i.e. objects of $\mathcal{C}$ with $\mathbb{O}_{\mathbb{G}}$-action.*

2. *If $\mathcal{C}$ is symmetric monoidal with colimits, define symmetric monoidal structure on $\underline{\mathcal{C}}_{\mathbb{G}}$ by $\mathcal{F} \otimes \mathcal{G} := \mu_*(\mathcal{F} \boxtimes \mathcal{G})$. Here if $\mathcal{F}, \mathcal{G}$ are objects with $\mathbb{O}_G$ action then $\mathcal{F} \boxtimes \mathcal{G}$ is $\mathcal{F} \otimes \mathcal{G}$ viewed as an object with $\mathbb{O}_G \otimes \mathbb{O}_G$-action and the functor μ_* pulls back the $\mathbb{O}_G \otimes \mathbb{O}_G$ action to $\mathbb{O}_G$ action via the comultiplication map $\mu^* : \mathbb{O}_G \to \mathbb{O}_G \times \mathbb{O}_G$.*

3. *Simiarly, if $\mathcal{C}$ is a module category over the symmetric monoidal category $\mathbf{a}$ then $\underline{\mathbf{a}}_{\mathbb{G}}$ acts on $\underline{\mathcal{C}}_{\mathbb{G}}$ in a similar way.*

Lemma 7. *Suppose G is a commutative discrete group scheme. Write*

$$\mathbb{G} := G^{\sim}.$$

Suppose that $\mathcal{C}$ is an Abelian category with all colimits and further, suppose every object of $\mathcal{C}$ can be filtered by compact subobjects. We view $\mathcal{C}$ as both a G-equivariant and a $\mathbb{G}$-equivariant category with trivial action.

1. *We have an equivalence of categories $\mathcal{C}^G \cong \underline{\mathcal{C}}_{\mathbb{G}}$. This equivalence of categories is compatible with monoidal structure if $\mathcal{C}$ is monoidal and with module category structure if $\mathcal{C}$ is a module category over the monoidal category $\mathbf{a}$.*

2. *We have an equivalence of categories $\mathcal{C}^{\mathbb{G}} \cong \mathcal{G}_{gr\ G}$. This equivalence of categories is compatible with monoidal structure if $\mathcal{C}$ is monoidal and with module category structure if $\mathcal{C}$ is a module category over the monoidal category $\mathbf{a}$.*

Proof. Part (1) follows from the definition. Part (2) can be rephrased as the following proposition:

Proposition 8. *Every representation of $\mathbb{G}$ is diagonalizable.*

To prove this, filter G by finitely-generated Abelian subgroups G_i (with i an index in some totally ordered set). Write $\mathbb{G}_i := G_i^{\sim}$, a projective system converging to $\mathbb{G}$. Suppose that V is a representation of $\mathbb{G}$, and let $V_i \subset V$ be the maximal subrepresentation on which the $\mathbb{G}$ action factors through $\mathbb{G}_i$. Now we know from standard algebraic geometry (and the classification of finitely generated abelian groups) that representations of $\mathbb{G}_i$ are diagonalizable. It suffices to show that the V_i filter V, equivalently that any compact $U \subset V$ is stabilized by the kernel of some map $\mathbb{G} \to \mathbb{G}_i$. Indeed, recall that the structure of a $\mathbb{G}$-representation is defined by a map $V \to V_{\mathbb{G}} = V \otimes k(G)$. By compactness, the image of $U \subset V$ will land in some finite sum $\oplus_{i=1}^N t^{g_i}$, which lies inside some finitely generated piece $V(G_i)$. This proves the proposition, and hence the lemma. The compatibility with tensor product is clear. $\square$

Corollary 9. *An algebra in the category of $\mathbb{G}$-representations is equivalent to a G-graded algebra and an algebra in the category of G-representations is equivalent to a $\mathbb{G}$-graded algebra (in the obvious sense: i.e. a sheaf* **a** *over $\mathbb{G}$ with algebra structure $\mathbb{A} \boxtimes \mathbb{A} \to \mathbb{A}$ lying over the map $\mu : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$).*

It follows from this result that the category $\mathrm{Vect}^{\mathbb{G}}$ is symmetric monoidally equivalent to the category of G-graded modules.

Definition 8. *In particualr, we have a special class of objects $\chi_g \in \mathrm{Vect}^{\mathbb{G}}$ for $g \in G$ corresponding to one-dimensional vector spaces in degree $g \in G$.*

Definition 9. *Similarly, the category Vect^G is equivalent to the category of sheaves over $\mathbb{G}$, and in particular we have a special sheaf $\chi_{\mathbb{G}}$ which we view as a family of one-dimensional representations fibered over $\mathbb{G}$.*

Note that the category Vect acts by tensor product on any abelian category over k with all colimits. It follows that Vect^G acts on the G-invariants $\mathcal{C}^G$ and, respectively, $\mathrm{Vect}^{\mathbb{G}}$ acts on the $\mathbb{G}$-invariants $\mathcal{C}^{\mathbb{G}}$ for $\mathcal{C}$ a G-equivariant, respectively, a $\mathbb{G}$-equivariant Abelian category with all colimits.

Lemma 10. *If $\mathcal{C}$ is a $\mathbb{G}$-equivariant category with all colimits then the category $\mathcal{C}^{\mathbb{G}}$ has G-action via the functors $\alpha_g : X \mapsto X \otimes \chi_g$ for $g \in G$. Simiarly, if $\mathcal{C}$ is a G-equivariant category with all colimits then the category $\mathcal{C}^G$ has $\mathbb{G}$-action via the functor $A_G : \mathcal{C} \to \mathcal{C}_{\mathbb{G}}$ given by $X \mapsto X \otimes \chi_{\mathbb{G}}$.*

Proof. Simple verification. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 11 (Categorical Pontrjagin Duality)**.**

1. *Suppose $\mathcal{C}$ is a compactly generated G-equivariant category with all colimits. Then we have an equivalence of categories $(\mathcal{C}^G)^{\mathbb{G}} \cong \mathcal{C}$*
2. *Suppose $\mathcal{C}$ is a compactly generated $\mathbb{G}$-equivariant category with all colimits. Then we have an equivalence of categories $(\mathcal{C}^{\mathbb{G}})^G \cong \mathcal{C}$.*

Proof. Suppose we are in case (1). Let $\mathbb{O} := k[G]$ be object of $(\mathrm{Vect}^G)^{\mathbb{G}}$ with G-action by multiplication and evident $\mathbb{G}$-equivariance. The functor $X \mapsto X \otimes \mathbb{O}$ gives us a functor $\mathcal{C} \to (\mathcal{C}^G)^{\mathbb{G}}$. In the other direction, note that given an object E of $(\mathcal{C}^G)^{\mathbb{G}}$ and forgetting the G action, we get an object $\mathrm{forg}_G(E)$ of $\mathcal{C}^{\mathbb{G}}$ with trivial action (as the action of $\mathbb{G}$ twists by characters of G, which are trivial when we forget the G action). By lemma 7, this gives the object of $\mathcal{C}$ underlying E a G-grading. It is a simple check that the functor $E \mapsto \mathrm{forg}_G(E)\langle 0 \rangle$ sending E to one of its graded components is an inverse functor to $X \mapsto X \otimes \mathbb{O}$.

Now suppose we are in case (2). Then (abusing notation) we can view the vector space $\mathbb{O} := k[G]$ as an object of $(\mathrm{Vect}^{\mathbb{G}})^G$ in an evident way, giving a functor $\mathcal{C} \to (\mathcal{C}^{\mathbb{G}})^G$ with $X \mapsto X \otimes \mathbb{O}$. In the other direction, observe once again that we have a forgetful functor $\mathrm{forg}_{\mathbb{G}} : (\mathcal{C}^{\mathbb{G}})^G \mapsto \mathcal{C}^G$ where the G-action is trivial. We have a fiber functor $\mathrm{fib} : \mathcal{C}^G \to \mathcal{C}$ given by taking the fiber at the point $0 \in \mathbb{G}$ of $\mathcal{C}$ viewed as an object of $\mathcal{C}_{\mathbb{G}}$. It is another simple verification that $\mathrm{fib} \circ \mathrm{forg}_{\mathbb{G}}$ is an inverse to $X \mapsto X \otimes \mathbb{O}$. $\square$

2.2 Short exact sequences

We prove a couple more technical results about the behavior of categorical equivariance with respect to short exact sequences of groups. Suppose $0 \to G'' \to G \to G' \to 0$ is a short exact sequence of commutative groups.

Proposition 12. *The sequence $(G')^{\sim} \to G^{\sim} \to (G'')^{\sim}$ is an exact sequence of affine commutative group schemes.*

Proof. Recall that a sequence $\mathbb{G}' \to \mathbb{G} \to \mathbb{G}''$ of affine commutative group schemes is a short exact sequence if and only if the map $\mathbb{G} \xrightarrow{\pi} \mathbb{G}''$ is flat and the map $i : \mathbb{G}' \to \mathbb{G}$ maps $\mathbb{G}'$ isomorphically to the fiber $\pi^{-1}(0)$.

Now write $\mathbb{G} := G^{\sim}, \mathbb{G}' := (G')^{\sim}, \mathbb{G}'' := (G'')^{\sim}$, with maps $\mathbb{G}' \xrightarrow{i} \mathbb{G} \xrightarrow{\pi} \mathbb{G}''$ dual to the corresponding maps of groups. The map π^* on functions is the embedding $k[G''] \to k[G]$ which evidently flat (in fact, free). The fiber $\pi^{-1}(0)$ is $\mathrm{Spec}(k[G] \otimes_{k[G'']} k)$ where the map of rings $k[G''] \to k$ is the augmentation map. Evidently, $k[G] \otimes_{k[G'']} k \cong k[G']$, completing the proof. □

Write $\mathbb{G}' \to \mathbb{G} \to \mathbb{G}''$ for the sequence of dual groups.

Lemma 13.
1. *Suppose $\mathcal{C}$ is a small G-equivariant category. Then $\mathcal{C}^{G''}$ is naturally equivalent to a G'-equivariant category and using this model we have an equivalence of categories $(\mathcal{C}^{G''})^{G'} \cong \mathcal{C}^{G}$.*
2. *Suppose $\mathcal{C}$ is a small $\mathbb{G}$-equivariant category. Then $\mathcal{C}^{\mathbb{G}'}$ is naturally equivalent to a $\mathbb{G}'$-equivariant category and using this model we have an equivalence of categories $(\mathcal{C}^{\mathbb{G}'})^{\mathbb{G}''} \cong \mathcal{C}^{\mathbb{G}}$.*

Proof. The statement is of model theoretic nature, and is proven most naturally by introducing a model category structure on G-equivariant (resp., $\mathbb{G}$-equivariant) categories with weak equivalences given by equivariant maps which are equivalences of categories. We will give a proof without explicitly introducing this formalism.

Suppose we are in the case (1). Then recall that $\mathcal{C}^{G''}$ is defined as the category of G''-equivariant functors $E \to \mathcal{G}''$, equivalently the category of strict G''-fixed objects in the category $\mathrm{Fun}(E_{G''}, \mathcal{C})$. Write $\widetilde{\mathcal{C}}$ for the category $\mathrm{Fun}(E_G, \mathcal{C})$ viewed combinatorially (a set of objects and morphisms), as a G-equivariant category. We can choose a G''-equivariant retract of the functor $E_{G''} \to E_G$, giving a pair of inverse G''-equivariant equivalences $E_{G''} \cong E_G$. This pair of functors gives us an equivalence of categories between $\mathcal{C}^{G''}$ and $\widetilde{\mathcal{C}}^{\text{strict } G''}$, where the notation strict in the superscript means we are taking invariants in a combinatorial sense (in terms of invariant objects and morphisms). Using this model, the (combinatorial) G-equivariant structure on $\widetilde{\mathcal{C}}$ restricts to an equivariant structure on $\widetilde{\mathcal{C}}^{\text{strict } G''}$ which factors through to a G'-equivariance. We have an equality of combinatorial objects $\left(\widetilde{\mathcal{C}}^{\text{strict } G''}\right)^{\text{strict } G'} = \widetilde{\mathcal{C}}^{\text{strict } G}$, and a natural functor $\widetilde{\mathcal{C}}^{\text{strict } G} \to \widetilde{\mathcal{C}}^{G}$ (pre-composing with the G-equivariant functor $E_G \to \mathrm{pt}$). Now we have

$$\left(\widetilde{\mathcal{C}}^{\text{strict } G''}\right)^{G'} \cong \quad \mathrm{Fun}\left(E_{G'}, \mathrm{Fun}\left(E_G, \mathcal{C}\right)^{\text{strict } G''}\right)^{\text{strict } G'} \qquad (11)$$

$$\cong \quad \mathrm{Fun}(E_{G\times G'}, \mathcal{C})^{\text{strict G}}, \qquad (12)$$

where G acts on $E_{G\times G'}$ diagonally and the isomorphism in the second row is a categorical adjunction result. Now $E_{G\times G'}$ is has a G-equivariant, equivariantly invertible category equivalence with E_G, so the last term is isomorphic to E_G.

Now we could prove part (2) of the lemma by reconstructing the argument above in an algebro-geometric context. (Note that in fact, the result holds for any short exact sequence of affine group schemes $\mathbb{G}' \to \mathbb{G} \to \mathbb{G}''$.) However, for simplicity we instead use an extension of Theorem 11 to prove it only for algebraic groups of the form $G^{\sim}$. Namely, we prove the following proposition.

Proposition 14. *We have an equivalence of categories $\mathcal{C}^{\mathbb{G}'} \cong (\mathcal{C}^{\mathbb{G}})^{G''}$. Here recall that G acts on $\mathcal{C}^{\mathbb{G}}$ by character twists, and G'' acts via the embedding $G'' \subset G$.*

Proof. Suppose X is an object of $(\mathcal{C}^{\mathbb{G}})^{G''}$. As seen in the proof of theorem 11, there is a G'', i.e. a $\mathbb{O}_{\mathbb{G}''}$, action on the nonequivariant object $\mathrm{forg}_{\mathbb{G}}(X) \in \mathcal{C}$. Write $\mathrm{fibforg}_{\mathbb{G}}(X)$ for the fiber of $\mathrm{forg}_{\mathbb{G}}(X)$ over $0 \in \mathbb{G}''$. Observe that this functor has $\mathbb{G}'$ action. This induces a pair of functors $\mathcal{C}^{\mathbb{G}'} \rightleftarrows (\mathcal{C}^{\mathbb{G}})^{G''}$, which are inverse by an argument similar to our proof of Theorem 11. □

□

3 Algebra of Cones

Define $M_{\mathbb{R}} := \mathbb{R}^n$. If $\Lambda \subset M_{\mathbb{R}}$ is a subset of $M_{\mathbb{R}}$ closed under addition (but not subtraction), the space $k(\Lambda)$ with basis Λ has the structure of an algebra, and $k(\Delta)$ has the structure of a $k(\Lambda)$-module if $\Delta \subset M_{\mathbb{R}}$ is a subset closed with respect to addition of elements of Λ. Furthermore, viewing $M_{\mathbb{R}}$ as a discrete group, we have $M_{\mathbb{R}}^{\sim}$ an algebraic group scheme with category of representations equivalent (in a monoidal way) to the category of $M_{\mathbb{R}}$-graded vector spaces. In particular, both $k(\Lambda)$ and $k(\Delta)$ for $\Lambda \subset M_{\mathbb{R}}, \Delta \subset M_{\mathbb{R}}$ as above are $M_{\mathbb{R}}$-graded, equivalently $M_{\mathbb{R}}^{\sim}$-equivariant.

We will be interested in this section in two cases: either when Λ is a (dual) polyhedral cone in $M_{\mathbb{R}}$ or when it is the intersection of a (dual) polyhedral cone with a lattice $M \subset M_{\mathbb{R}}$.

3.1 Direct limits

Recall from basic category theory that the functor $\mathrm{Free} : \mathrm{Sets} \to \mathrm{Vect}$ with $X \mapsto k(X)$ commutes with direct limits. It follows easily that its graded analogue $\mathrm{Free}_{\mathrm{gr}\ M_{\mathbb{R}}} : \mathrm{Sets}_{M_{\mathbb{R}}} \to \mathrm{Vect}_{M_{\mathbb{R}}}$ commutes with direct limits as well. We deduce the following two results.

Proposition 15. *1. Suppose that $U_1 \subset \cdots \subset U_i \subset \dots$ is a collection of open sets indexed by a totally ordered index set $i \in I$. Write $U := \cup_i U_i$. Then* $\operatorname{colim} k(U_i) \cong k(U)$*.*

2. Suppose that $\{U_i\}$ is a collection of (arbitrary) subsets $U_i \subset M_{\mathbb{R}}$ indexed by an arbitrary set $i \in I$. For $i, j \in I$ write $U_{ij} := U_i \cap U_j$. Then the diagram $\bigoplus_{i,j} k(U_{ij}) \rightrightarrows \bigoplus_i k(U_i) \to k(U)$ is a coequalizer diagram.

From part (1) of the above, we deduce the following result.

Proposition 16. *Suppose that $\Lambda \subset M_{\mathbb{R}}$ is a sub-semigroup of $M_{\mathbb{R}}$ and Δ is a Λ-equivariant subset. Let $\lambda \in \Lambda$ be a point. Write t^λ for the generator corresponding to the point $\lambda \in M_{\mathbb{R}}$. Then we have localizations $k(\Lambda)[(t^\lambda)^{-1}] \cong k(\Lambda - \mathbb{N}\lambda)$ and $k(\Delta)[(t^\lambda)^{-1}] \cong k(\Delta - \mathbb{N}\lambda)$. Here $\mathbb{N}\lambda$ is the set $0, \lambda, 2\lambda, \dots$.*

Inspired by this fact, we make the following definition. Suppose $\lambda \in M_{\mathbb{R}}$ is a vector and $\Delta \in M_{\mathbb{R}}$ is a subset with $\Delta + \lambda \subset \Delta$. Then we define the *localization* Δ_λ of Δ by λ to be

$$\Delta_\lambda := \Delta - \mathbb{N}\lambda.$$

3.2 Polyhedral cones

Write $N_{\mathbb{R}} := M_{\mathbb{R}}^\vee$.

Definition 10. *A subset $\Lambda \subset N_{\mathbb{R}}$ is called a* cone *if Λ is closed, $0 \in \Lambda$ it is invariant with respect to positive dilations $\mathbb{R}^+ \cdot \Lambda \subset \Lambda$ and Λ is convex. (Note that some authors use "cone" for something which is not necessarily closed and convex: a more consistent terminology might be "closed, convex cone".)*

Convexity implies that for any $x, y \in \Lambda$, the midpoint $\frac{x+y}{2}$ is also in Λ. Dilation-invariance implies that $x + y \in \Lambda$ as well, so that Λ is a unital semigroup.

We write down some standard definitions for cones.

Definition 11. *1. $\langle\Lambda\rangle$ is the vector space spanned by Λ, equivalently (by the semigroup property and dilation invariance,) $\Lambda + (-\Lambda)$.*

2. $\Lambda_\pm$ is the maximal vector space contained in Λ, equivalently $\Lambda \cap -\Lambda$.

3. The "dual cone" to $\Lambda \subset M_{\mathbb{R}}$ is the cone in the dual vector space $N_{\mathbb{R}}$ of vectors that pair nonnegatively with Λ, i.e.

$$\Lambda^\vee := \{y \in M_{\mathbb{R}} \mid \langle \lambda, y\rangle \geq 0 \forall \lambda \in \Lambda\} \subset M_{\mathbb{R}}.$$

4. The "orthogonal vector space" to a cone Λ is the orthogonal space to its span, $\Lambda^\perp := \langle\Lambda\rangle^\perp$.

5. We say that Λ is sharp *(also called "salient") if $\Lambda_\pm = \{0\}$.*

6. We say that Λ is full *if it contains a non-empty open, equivalently if $\langle\Lambda\rangle = M_{\mathbb{R}}$.*

Note that we allow cones which are not full, i.e. which are contained in a proper sub-vector space. For such cones we will slightly abuse notation as follows.

Definition 12. *Given a cone* $\Lambda \subset M_{\mathbb{R}}$*, write* $\mathring{\Lambda}$*, the "interior of* Λ*" for the maximal open set* of $\langle\Lambda\rangle$ *contained in* Λ*.*

We collect here some simple combinatorial results about cones.

Lemma 17.

1. *The duality functor is contravariant with respect to inclusion.*
2. *For any cone* Λ*, the double dual* $(\Lambda^\vee)^\vee = \Lambda$*.*
3. $\lambda^\perp = \langle\Lambda\rangle^\perp = (\Lambda^\vee)_\pm$
4. Λ *is full if and only if* $\Lambda^\vee$ *is sharp.*
5. $\Lambda = M_{\mathbb{R}}$ *if and only if it contains an open neighborhood of* 0*.*

Definition 13. *We say that a cone* Λ *is* polyhedral *if it is the set of solutions to a finite set of inequalities, equivalently an intersection of closed half-spaces.*

Definition 14. *Given a lattice* $M \subset M_{\mathbb{R}}$*, we say that a polyhedral cone* Λ *is* rational *if it is the set of solutions to a finite set of inequalities with rational coefficients relative to a basis of* M*.*

Definition 15. *We say that a polyhedral cone is* regular *if it is the positive quadrant* $\sum_{m_i \in \mathbb{Z}^+} m_i x_i$ *for* $x_i \in M' \subset M$ *is the basis of some (possibly lower-dimensional) sublattice* $M' = M \cap M'_{\mathbb{R}}$*.*

Given a polyhedral cone Λ and a vector $\lambda \in \Lambda$, we can consider the localization (defined above) $\Lambda_\lambda := \Lambda - \mathbb{N}\lambda$.

Proposition 18.

1. $\Lambda_\lambda = \Lambda + \mathbb{R}\lambda$*, and in particular is also a cone.*
2. *The dual cone* $\Lambda_\lambda^\vee$ *is the closed sub-cone*

$$\Lambda^\vee \cap (\mathbb{R} \cdot \lambda)^\perp.$$

Proof. The first part is obvious. For the second, note that since $\Lambda \cup \pm\lambda \in \Lambda_\lambda$, contravariance implies that $\Lambda_\lambda^\vee \subset \Lambda^\vee \cap (\mathbb{R}\cdot\lambda)^\vee$. In the other direction, if we have some $x \in \Lambda^\vee \cap (\mathbb{R}\cdot\lambda)^\perp$ then $\langle x, y\rangle \geq 0$ for $y \in \Lambda$ and $\langle x, -\lambda\rangle = 0$ so $\langle x, y - r\lambda \geq 0$ for any $r \in \mathbb{R}$. □

If Λ is a cone, we say that $\Lambda' \subset \Lambda$ is a *face* of Λ either if $\Lambda' = \Lambda$ or if $\Lambda' = H \cup \Lambda$, for H the boundary of a half-space which contains Λ. (For us, a face is allowed to have arbitrary codimension.)

Definition 16. *Write* $\Lambda' \preccurlyeq \Lambda$ *if* Λ' *is a face of* Λ*.*

From the above proposition, we get the following useful result.

Lemma 19. Λ' *is a face of* Λ *if and only if* $(\Lambda')^\vee$ *is a localization of* $\Lambda^\vee$*. Further, if* Λ *is a* rational *polyhedral cone with respect to some lattice* $M \subset M_\lambda$ *then* λ *can be taken to be an integral vector* $\lambda \in M \cap \Lambda$*.*

Proof. Suppose $0 \neq \lambda \in \Lambda$. Consider the hyperplane $(\mathbb{R}^+\lambda)^\vee$. Then on the one hand, $\Lambda \subset (\mathbb{R}^+\lambda)^\vee$, on the other hand, by Proposition 18, we see that the dual to the localization $\Lambda_\lambda^\vee = \Lambda^\vee \cap (\mathbb{R}\lambda)^\perp$ is the intersection of the localization with the boundary of $(\mathbb{R}^+\lambda)^\vee$. Thus the intersection will be a face of $\Lambda^\vee$. In the other direction, by convexity, for any face $\sigma \subset \Lambda^\vee$ there is a nonnegative linear function on $\Lambda^\vee$ which is equal to zero exactly on σ. Writing $\lambda \in N_{\mathbb{R}}^\vee$ for a linear extension of this function to all of $N_{\mathbb{R}}$, we have $\lambda \in \Lambda$ by nonnegativity of $\lambda \mid \Lambda^\vee$ and $\Lambda_\lambda = \sigma^\vee$. Obviously, if Λ is rational (with respect to a lattice), λ can be taken to be rational. □

Corollary 20. *Suppose that Λ is a cone, and $\lambda_1, \lambda_2 \in \Lambda$ are vectors. Let $\Lambda_i := \Lambda_{\lambda_i}$. Then $(\Lambda_{\lambda_1+\lambda_2})^\vee = \Lambda_1^\vee \cap \Lambda_2^\vee$*

Since the $\{\Lambda' \mid \Lambda' \preccurlyeq \Lambda\}$ give a full set of faces of the polytope Λ in a combinatorial sense, we have the following result:

Proposition 21. *For any polyhedral cone Λ, we have*

$$\Lambda = \bigsqcup_{\Lambda' \preccurlyeq \Lambda} \mathring{\Lambda}'.$$

Now we can define a *fan*.

Definition 17. *A* fan Σ *in $N_\mathbb{R}$ is a finite nonempty set $\sigma \in \Sigma$ of* sharp *cones in $N_\mathbb{R}$ such that*

1. *$\bigsqcup_{\sigma \in \Sigma} \mathring{\sigma} \subset N_\mathbb{R}$ (i.e., the cones are disjoint) and*
2. *if $\sigma \in \Sigma$ and $\sigma' \preccurlyeq \sigma$ is a face then $\sigma' \in \Sigma$.*

Any fan Σ comes naturally with the structure of a partially ordered set by boundary containment, with $\sigma \preccurlyeq \sigma'$ if $\sigma \subset \sigma'$. Note since our cones are sharp and Σ is nonempty, the zero cone $\{0\} \subset N_\mathbb{R}$ will be the initial element of Σ. Given a pair σ, σ' of cones in Σ, their intersection $\sigma \cap \sigma'$ is also in Σ. Thus, Σ is a finite poset with arbitrary intersections (i.e., a meet-semilattice). We will use the notation $\Sigma^k \subset \Sigma$ for the subset of k-dimensional cones in Σ.

Most fans do not have a terminal element (when viewed as posets). Those that do are called *affine*. For any sharp cone σ, there is a unique fan with terminal element σ. Namely, it follows from Proposition 21 that the set defined in the following definition is a fan:

Definition 18. *Given a sharp polyhedral cone $\sigma \subset N_\mathbb{R}$, write Σ_σ, the "affine fan associated to Σ", for the set of all faces $\sigma' \preccurlyeq \sigma$.*

Definition 19. *We say that a fan Σ is* complete *if $\bigcap_{\sigma \in \Sigma} \mathring{\sigma} = M_\mathbb{R}$.*

Definition 20. *We say that a fan is rational, resp., regular, if all its cones are rational, resp., regular.*

A one-dimensional cone is called a *ray*. If $\rho \subset M_\mathbb{R}$ is a ray and $M \subset M_\mathbb{R}$ is a designated lattice ρ is rational with respect to M if and only if it contains a nonzero integral point of M. In this case, the intersection $\rho \cap M$ will be a semigroup isomorphic to $\mathbb{N}$.

Definition 21. *For rational rays ρ, we use the notation g_ρ for the (unique) generator of $\rho \cap M$.*

4 Toric Novikov Varieties and Almost Local Sheaves

Suppose that Σ is a fan. For $\sigma \in \Sigma$ write

$$\mathbb{O}_\sigma^{Nov} := k(\sigma^\vee).$$

If σ is rational with respect to a lattice $N \subset N_\mathbb{R}$, we also define

$$\mathbb{O}_\sigma := k(\sigma^\vee \cap \mathcal{M}),$$

with $M := N^\vee$ the dual lattice. If $\sigma \preccurlyeq \sigma'$ we have a containment of rings $\mathbb{O}_{\sigma'}^{Nov} \subset \mathbb{O}_\sigma^{Nov}$ and $\mathbb{O}_{\sigma'} \subset \mathbb{O}_\sigma$ of both of these are defined.

Lemma 22. *1. The $\mathbb{O}_\sigma^{Nov}$, as well as (when defined) the $\mathbb{O}_\sigma$, together with the maps above give a representation of the poset $\Sigma_{\preccurlyeq}^{op}$ in rings.*

2. *For each pair $\tau \preccurlyeq \sigma$, the map of rings $\mathbb{O}_\sigma^{Nov} \to \mathbb{O}_\tau^{Nov}$, as well as (when defined) the map $\mathbb{O}_\sigma \to \mathbb{O}_\tau$, are localization maps.*
3. *If we have a quadruple of cones*

$$\begin{tikzcd}
& \sigma \arrow[dr] & \\
\tau \arrow[ur] \arrow[dr] & & \kappa \\
& \sigma' \arrow[ur] &
\end{tikzcd}$$

such that $\tau = \sigma \cap \sigma'$ then $\mathbb{O}_\sigma^{Nov} \otimes_{\mathbb{O}_\kappa^{Nov}} \mathbb{O}_{\sigma'}^{Nov} \cong \mathbb{O}_\tau^{Nov}$, i.e. on the level of open subsets in $\operatorname{Spec} \mathbb{O}_\kappa^{Nov}$, we have

$$\operatorname{Spec} \mathbb{O}_\sigma^{Nov} \cap \operatorname{Spec} \mathbb{O}_{\sigma'}^{Nov} = \operatorname{Spec} \mathbb{O}_\tau^{Nov}.$$

If the fan Σ is rational, we have an analogous result with $\mathbb{O}_*^{Nov}$ replaced by $\mathbb{O}_*$.

The lemma above is precisely what is needed to show that the $\operatorname{Spec} \mathbb{O}_\sigma^{Nov}$, respectively, when defined, the $\operatorname{Spec} \mathbb{O}_\sigma$, glue together to produce a variety, which we call X_Σ^{Nov} (resp., for rational Σ, the variety X_Σ). In the special case where the fan is affine, $\Sigma = \Sigma_\sigma$, the glueing diagram has a terminal element, σ, and so the variety is simply the corresponding affine $\operatorname{Spec} \mathbb{O}_\sigma$ or $\operatorname{Spec} \mathbb{O}_\sigma^{Nov}$. We introduce notation:

Definition 22.

$$X_\sigma := \operatorname{Spec} \mathbb{O}_\sigma,$$
$$X_\sigma^{Nov} := \operatorname{Spec} \mathbb{O}_\sigma^{Nov}.$$

Since all our rings as well as the maps between them, are $M_\mathbb{R}$ graded (resp., in the rational case, M-graded), we get a T_{Nov} (respectively, in the rational case, a T)-graded structure on X_Σ^{Nov} (resp., in the rational case, X_Σ). The variety X_Σ is the standard toric variety associated to the rational fan Σ, with standard T action. We call the T_{Nov}-equivariant variety X_Σ^{Nov} the *(toric) Novikov variety* associated to the fan Σ.

With the glueing data above in hand, the following definition is standard:

Definition 23. *A coherent sheaf $\mathcal{F}$ on X_Σ^{Nov} is uniquely determined by a collection of modules $\mathcal{F}_\sigma$ over $\mathbb{O}_\sigma^{Nov}$ with coherent maps $res_{\sigma\tau}\mathcal{F}_\sigma \to \mathcal{F}_\tau$ for $\sigma \subset \tau$ indexed by the poset $\Sigma_{\preccurlyeq}^{op}$, such that the adjoint map*

$$\mathcal{F}_\sigma \otimes_{\mathbb{O}_\sigma^{Nov}} \mathbb{O}_\tau^{Nov} \to \mathcal{F}_\tau$$

is an isomorphism. Similarly, for rational Σ, a coherent sheaf on X_{Nov} is determined by a diagram $\mathcal{F}_\sigma$ of modules over $\mathbb{O}_\sigma$ indexed by Σ with an analogous localization requirement.

Now from Corollary 20, we deduce the following result.

Lemma 23. *A T^{Nov}-equivariant coherent sheaf on X_Σ^{Nov} is uniquely determined by a colletion $\mathcal{F}_\sigma$ as in definition 23 together with $M_\mathbb{R}$-gradings of each $\mathcal{F}_\sigma$ which are compatible with the maps $res_{\sigma\tau}$.*

4.1 The poset M_σ

Suppose σ is a cone.

Definition 24. *Let $M_\sigma := M/\sigma^\perp$ be the dual vector space to $\langle\sigma\rangle$.*

On M_σ the cone $\sigma^\vee/\sigma^\perp$ is sharp, and thus defines a partially ordered set structure as follows.

Definition 25. *For $m, m' \in M_\sigma$, we say that $m \preccurlyeq_\sigma m'$ if $m' - m \in \sigma^\vee/\sigma^\perp$.*

Note that if $m, m' \in M$ we can also define a category structure on M itself with a single morphism $\sigma \to \sigma'$ if $\sigma' - \sigma \in \sigma^\perp$. This will be a *preorder* (poset with isomorphisms), and quotienting out by the isomorphisms will give us back the poset M_σ. Sometimes we will abuse notation slightly and use M_σ for the above preorder on M.

4.2 Almost locality

?? In the Novikov setting, we will consider a certain full subcategory $\mathrm{Qcoh}^a(X_\Sigma^{Nov}) \subset \mathrm{Qcoh}(X_\Sigma^{Nov})$ of "almost local" objects. We begin by defining this category for the affine case, $X_\sigma^{Nov} = \mathrm{Spec}\,\mathbb{O}_\sigma$ for a cone σ. Write $\mathbf{a}_\sigma \subset \mathbb{O}_\sigma$ for the homogeneous ideal

$$\mathbf{a}_\sigma := k(\mathring{\sigma}^\vee) \subset k(\sigma).$$

The ideal $\mathbf{a}_\sigma$ has the wonderful property

$$\mathbf{a}_\sigma^2 = \mathbf{a}_\sigma$$

(the corresponding property fails for ideals in the classical toric case), and so we can apply a small piece of the theory of "almost Mathematics" of [Faltings].[1] First, a definition.

Definition 26. *1. Define the category of* almost local *representations, $\mathrm{Rep}^a(\mathbb{O}_\sigma)$ to be the full subcategory of $\mathrm{Rep}(\mathbb{O}_\sigma)$ consisting of representations V such that the natural map $V \to \mathrm{Hom}_{\mathbb{O}_\sigma}(\mathbf{a}_\sigma, V)$ is an isomorphism.*

2. Write $\mathrm{Rep}_\partial(\mathbb{O}_\sigma)$ for the full subcategory of representations of V on which a_σ acts by 0.

Lemma 24. *1. The category $\mathrm{Rep}_\partial(\mathbb{O}_\sigma^{Nov})$ is a Serre subcategory. (Note that the corresponding statement does not hold in the classical toric case!)*

[1] Faltings considers a slightly different context: that of non-discretely normed valuation fields. In the case where the cone σ is a one-dimensional ray in $\mathbb{R}^1$, imposing the almost locality condition on sheaves is equivalent to imposing the condition in [Faltings] on their adic completion, the ring of Novikov *series* of positive valuation.

2. *The functor* $V \mapsto \mathrm{Hom}(\mathbf{a}_\sigma, V)$ *takes an object of* $\mathrm{Rep}\,\mathbb{O}_\sigma^{Nov}$ *to an object of* $\mathrm{Rep}^a\,\mathbb{O}_\sigma^{Nov}$ *and is right adjoint to the inclusion of categories* $\mathrm{Rep}^a\,\mathbb{O}_\sigma^{Nov} \to \mathrm{Rep}\,\mathbb{O}_\sigma^{Nov}$.
3. *The triple of functors* $\mathrm{Rep}_\partial(\mathbb{O}_\sigma^{Nov}) \to \mathrm{Rep}(\mathbb{O}_\sigma^{Nov}) \to \mathrm{Rep}^a(\mathbb{O}_\sigma^{Nov})$ *is a Serre quotient diagram.*

All of these are proven in [Gabber-Ramero]. From this we deduce that we have a short exact sequence of categories (in the sense of Serre)

$$\mathrm{Rep}_\partial(\mathbb{O}_\sigma) \to \mathrm{Rep}(\mathbb{O}_\sigma) \to \mathrm{Rep}^a(\mathbb{O}_\sigma).$$

Given a representation $V \in \mathrm{Rep}(\mathbb{O}_\sigma)$, write $V^a := \mathrm{Hom}_{\mathbb{O}_\sigma}(\mathbf{a}_\sigma, V)$. We say V^a is the "almost local representation associated to V".

Now for a fan Σ which is not necessarily affine, write $\mathrm{Qcoh}^a(X)$ for the category of sheaves of modules $\mathcal{F}$ such that $\mathcal{F} \mid X_\sigma$ is almost local. Alternatively, define $X_\partial \subset X$ to be the variety glued out of $\mathrm{Spec}\,\mathbb{O}_\sigma/\mathbf{a}_\sigma$, a union of toric Novikov varieties of lower dimension. Define $\mathbf{a}_\partial$ to be the sheaf of ideals defining this variety. The condition of $\mathcal{F}$ being almost local is equivalent to the condition that $\mathcal{F} \to \mathrm{Hom}(\mathbf{a}_\partial, \mathcal{F})$ is locally an isomorphism. Note that once again contrary to our expectations from classical algebraic geometry, this category is different from requiring $\mathcal{F}$ to be a pushforward of a sheaf on the open orbit T_{Nov}: the latter condition can be thought of as the vanishing of $\mathcal{F}$ on a "large" formal neighborhood, $X_{\widehat{\partial}}$ of the boundary, whereas this is vanishing on the boundary itself, with no thickening.

If V is a representation of $\mathbb{O}_\sigma$ and $\tau \preccurlyeq \sigma$ is a subcone, then

$$(V \otimes_{\mathbb{O}_\sigma} \mathbb{O}_\tau)^a \cong V^a \otimes_{\mathbb{O}_\sigma} \mathbb{O}_\tau.$$

This means that the functor $V \mapsto V^a$, as well as the functor $V \mapsto V_\partial$ glue to functors on sheaves, $\mathcal{F} \mapsto \mathcal{F}^a$ and $\mathcal{F} \mapsto \mathcal{F}_\partial$.

4.3 Quasidivisors and $\mathbb{R}$-quasidivisors

Suppose that $X = X_\sigma^{Nov}$ is a toric variety.

Definition 27. *We define*

$$\mathrm{Div}_\sigma^{\mathbb{R}} := \langle \sigma \rangle^*$$

and, if σ is rational, we define the lattice

$$\mathrm{Div}_\sigma^{\mathbb{Z}} := (\langle \sigma \rangle \cap N)^* \subset \langle \sigma \rangle^*.$$

We view elements of $\mathrm{Div}_\sigma^{\mathbb{R}}$ as linear functions on σ (which extend uniquely to $\langle \sigma \rangle$). In particular, we say that a divisor (or $\mathbb{R}$-divisor) α is *effective* if it is nonnegative as a function on σ. Note that for a rational fan, divisors $\alpha \in \mathrm{Div}_\sigma$ are in bijection with equivariant Cartier divisors on the affine toric variety X_σ.

If Σ is a general fan, we make the following definition. Write $V_\Sigma := \bigcup_{\sigma \in \Sigma} \sigma \subset N$ for the "support" of a fan. Write

Definition 28. $\mathrm{Div}_\Sigma^{\mathbb{R}} := \{\alpha : V_\Sigma \to \mathbb{R} \mid \alpha|_\sigma \in \mathrm{Div}_\sigma^{\mathbb{R}}\}$. *Similarly, when Σ is rational,*

$$\mathrm{Div}_\Sigma^{\mathbb{Z}} := \{\alpha \in \mathrm{Div}_\sigma^{\mathbb{R}} \mid \alpha|_\sigma \in \mathrm{Div}_\sigma^{\mathbb{Z}}\}.$$

We say that a ($\mathbb{R}$-)divisor is *effective* if α is nonnegative and *principal* if $\alpha = \lambda \mid V_\sigma$ for some $\lambda \in M_{\mathbb{R}} = N_{\mathbb{R}}^*$ (viewed as functions on $N_{\mathbb{R}}$).

Note that if $X = X_\Sigma$ is a toric variety, elements of $\mathrm{Div}_X^{\mathbb{Z}}$ are in bijection with T-equivariant Cartier divisors on X. Indeed, recall that every irreducible equivariant variety in X is the closure of a torus orbit. In particular, irreducible codimension-one equivariant subvarieties are $Y_\sigma := \overline{X_\sigma \setminus T}$ classified by rays $\eta \in \Sigma^1$. The divisor $\alpha \in \mathrm{Div}_\Sigma^{\mathbb{Z}}$ in the sense above then corresponds to the Weyl divisor $\sum \alpha(g_\eta) Z_\eta$, where recall that $g_\eta \in \eta$ is the unique generator of the lattice subgroup. It is a standard result in the geometry of toric varieties, which we will not use, that a Weyl divisor $\sum d_\eta Z_\eta$ is a Cartier divisor (and hence corresponds to an equivariant line bundle) if and only if the $d_\eta = \alpha(g_\eta)$ for some function $\alpha \in \mathrm{Div}_\Sigma^{\mathbb{Z}}$. Further, equivariant line bundles are in bijection with equivariant divisors, and non-equivariant divisors are in bijection with the quotient $\frac{\mathrm{Div}_\Sigma^{\mathbb{Z}}}{M}$ of equivariant Cartier divisors by principal equivariant divisors.

Recall that to every *Weyl* divisor d on X we can associate a sheaf, consisting of rational functions on X with degree of pole (or zero) along any irreducible codimension-one subvariety $Z \subset X$ bounded locally by d_η. When d is an equivariant divisor, this sheaf is naturally equivariant and when d is a Cartier divisor, this sheaf is a line bundle. Note that if we formally set some of the $d_\eta = +\infty$, there will still be a quasicoherent (but not coherent!) sheaf associated to d, where if $d_\eta = +\infty$, we consider functions with locally arbitrary poles allowed along Z_η.

It will be useful for us to partially compactify the groups $\mathrm{Div}_\Sigma^{\mathbb{R}}$ and $\mathrm{Div}_\Sigma^{\mathbb{Z}}$ to a larger semigroup, called *quasidivisors*, which (in the rational case) precisely corresponds to extending the set of (Cartier) divisors by allowing some of the divisor coefficients to be $+\infty$. To this end, we make the following definitions.

Definition 29. *Define* $\overline{\mathbb{R}} := \mathbb{R} \cup +\infty$ *and* $\overline{\mathbb{Z}} := \mathbb{Z} \cup +\infty$.

These have the structure of ordered semigroups in an obvious way (with $\infty + k = \infty$ for any $k \in \overline{\mathbb{R}}$), and we can rescale elements of $\overline{\mathbb{R}}$ by $\mathbb{R}^+$, with $0 \cdot +\infty := 0$. Now for $\Sigma = \Sigma_\sigma$ an affine fan, define $\mathrm{QDiv}_\sigma^{\mathbb{R}}$ to be the set of maps of semigroups $\sigma \to \overline{\mathbb{R}}$ which commute with scaling. This is equivalent to the following definition.

Definition 30. $\mathrm{QDiv}_\sigma^{\mathbb{R}}$ *is the set of functions* $\alpha : \sigma \to \overline{\mathbb{R}}$ *which take finite values on a single face* $\tau \subset \sigma$ *(of arbitrary dimension* ≥ 0*) and are linear on that face.*

We also define

Definition 31. *If* σ *is rational* $\mathrm{QDiv}_\sigma^{\mathbb{Z}} \subset \mathrm{QDiv}_\sigma^{\mathbb{R}}$ *are those functions which take values in* $\overline{\mathbb{Z}} \subset \overline{\mathbb{R}}$ *on* $\sigma \cap N$.

If Σ is not affine, we define $\mathrm{QDiv}_\Sigma^{\mathbb{R}}$ and $\mathrm{QDiv}_\Sigma^{\mathbb{Z}}$ to be functions $V_\Sigma \to \overline{\mathbb{R}}$ which satisfy the conditions above on each (closed) cone σ:

Definition 32. $\mathrm{QDiv}_\Sigma^{\mathbb{R}}$*, resp. (if* Σ *is rational),* $\mathrm{QDiv}_\Sigma^{\mathbb{Z}}$*, is the set of functions* $\alpha : V_\Sigma \to \overline{\mathbb{R}}$ *such that* $\alpha \mid \sigma \in \mathrm{QDiv}_\sigma^{\mathbb{R}}$ *for each* $\sigma \in \Sigma$*, respectively, such that* $\alpha \mid \sigma \in \mathrm{QDiv}_\sigma^{\mathbb{Z}}$ *for each* $\sigma \in \Sigma$.

We say that a ($\mathbb{R}$-)quasidivisor is

1. *effective*, if α takes values in $\mathbb{R}^+ \cup \infty$;
2. *finite*, if $\alpha \in \mathrm{QDiv}^{\mathbb{R}} \subset \mathrm{QDiv}$;
3. *principal*, if $\alpha = \langle \lambda, \cdot \rangle$ for some $\lambda \in \Lambda$ (in particular, such quasidivisors are finite),
4. *affine*, if $\alpha = +\infty$ outside a single cone $\sigma \in \Sigma$;
5. *semiample*, if α is convex up, i.e. if $\alpha(x+y) \leq \alpha(x) + \alpha(y)$ (in an ordered semi-group sense), and
6. *ample*, if α is semiample and $\alpha(x+y) = \alpha(x) + \alpha(y)$ if and only if either $\alpha(x+y) = \infty$ or x, y both belong to some cone σ.

4.4 Sheaves associated to quasidivisors and $\mathbb{R}$-quasidivisors

We make the following definition.

Definition 33. *Given an $\mathbb{R}$-quasidivisor α on an affine fan Σ_σ, define the set*

$$\Delta_\alpha (= \Delta_\alpha^\sigma) := \{\lambda \in M_{\mathbb{R}} \mid \langle \lambda, y \rangle \leq -\alpha(y) \forall y \in \sigma.\}$$

Note the negative sign $-\alpha$ in the definition. This is the same sign that comes in the definition of a line bundle associated to a divisor: the larger the multiplicity of D in a divisor, the more *negative* the (I_D-adic) valuation of functions allowed in the ideal.

All Δ_α for α *finite* are shifts of $\sigma^\vee$. Indeed, there is some λ_α such that $\alpha = \langle \lambda_\alpha, \cdot \rangle \mid \sigma$, and then $\Delta_\alpha = \sigma^\vee - \lambda_\alpha$. Note that the shift vector λ_α is not in general unique, as $\sigma^\vee$ has additive symmetry by $\sigma^\perp$: rather, it is unique up to an additive factor of $\sigma^\perp$. Now define $a_\alpha^\sigma := k(\mathring{\Delta}_\alpha)$. This is a graded $\mathbb{O}_\sigma$-module isomorphic in a non-graded way to a_σ. Recall that a general quasidivisor α is finite precisely on some face $\tau \subset \sigma$, and then Δ_α is a shift of $\tau^\vee$.

Definition 34. *Given an $\mathbb{R}$-quasidivisor $\alpha \in \mathrm{QDiv}_\Sigma^{\mathbb{R}}$, define the T_{Nov}-equivariant sheaf $\mathbb{O}(-\alpha) \in \mathrm{Qcoh}^a(X_\Sigma^{Nov})^{T_{Nov}}$ to be the sheaf which on the affine $X_\sigma^{Nov} \subset X_\Sigma^{Nov}$ has sections $k(\Delta_\alpha)$, and with obvious transition maps. Similarly if Σ is a rational fan and $\alpha \in \mathrm{QDiv}_\Sigma^{\mathbb{Z}}$ is an integral quasidivisor, define the T-equivariant sheaf $\mathbb{O}(\alpha)$ to be the sheaf which on the affine X_σ has sections $k(\Delta_\alpha \cap M)$.*

It is immediate to check that these indeed glue to give equivariant and, in the Novikov case, almost local, sheaves. These sheaves will be the objects corresponding to opens of the Dolan topology, defined in the next section.

5 The Dolan Topology and the Abelian Coherent-Constructible Correspondence

Associated to any fan Σ we will define a Grothendieck topology $\Theta_{\mathbb{R}}$ on the poset $\mathrm{QDiv}_{\mathbb{R}}$ which we call the "Dolan-Novikov topology". When Σ is rational, we also define an integral subtopology on $\mathrm{QDiv}_{\mathbb{Z}}$, the "Dolan topology". The main result of this section will be the Abelian coherent-constructible correspondence, i.e. the equivalence of the equivariant almost coherent category $\mathrm{Qcoh}^a(X_{Nov})^{T_{Nov}}$ with sheaves on $\Theta_{\mathbb{R}}$. Similarly, we will show that when Σ is **regular**, we have a an equivalence in the non-Novikov (classical algebro-geometric) context $\mathrm{Qcoh}(X)^T \cong \mathrm{Shv}\,\Theta_{\mathbb{Z}}$.

Remark 2. *Assuming regularity in the rational case allows us to avoid a few topos-theoretic difficulties that come up in the rational non-regular case (since the Dolan site is better behaved, in particular having all intersections). With a little more work, we can recover the equivalence of Abelian categories* $\mathrm{Qcoh}(X)^T \cong \mathrm{Shv}(\Theta_\Sigma^{\mathbb{Z}})$ *for arbitrary rational* Θ*, and this, along with a proof of the classical coherent-constructible correspondence for non-smooth varieties will be done in [Appendix:non-smooth]. Surprisingly, none of the issues arising for non-regular fans in the rational context come up in the Novikov context.*

5.1 Grothendieck topology on a poset

In this section, some reminders about Grothendieck topologies. The categories $\mathrm{QDiv}_{\mathbb{Z}}, \mathrm{QDiv}_{\mathbb{R}}$ on which our topologies are defined are particularly nice in some of the same ways in which topoi coming from a topological space are nice: the underlying categories are posets, they have all intersections (i.e., they are "meet semi-lattices") and the covering families we introduce can be easily checked to be "strongly epic". In this situation, the theory simplifies and we use full advantage of these special simplifications in our definitions and results. In [Appendix on topoi] we will upgrade some of these results to be able to deal with the M-equivariant analogues $\overline{\tau}$ of τ.

Suppose a category $\mathcal{C}$ is a *poset*. Then a Grothendieck topology on $\mathcal{C}$ consists of the following data satisfying the following conditions.

Definition 35. *Suppose* $\mathcal{C}$ *is a poset. A* Grothendieck topology Θ *on* $\mathcal{C}$ *is a collection,* Cov_X*, for every object* $X \in \mathcal{C}$*, of sets of subobjects* $\mathcal{U} \preccurlyeq X$*. The poset* $\mathcal{C}$ *and the set of covering subcategories is required to satisfy the following conditions.*

1. *(Identity) The set* $\{X\}$ *is a covering of* X*.*
2. *(Existence of pullbacks) Given any map* $U \preccurlyeq X$ *with an element of some* $\mathcal{U} \in Cov_X$*, and any map* $Y \to X$*, the pullback* $U \times_X Y$ *exists (for a poset, a pullback is also called a "meet").*
3. *(Basechange) Given a* $\mathcal{U} \in Cov_X$ *and any map* $Y \to X$*, the set of basechanged maps* $U \times_X Y \xrightarrow{\iota \times_X Y} Y$ *is a covering of* Y *as* $U \xrightarrow{\iota} Y$ *goes over objects of* $\mathcal{U}$*.*
4. *(Subdivision) If* $\mathcal{U}_X \in Cov_X$ *is a covering of* X *and* $\mathcal{U}_U \in Cov_U$ *is a collection, for each* $U \in \mathcal{U}_X$*, of coverings, the "subdivided covering"*

$$\bigcup_{U \in \mathcal{U}} \mathcal{U}_U$$

is a covering of X*.*

All the Grothendieck topologies we consider will be "strong", i.e. have finite limits, and covering families will be strong epimorphisms (see [Appendix: sites]).

Definition 36. *A map of strong Grothendieck topologies $\alpha : \Theta \to \Theta'$ is a functor of underlying categories in the opposite direction,*[2] $\alpha^{-1} : \mathcal{C}' \to \mathcal{C}$, *such that*

1. *(Covering.) If $\mathcal{U}_Y$ is a covering of Y then $f^{-1}(\mathcal{U}_Y)$ is a covering of $f^{-1}(Y)$ and*
2. *(Continuity.) The functor f^{-1} respects pullbacks*

Note that for Grothendieck topologies which are not strong, the covering and continuity properties need to be replaced by slightly more subtle properties.

5.2 The Dolan and Novikov-Dolan topologies

Definition 37. *Let $\mathrm{QDiv}_{\mathbb{R}}, \preccurlyeq$ be the category of quasidivisors with morphisms given by the partial order $d \preccurlyeq d'$ if $d(x) \leq d(x)$ pointwise. We say that the set d_i cover d if $d_i \preccurlyeq d$ and the supremum $\sup(d_i(x)) = d(x)$ for any $x \in N_{\mathbb{R}}$. Write $\Theta(= \Theta_\Sigma)$ for QDiv with this notion of open cover. Define $\Theta^{\mathbb{Z}}(= \Theta_\Sigma^{\mathbb{Z}})$ for the category $\mathrm{QDiv}_{\mathbb{Z}} \subset \mathrm{QDiv}_{\mathbb{R}}$ with induced partial order and collection of open covers.*

Lemma 25. *The assignments of open cover $\Theta, \Theta_{\mathbb{R}}$ define Grothendieck topologies.*

Proof. Note that Θ and $\Theta_{\mathbb{Z}}$ have all limits. Checking this is equivalent to checking the existence of a terminal element and fibered product. The terminal element is the infinite quasidivisor,

$$d_\infty : x \mapsto \begin{cases} 0, & x = 0 \\ \infty, & x \neq 0 \end{cases}$$

Existence of fibered products in a poset is equivalent to existence of meet (highest lower bound), which is given by $d \cap d' := \inf(d, d')$. Existence of pullback follows. Identity, basechange and subdivision are obvious. $\square$

Now we are finally able to state the main result of this section.

Theorem 26 (Abelian coherent-constructible correspondence)**.** *Suppose that Σ is a fan. Then we have equivalences of Abelian categories*

1. $\mathrm{Shv}(\Theta_\sigma) \cong \mathrm{Shv}^a(X_\Sigma^{Nov})^{T_{Nov}}$,
2. *if Σ is rational, we have an equivalence of Abelian categories* $\mathrm{Shv}(\Theta_\sigma^{\mathbb{Z}}) \cong \mathrm{Shv}(X_\Sigma)^T$.

[2]This is to model the "pullback of opens" functor $U_Y \mapsto f^{-1}(U_Y) \subset X$ associated to a map of topological spaces $f : X \to Y$

5.3 Push-pull functors for fan inclusions

Suppose $\Sigma' \subset \Sigma$ is a sub-fan (we will mostly be using affine $\Sigma' = \Sigma_\sigma$ for $\sigma \in \Sigma$ a cell). We define two maps $\iota_{\sigma\sigma'} : \Theta_{\Sigma'} \to \Theta_\Sigma$ and $\pi_{\sigma\sigma'} : \Theta_\Sigma \to \Theta_{\Sigma'}$ as follows. Suppose $d : V_\Sigma \to \mathbb{R}$ is an $\mathbb{R}$-quasidivisor and $d' : V_{\sigma'} \to \mathbb{R}$ is an $\mathbb{R}$-quasidivisor.

Definition 38. *We define*

$$\iota^{-1}(d) := \iota \mid V'_\Sigma$$

and

$$\pi^{-1}(d) : y \mapsto \begin{cases} d(y), & y \in V_{\Sigma'} \\ \infty, & \textit{else.} \end{cases}$$

We write ι_σ for $\iota_{\Sigma_\sigma,\Sigma}$, and similarly for π_σ.

Proposition 27. *The functors $\iota_{\Sigma\Sigma'}, \pi_{\Sigma\Sigma'}$ are morphisms of Grothendieck topologies.*

We see this by checking compatibility with covers and intersections. Fixing a map $\Sigma \subset \Sigma'$, write $\iota^*, \iota_*, \pi^*, \pi_*$ for the pullback and pushforward functors between $\mathrm{Shv}(\Sigma), \mathrm{Shv}(\Sigma')$. These sheaves have an extremely unusual relation (compared to what one would expect for topological or étale sites):

Lemma 28. *There is a natural isomorphism between the functors $\pi^*, \iota_* : \mathrm{Shv}(\Sigma) \to \mathrm{Shv}(\Sigma')$.*

Proof. Given a sheaf $\mathcal{V}$ on Σ, denote $\mathcal{V}_d$ for $\Gamma(\mathcal{V}, d)$ for $d \in \mathrm{QDiv}(\Sigma)$. Suppose $\mathcal{V}$ is a sheaf on Σ and d' is a divisor in $\mathrm{QDiv}_{\Sigma'}$. Then $\iota^\sigma_*(\mathcal{V})_{d'} := \mathcal{V}_{\iota^{-1}(d')} \cong \mathcal{V}_{d'|\Sigma}$, where $d' \mid \Sigma$ is a restriction of d' viewed as a function on the support of Σ' to the support of Σ'. On the other hand, $\pi^*_\sigma(\mathcal{V})_{d'} := \varinjlim \mathcal{V}'_d \mid \pi^{-1}(d) \succcurlyeq d'$. But $\pi^{-1}(d) \succcurlyeq d'$ if and only if $d \succcurlyeq \iota^{-1}(d')$, and in particular the poset $d' \mid \pi^{-1}(d') \succcurlyeq d$ has an initial element, $d = \iota^{-1}(d')$. Thus $\iota_*(\mathcal{V})^*_\sigma \cong \pi^*(\mathcal{V})_\sigma$, and it is an immediate check that this isomorphism is compatible with restriction maps and functorial. $\square$

This fact will be extremely useful for us, and implies in particular that π^* has an easily accessible left adjoint ι^* (such a functor can with some legitimacy be called $\pi_!$). Further, since $\iota_* = \pi^*$, both ι^* and ι_* are exact (this corresponds to the fact that both pullback and pushforward from an open subvariety is exact).

5.4 Points

Recall that a Grothendieck topology has a notion of *point* which generalizes the notion of a point of a variety, and that for general Grothendieck topologies points form a *category* rather than just a set, and the functor of "stalks" produces a representation of this category starting from any sheaf $\mathcal{V}$. When there are "enough points" (as we shall see to be the case for the Novikov-Dolan topology $\Theta_\mathbb{R}$ and its relatives), we can check isomorphism of sheaves on stalks. As we shall see, stalks at points in the Dolan topology correspond to (graded components of) restrictions to affine opens of X^{Nov}.

Definition 39. *A point of a Grothendieck topology $(\mathcal{C}, \Theta)$ is a functor $\mathcal{C} \to Op_{\mathrm{pt}}$, where Op_{pt} is the Boolean category $\{\emptyset \to \mathrm{pt}\}$, which is compatible with intersections and open covers*[3].

Now suppose $\Sigma = \Sigma_\sigma$ is an affine fan, and say $m \in M_{\mathbb{R}}/\sigma^\perp$ is a point of the dual poset associated to σ. Then we define the following functor from QDiv_σ to Sets:

$$Þ_m(d) := \begin{cases} \mathrm{pt}, & \Delta_d \ni m \\ \emptyset, & \text{otherwise.} \end{cases}$$

Write $\mathrm{St}Þ^\sigma_m(\mathcal{V})$ for the stalk of $\mathcal{V}$ at the point $Þ^\sigma_m$ Now any map of Grothendieck topologies sends points to points (in a functorial way). If Σ is an arbitrary flag and $\sigma \in \Sigma$ is a cone with $m \in M/\sigma^\perp$ as before, write $Þ^\sigma_m := \iota^\sigma(Þ_m)$, for $Þ_m$ the point of Θ_σ as above. Note that this construction gives new points in Θ_σ itself, corresponding to pushforward of points from Θ_τ for $\tau \preccurlyeq \sigma$.

Lemma 29. *Let Σ be a fan. The collection $Þ^\sigma_m$ is a separating collection of points for Θ_Σ.*

Proof.

Lemma 30. *Let $d \in \mathrm{QDiv}$ be a divisor and $\mathcal{V} \in \mathrm{Shv}_\mathcal{T}$ be a sheaf. Then $\Gamma(d, \mathcal{V}) \cong \underset{\to \, \sigma \in \Sigma^{op}_{\preccurlyeq}}{lim} St(Þ^{d_\sigma}_\sigma)$, via the evident diagram of inclusions $Þ^{d_\tau}_\tau \subset Þ^{d_\sigma}_\sigma$ for $\tau \subset \sigma$ cones, compatible with the inclusions $Þ^{d_\sigma}_\sigma \subset d$.*

It is obvious that the characteristic sheaves of open sets k_d are a separating collection of sheaves for $d \in \mathrm{QDiv}$. Now the colimit of all skyscraper sheaves $\delta_{Þ^m_\sigma}$ for $Þ^m_\sigma \in d$ is k_d, and so the $Þ^m_\sigma$ separate. ∎

5.5 The Abelian coherent-constructible correspondence

We are ready to define a pair of inverse functors between $\mathrm{Qcoh}^a(X^{Nov}_\Sigma)^{T_{Nov}}$ and $\mathrm{Shv}(\Theta)$. Namely, consider the collection of equivariant sheaves $\mathbb{O}_d$ for $d \in \mathrm{QDiv}$. We have obvious maps $\mathbb{O}_d \to \mathbb{O}_{d'}$ for $d \preccurlyeq d'$, which give a representation of the coset QDiv .

Lemma 31. *The $\mathbb{O}_d$ form a cosheaf of objects of $\mathrm{Qcoh}^a(X^{Nov}_\Sigma)$ over Θ.*

Proof. Working locally, it is sufficient to show this for an affine open $X_\sigma \subset X_\Sigma$. Using almost locality, $\mathrm{Hom}(\mathbb{O}_d, \mathcal{F}) = \mathrm{Hom}(\mathbb{O}_d \otimes \mathbf{a}_\partial, \mathcal{F})$, with $\mathbb{O}_d \otimes \mathbf{a}_\partial$ locally given by $k(\mathring{\Delta}_d)$ Thus, using Proposition 15, it is enough to show that the shifted open cones $\mathring{\Delta}_d$ satisfy the sheaf property for $d \in \mathrm{QDiv}(\Sigma_\sigma)$, which is obvious. ∎

Now associated to the cosheaf $\mathbb{O}_{\mathrm{QDiv}}$, we get a pair of functors: $F : \mathrm{Qcoh}^a(\mathbb{O}_{\mathrm{QDiv}})^{T_{Nov}} \leftrightarrows \mathrm{Shv}(\Theta_\Sigma) : G$, defined as follows.

$$\Gamma\big(F(\mathcal{F}), d\big) := \mathrm{Hom}(\mathbb{O}_d, \mathcal{F})$$

[3]Here again we are using the strong surjectivity property

$$\Gamma(G(\mathcal{V}), X_\sigma)\langle m\rangle := \mathrm{St}\text{Þ}^\sigma_m(\mathcal{V}).$$

We refine the Abelian ccc to the following lemma.

Lemma 32. *The functors F, G defined above are mutually inverse equivalences, as are their integral analogues $F_\mathbb{Z}, G_\mathbb{Z}$.*

We will prove this lemma bit by bit in the remainder of the section. We work mostly with the Novikov case and functors F, G and only treat the classical case $F_\mathbb{Z}, G_\mathbb{Z}$ separately when the arguments from the Novikov case do not translate in an obvious way. Recall that we have sub-posets $\mathrm{Div}_\mathbb{R} \subset \mathrm{QDiv}_\mathbb{R}$ and $\mathrm{Div}_\mathbb{Z} \subset \mathrm{QDiv}_\mathbb{Z}$ of "finite" divisors. Since these posets are cofiltering, any cover of an element of $\mathrm{Div}_\mathbb{R}$ consists of other elements of $\mathrm{Div}_\mathbb{R}$ and intersections in $\mathrm{Div}_\mathbb{R}$ agree with intersections in $\mathrm{QDiv}_\mathbb{R}$. Write $\Theta^f_\mathbb{Z}, \Theta^f_\mathbb{R}$ for the induced Grothendieck topology on $\mathrm{Div}_\mathbb{Z}, \mathrm{Div}_\mathbb{R}$. We have the following result:

Proposition 33. *For any fan Σ, we have an equivalence of Abelian categories* $\mathrm{Shv}(\Theta^f_\mathbb{R}) \cong \mathrm{Shv}(\Theta_\mathbb{R})$ *and, if Σ is rational,* $\mathrm{Shv}(\Theta^f_\mathbb{Z}) \cong \mathrm{Shv}(\Theta_\mathbb{Z})$.

Proof. This follows from the fact that divisors are cofiltering (i.e. closed with respect to taking subobjects) and generating in quasidivisors (i.e. any quasidivisor has a covering by divisiors). □

Now we proceed in stages.

5.5.1 Affine, rational fan Σ_σ

Assume $\Sigma = \Sigma_\sigma$ is affine and rational, and we are considering the site $\Theta_\mathbb{Z}$. Then note that the topology $\Theta^f_\mathbb{Z}$ is trivial — i.e. any open cover of a finite divisor d contains d itself as a terminal element. This means that $\mathrm{Shv}(\Theta_\mathbb{Z}) \cong \mathrm{Shv}(\Theta^f_\mathbb{Z}) \cong \mathrm{Rep}(\mathrm{Div}_\mathbb{Z})$, where $\mathrm{Rep}(\mathrm{Div}_\mathbb{Z})$ is the category of representations of the poset Div_Σ . On the other hand, an object $\mathcal{F}$ of $\mathrm{Qcoh}(X_\Sigma)^T$ is an M-graded representation of the graded semigroup $\Lambda_\sigma \cap M$. Let $\mathcal{F}\langle m\rangle$ be the m-weight component of $\mathcal{F}$, for $m \in M$. Then the action t^λ takes $\mathcal{F}\langle m\rangle \mapsto \mathcal{F}\langle m+\lambda\rangle$. The relations between $\lambda \in \Lambda$ amount to the statement that all maps $\mathcal{F}\langle m\rangle \to \mathcal{F}\langle m+\lambda\rangle$ obtained as compositions of ρ_{λ_i} for $\lambda_1 + \cdots + \lambda_k = \lambda$ are equal, i.e. that $\mathcal{F}\langle\cdot\rangle$ is a representation of the poset M_σ, equivalently a presheaf on $\Theta^f_\mathbb{Z}$.

5.5.2 Affine, Novikov fan Σ_σ

Now consider the Novikov case. An object of $\mathrm{Qcoh}(X^{Nov}_\Sigma)^{T_{Nov}}$ is an $M_\mathbb{R}$-graded representation of Λ_σ, equivalently as above a presheaf on $\Theta^f_\mathbb{R}$. However, in this case while $\Theta^f_\mathbb{R}$ is still cofiltering and generating (and hence has the same category of representations as $\Theta_\mathbb{R}$), its category of sheaves and presheaves are not the same, as we have nontrivial covers. For example for $M = \mathbb{R}$ and $\sigma = \mathbb{R}^+$, where $\mathrm{Div} \cong \mathbb{R}$, any upper-bounded collection of numbers which never attains its supremum s, such as $-1, -1/2, -1/3, \ldots$, will give a nontrivial cover of s. For general affine $\Sigma = \Sigma_\sigma$, we once again identify $\mathrm{Div}^\Sigma_\mathbb{R}$ with the poset $M^\sigma_\mathbb{R}$. Covers of the form $m_1 \preccurlyeq m_2 \preccurlyeq \ldots$ converging to some $m \in M^\sigma_\mathbb{R}$ will generate the topology. The sheaf condition on a presheaf $\mathcal{F}$ is then that

$\lim \mathcal{F}\langle m_i \rangle \cong \mathcal{F}\langle m \rangle$ for $m_1 \preccurlyeq m_2 \preccurlyeq \ldots \preccurlyeq m$ as above. Now note that $\mathcal{F}\langle m \rangle = \mathrm{Hom}_{gr}(k[\Delta_m], \mathcal{F})$, so by the exactness property of Hom, we have $\lim \mathcal{F}\langle m_i \rangle = \mathrm{Hom}_{gr}(k[\cup_i \Delta_{m_i}], \mathcal{F})$. Now there is a sequence of m_i such that $\cup \Delta_{m_i} = \mathring{\Delta}_{m_i}$, so that $\mathrm{Hom}_{gr}(k[\mathring{\Delta}_m], \mathcal{F}) = \mathrm{Hom}_{gr}(\mathbf{a} \otimes \mathbb{O}_m, \mathcal{F})$, so the sheaf condition implies the map $\mathcal{F} \to \mathrm{Hom}(\mathbf{a}, \mathcal{F}) = \bigoplus_{m \in M_{\mathbb{R}}} \mathrm{Hom}(\mathbf{a} \otimes \mathbb{O}_m, \mathcal{F})$ is an isomorphism, i.e. $\mathcal{F}$ is almost local. Conversely, suppose $\mathcal{F} \cong \mathrm{Hom}(\mathbf{a}, \mathcal{F})$. Then $\mathrm{Hom}(\mathbf{a}', \mathcal{F}) \cong \mathrm{Hom}(\mathbf{a}', \mathrm{Hom}(\mathbf{a}, \mathcal{F})) \cong \mathrm{Hom}(\mathbf{a} \otimes \mathbf{a}', \mathcal{F}) \cong \mathrm{Hom}(\mathbf{a}, \mathcal{F}) \cong \mathcal{F}$.

5.5.3 General Σ

Suppose Σ is a Novikov fan and $\mathcal{F} \in \mathrm{Qcoh}^a(X_{Nov})^{T_{Nov}}$ is an equivariant quasicoherent sheaf. Suppose that $Þ^m_\sigma$ is a point. Then

$$\mathrm{St}Þ^m_\sigma F(\mathcal{F}) := \varinjlim_{d \to Þ^m_\sigma} F(\mathcal{F})_d \cong \varinjlim_{d \to Þ^m_\sigma} \mathrm{Hom}(\mathbb{O}_d, \mathcal{F}) \cong \Gamma(X, \mathcal{F} \otimes \mathbb{O}_\sigma)\langle m \rangle.$$

Further, these identifications are functorial and compatible with restriction maps, establishing that $G \circ F \cong \mathbb{I}\mathrm{d}$. In the other direction, suppose that $\mathcal{V}$ is a sheaf on Θ. We have a restriction map $\mathcal{V}_d \to \oplus_{Þ^m_\sigma \in d} \mathrm{St}Þ^m_\sigma \mathcal{V}$ which glues to a map $\mathcal{V}_d \to \mathrm{colim}_{Þ^m_\sigma \in d} \mathrm{St}Þ^m_\sigma \mathcal{V}$. Now using the definition of the functor G and the affine Čech resolution for maps of quasicoherent sheaves, this colimit is canonically $\mathrm{colim}_{Þ^m_\sigma \in d} \mathrm{St}Þ^m_\sigma \mathcal{V} \cong \mathrm{Hom}_{gr}(\mathbb{O}_m, \mathcal{V}_d)$. This gives a map $\mathbb{I}\mathrm{d} \to F \circ G$ of endofunctors of $\mathrm{Shv}\, \Theta$. To check it is a natural isomorphism, it is sufficient to see this on a separating collection of stalks, specifically on the $\mathrm{St}Þ^m_\sigma$, which is obvious. An analogous argument shows that $F_{\mathbb{Z}}, G_{\mathbb{Z}}$ are inverse functors. This concludes the proof of Lemma 32 and hence of the Abelian coherent-constructible correspondence (Theorem 26).

6 The Equivariant Novikov Coherent-Constructible Correspondence

In previous sections, we have used again and again the geometric nature of the category $\mathrm{QDiv}^{\mathbb{R}}_\Sigma$ and the polyhedra Δ_d as combinatorial bookkeeping tools. In this section this geometric nature comes to the forefront. We will use this geometry to define the coherent-constructible correspondence (dg) functors, B^* and B_*. It then turns out to be almost immediate to check, using the formalism of points we have developed and a result in [FLTZ], that these functors are mutually inverse. From now on, we work with the formalism of DG categories, and all limits, pushforwards and pullback functors we will work with will be replaced by their derived analogues (using the usual derived convention of taking injective resolutions for computing pushforwards). We work with the injective derived categories $D^+(\mathrm{Shv}(\Theta_{\mathbb{R}}))$, etc. using that all the categories we are working with have enough injectives; we think of D^b as objects of D^+ with finitely many nonzero homology groups.

6.1 Affine resolution of sheaves

Before defining the mirror symmetry functors, we write down a result which will be very useful for us for deducing results for sheaves on Σ from affine analogues.

Lemma 34. *Suppose the fan Σ is complete. Then the functor $\mathbb{I}\mathrm{d}$: $\mathrm{Shv}(\Theta_{\mathbb{R}}) \to \mathrm{Shv}(\Theta_{\mathbb{R}})$ is the homotopy inverse limit in the derived category of the diagram of exact functors $\iota_*^\sigma \iota_\sigma^*$ indexed by $\Sigma_{\preccurlyeq}^{op}$, and with connecting morphisms given by adjunction.*

Proof. It is enough to show the natural map is an isomorphism on stalks. Now note the equivalence of exact functors

$$\mathrm{St}\text{Þ}_m^\sigma \circ \iota_*^{\sigma'} \circ \iota_\sigma^*) \cong \mathrm{St}\text{Þ}_m^\sigma \circ \pi_{\sigma'}^* \circ \iota_\sigma^* \cong \mathrm{St}\text{Þ}_m^{\sigma\cap\sigma'}.$$

It follows that the diagram $\lim\limits_{\leftarrow\ \sigma'} \mathrm{St}\text{Þ}_m^\sigma(\lim\limits_{\leftarrow} \iota_*^{\sigma'} \iota_{\sigma'}^*) \cong \lim\limits_{\leftarrow\ \sigma' \in \Sigma_{\preccurlyeq}^{op}} \mathrm{St}\text{Þ}_m^{\sigma\cap\sigma'}$ collapses to the subdiagram $\{\sigma' \mid \sigma' \preccurlyeq \sigma\}^{op}$, which has an initial element, σ. $\square$

6.2 Defining the Bondal functors B_*, B^*.

Let $\mathrm{Op}(M_{\mathbb{R}})$ be the Grothendieck topology of open sets in $M_{\mathbb{R}}$. For $\sigma \in \Sigma$ a cone, we make the following definition.

Definition 40. *We define a map of Grothendieck topologies $B_\sigma : Op(M_{\mathbb{R}}) \to \Theta_\sigma^{\mathbb{R}}$, with $B^{-1}(d) := \mathring{\Delta}_d$.*

From this we get a pair of adjoint functors $B_\sigma^* : \mathrm{Shv}(\Theta_\sigma) \to \mathrm{Shv}(M_{\mathbb{R}})$ and $B_*^\sigma : \mathrm{Shv}(M_{\mathbb{R}}) \to \mathrm{Shv}(\Theta_\sigma)$, whose derived functors constitute the full coherent-constructible correspondence for affine toric varieties.

Now if Σ is not affine, we can still get functors from $\mathrm{Shv}(\Theta_\Sigma)$ to and from $\mathrm{Shv}(M_{\mathbb{R}})$ by pushing and pulling along the composed map of topologies

$$B_{\sigma,\Sigma} := \iota_\sigma \circ B_\sigma : M_{\mathbb{R}} \to \Theta.$$

Now for general topological spaces, maps of topologies themselves form a category (which should be considered as a full subcategory of functors of topoi). In order to get our derived coherent-constructible functor, we will put the maps $B_{\sigma,\Sigma}$ into a diagram and take its colimit; this will give us an object of some category of derived correspondences. For the sake of brevity, we will not define any such category, but rather explicitly define its pushforward and pullback functors B_*, B^* as homotopy limits and colimits (which we will write as explicit chain complexes) of suitable diagrams with entries $B_*^{\sigma,\Sigma}$ and $B_{\sigma,\Sigma}^*$, respectively.

Recall that $\pi^* = \iota_*$, so ι^* is left adjoint to π^*. In particular, we have adjunction maps $\mathbb{I}\mathrm{d} \to \iota^*\pi^*, \pi^*\iota^* \to \mathbb{I}\mathrm{d}$ and, adjointly, $\mathbb{I}\mathrm{d} \to \iota_*\pi_*, \pi_*\iota_* \to \mathbb{I}\mathrm{d}$. Now write $B_{\tau\Sigma} = \iota_{\tau\Sigma} B_\tau = \iota_{\sigma\Sigma}\iota_{\tau\sigma}\pi_{\tau\sigma}B_\sigma$. Then replacing each map by its pushforward, we can compose it with the map $\mathbb{I}\mathrm{d}_\sigma \to \iota_*^{\tau\sigma}\pi_*^{\tau\sigma}$ to get a map $r_{\sigma\tau} : B_*^{\sigma\Sigma} \to B_*^{\tau\Sigma}$ of functors $\mathrm{Shv}(M_{\mathbb{R}}) \to \mathrm{Shv}(\Theta_{\mathbb{R}})$. Conversely, on pullbacks we compose with the counit map $\pi_{\tau\sigma}^* \circ \iota_{\tau\sigma}^* \to \mathbb{I}\mathrm{d}_\sigma$ to get a natural transformation, $q_{\sigma\tau} : B_{\tau\Sigma}^* \to B_{\sigma\Sigma}^*$. Note that the functors $q_{\sigma\tau}$ form a representation in functors of the poset $\Sigma_{\preccurlyeq}$ and the functors $r_{\sigma\tau}$ form a representation of its opposite. We want to define the Bondal functor B_* as the homotopy limit of the $B_*^{\sigma\Sigma}$ along the r maps and, adjointly, B^* as the homotopy colimit of the $B^*_{\sigma\Sigma}$ along the q maps. Later on, we will give an explicit model for this colimit as a complex of functors applied to an injective resolution of $\mathcal{V}$.

6.3 Full faithfulness of B^*

From now on, assume that the Novikov fan Σ is complete. We use a category-theoretic trick (explained to the author by Roman Bezrukavnikov) to prove the following Lemma.

Lemma 35. *The functor $B^*_{\mathbb{R}} : \mathrm{Shv}(\Theta_{\mathbb{R}}) \to \mathrm{Shv}(M_{\mathbb{R}})$ is fully faithful (as a functor of DG categories).*

Recall that an object X of a dg category $\mathcal{C}$ is *compact* if for any diagram of objects $D : I \to \mathcal{C}$, taking RHom with X preserves (homotopy) colimit along I. Specifically, for small index category I, we require:

$$\forall D : I \to \mathcal{C} \text{ admitting a direct limit, we have}$$
$$\mathrm{RHom}\,(X, \mathrm{colim}_{i\in I} D(i)) = \mathrm{colim}_{i\in I} \mathrm{RHom}(X, D(i)).$$

Recall also that a collection of objects X_i *colimit generate* a DG category $\mathcal{C}$ if every object of $\mathcal{C}$ can be expressed as a colimit of the X_i. (Note that there are different notions of DG generation and this is the strongest). Then we have the following categorical proposition.

Proposition 36. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor of DG categories that commutes with colimits and X_i is a collection of compact objects that colimit generate $\mathcal{C}$, such that $F(X_i)$ are also compact. Then F is fully faithful if and only if $F_{ij} : \mathrm{RHom}_{\mathcal{C}}(X_i, X_j) \to \mathrm{RHom}_{\mathcal{D}}(F(X_i), F(X_j))$ is a quasiisomorphism for all pairs i, j.*

Proof. For any pair of objects $X, Y \in \mathcal{C}$, we can express $X = \lim_I X_i$ and $Y = \lim_J X_j$. Then by compactness, $\mathrm{RHom}(F(X), F(Y))$

$$\cong \mathrm{colim}_{I^{\mathrm{op}}} \mathrm{RHom}(F(X_i), F(Y)) \text{ (by colimit compatibility)} \tag{13}$$
$$\cong \mathrm{colim}_{I^{\mathrm{op}}} \mathrm{colim}_J \mathrm{RHom}(F(X_i), F(Y_j)) \text{ (by compactness of } F(X_i)\text{)} \tag{14}$$
$$\cong \mathrm{colim}_{I^{\mathrm{op}}\times J} \mathrm{colim}\, \mathrm{RHom}(X_i, Y_j) \text{ (by faithfulness on generators)} \tag{15}$$
$$\cong \mathrm{RHom}(X, Y) \text{ (by reverse arguments in } \mathcal{C}\text{).} \tag{16}$$

□

We apply this proposition with F the functor B^* (which commutes with colimits because it is a left adjoint) and take for the collection of generators the sheaves $\mathbb{O}_d$, using the following proposition.

Proposition 37. *The objects $\mathbb{O}_d$ for d finite quasidivisors are a collection of compact generators for $D^- \mathrm{Shv}(Theta_{\mathbb{R}})$.*

Proof. To see the $\mathbb{O}_d$ are compact, using lemma 34 (or the affine-local nature of computing RHom on X_Σ^{Nov}), it suffices to check that $\mathbb{O}_d$ for finite d is compact in the affine case $\Sigma = \Sigma_\sigma$. But in this case the $\mathbb{O}_d$ are isomorphic to one-dimensional free modules, hence compact in $\mathrm{Rep}(\mathbb{O}_\sigma)$ itself (and therefore, a fortiori, in $\mathrm{Rep}^a(\mathbb{O}_\sigma)$). Now any $\mathbb{O}_d$ for d not necessarily finite is a direct limit of finite $\mathbb{O}_d$. Thus in order to show the finite $\mathbb{O}_d$ colimit-generate, it suffices to show all $\mathbb{O}_d$ colimit-generate. Once again using Lemma 34, we have $\mathcal{V} \cong \lim \iota_*^\sigma \mathcal{V}_\sigma$ together with the fact that finite inverse limits are (shifted) direct limits in the derived category, we reduce to the affine case, $\Sigma = \Sigma_\sigma$. Now since the forgetful functor from sheaves to presheaves is left exact, it commutes with direct limits and hence it suffices to show the $\mathbb{O}_\sigma$ colimit-generate the category of presheaves, i.e. equivariant modules over $\mathbb{O}_\sigma$. But this is standard: any module has a left resolution by free modules. (Note: we are using that the category $\mathrm{PreSh}(\Theta)$ has both enough projectives and enough injectives, and so for objects of the bounded derived category, Ext can be computed using either definition). $\square$

It follows that in order to check derived full faithfulness, it suffices to check it on the full subcategory of $\mathbb{O}_d$. In fact, we can do one better using Lemma 34 to reduce to pushforwards $\iota_* \mathbb{O}_d^\sigma$ from affines. Now

$$\begin{aligned}
\mathrm{RHom}(\iota_*^\sigma \mathbb{O}_d^\sigma, \iota_*^{\sigma'} \mathbb{O}_{d'}^{\sigma'}) &\cong \mathrm{RHom}_{\Theta_{\sigma'}}(\iota_{\sigma'}^* \iota_*^\sigma \mathbb{O}_{\iota^* d}) \\
\cong \mathrm{RHom}_{\Theta_{\sigma'}}(\pi_*^{\sigma'} \iota_*^\sigma \mathbb{O}_{\iota^*(d)}, \mathbb{O}_{\iota'^*(d')}) &\cong \mathrm{RHom}_{\Theta_{\sigma'}}(\mathbb{O}_{\iota^*_{\sigma\cap\sigma'}(d)} \mathbb{O}_{\sigma'^* d'}) \\
&\cong \begin{cases} \mathbb{C}, & \sigma \subset \sigma' \text{ and } m \preccurlyeq_\sigma m' \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

(We know how to RHom out of $\mathbb{O}_\sigma$ in $\mathrm{Rep}(\mathbb{O}_\sigma)$, and view $\mathbb{O}_{\sigma' \cap \sigma'} \in \mathrm{Rep}(\mathbb{O}_\sigma)$ as a direct limit of shifts of $\mathbb{O}_\sigma$, to get a colimit expression (with all connecting maps isomorphisms) to compute the derived Hom's above. It follows by Proposition 3.3 of [FLTZ] that the functor B^* is indeed faithful on the $\mathbb{O}_{\pi_\sigma^{-1}(d)}$, from which we deduce faithfulness on finite $\mathbb{O}_d$ and thence by finite generation on all of $\mathrm{Shv}(\Theta_\mathbb{R})$, completing the proof of faithfulness of B^*.

6.4 Essential surjectivity

It remains to check B^* is essentially surjective, or equivalently the following lemma.

Lemma 38. *The counit map* $B^* B_* : \mathrm{Shv}(M_\mathbb{R}) \to \mathrm{Shv}(M_\mathbb{R})$ *is the identity.*

Proof. Let $\mathcal{S}$ be a sheaf on $M_\mathbb{R}$. It suffices to check the map $B^* B_* \mathcal{S} \to \mathcal{S}$ is an isomorphism on stalks St_m for $m \in M_\mathbb{R}$, or, equivalently on dual stalks, (taking δ_m fo the skyscraper sheaf on a point m) that $\mathrm{Hom}(\mathcal{S}, \delta_m) \to \mathrm{Hom}(B^* B_* \mathcal{S}, \delta_m)$ is an isomorphism. Now we write $\mathrm{Hom}(B^* B_* \mathcal{S}, \delta_m) \cong \mathrm{Hom}(B_* \mathcal{S}, \mathcal{B}_* \delta_m)$. In the affine case where B is a genuine map of Grothendieck topologies, $B_*^\sigma \delta_m \cong \delta_{B(m)} \cong \delta \text{Þ}_m^\sigma$. Since B_* is glued out of the B_*^σ, we get $B_* \delta_m \cong \varprojlim \delta \text{Þ}_m^\sigma$. Now the stalk at $\mathrm{St}\text{Þ}_\sigma^m$ of $B_*^\sigma \mathcal{S}$ is equivalent to the global sections of $\mathcal{S}$ pulled back to $\pi_\sigma^{-1}(\mathrm{St}Þ_\sigma^m)\big(=\cap_{\mathrm{St}Þ_\sigma^m\in\pi^{-1}(\mathring{\Delta}_d^\sigma)}\pi_\sigma^{-1}(\mathring{\Delta}_d^\sigma)\big)$. We write this as $\mathrm{St}_{\Delta_\sigma^m}(\mathcal{S})$. Using a stalkwise argument, we see that $\lim_\sigma \mathrm{St}_{\Delta_\sigma^m}(\mathcal{S})$ is the shifted stalk functor St_m, and the Lemma follows.

$\square$

This concludes the proof of the equivariant Novikov ccc.

7 Singular Support Conditions and Subtopologies

Now suppose that Σ is a regular fan. Recall that we have a sub-category $\mathrm{QDiv}_{\mathbb{Z}} \subset \mathrm{QDiv}_{\mathbb{R}}$ with induced topology, which corresponds to a projection map on topologies, $\pi_{\mathbb{RZ}} : \Theta_{\mathbb{R}} \to \Theta_{\mathbb{Z}}$. We define adjoint "integral Bondal functors"

$$B_{\mathbb{Z}}^* := B^* \circ \pi_{\mathbb{RZ}}^*$$

and

$$B_*^{\mathbb{Z}} := \pi_*^{\mathbb{RZ}} \circ B_*.$$

Since $\pi_*^{\mathbb{RZ}}\pi_{\mathbb{RZ}}^* \cong \mathbb{I}\mathrm{d} : \mathrm{Shv}(\Theta_{\mathbb{Z}}) \to \mathrm{Shv}(\Theta_{\mathbb{Z}})$, we see that $\pi_{\mathbb{R}}^*$ is fully faithful, and hence so is $B_{\mathbb{Z}}^*$. It follows from [FLTZ] that the image of $B_{\mathbb{Z}}^*$ has singular support contained in Λ. Thus it remains to check that if $\mathcal{S}$ is a complex of sheaves with singular support contained in Λ, then $B_{\mathbb{Z}}^* B_*^{\mathbb{Z}}\mathcal{S} \cong \mathcal{S}$ (via the natural adjunction counit). Using Lemma **??**, it is enough to check that in the affine case $\Sigma = \Sigma_\sigma$, we have the identity $B_*^{\mathbb{Z}}(\mathcal{S}) \cong B_*^{\mathbb{R}}(\mathcal{S})$ (under singular support restrictions on $\mathcal{S}$). Equivalently, this is asking that the derived sections functor $R\Gamma(\Delta_d, \mathcal{S})$ does not change for $d \in \mathrm{QDiv}_{\mathbb{R}}$ so long as the "highest lower bound" $\lfloor d \rfloor \in \mathrm{QDiv}_{\mathbb{Z}}$ does not change. This is precisely guaranteed by the singular support condition. This completes the proof of Theorem 1, and using Theorem 11 we deduce the coherent-constructible correspondence **??** for smooth, proper varieties. (See Appendix [Appendix:non-smooth] for a natural extension to non-smooth and non-affine varieties, both of the classical and the Novikov type).

8 Log-Perfectoid Mirror Symmetry Interpretation

8.1 From Novikov varieties to perfectoids

Suppose $A \subset \mathbb{R}^n$ is an arbitrary *dense* Abelian group containing a lattice M. Define T_{Nov-A} for the algebraic group $\mathrm{Spec}\, k(A)$ (if $A \subset \mathbb{Q}\otimes M$, this is a profinite cover of the torus T). Write $\Gamma := (A/M)^\sim (:= \mathrm{Spec}\, k[A/M])$ this should be thought of as an algebraic version of the Galois group of the cover $T_{Nov-A} \to T$. For a rational and regular[4] fan Σ, write X_Σ^{Nov-A} for the space glued out of the affine Novikov pieces $\mathrm{Spec}\, k[A \cap \sigma^\vee]$ (just

[4]one can weaken both the rationality and the regularity conditions

replacing $M_\mathbb{R}$ by A in the definitions of X_Σ^{Nov}). The category of quasi-coherent sheaves on X_Σ^{Nov-A} has a well-defined almost-local subcategory, $\operatorname{Qcoh} X_\Sigma^{Nov-A}$ (defined locally on affines as an idempotence condition for the ideal a_σ, as for X_Σ^{Nov}). By repeating the arguments of sections 1-8 above, we get the following results.

The category $D^b \operatorname{Qcoh}^a X_\Sigma^{Nov} \cong D^b \operatorname{Shv}(\Theta_A)$, where $\Theta_A (= \operatorname{QDiv}_A, \Theta_A)$ is the sub-topos of Θ on quasidivisors $d \in \operatorname{QDiv}_\sigma^\mathbb{R}$ such that for each $\sigma \in \Sigma$, the restriction $d \mid \sigma$ is the restriction to σ of an element of $A \subset M_\mathbb{R}^*$ (viewed as a function on $\sigma \subset M_\mathbb{R}$). Consider the map of Grothendieck topologies $\pi_{A\mathbb{R}} : \Theta_\mathbb{R} \to \Theta_A$ (with $\pi^{-1}(d) = d$ for d a quasidivisor).

We use the following lemma.

Lemma 39. *The pushforward and pullback functors along $\pi_{A\mathbb{R}}$ are mutually inverse isomorphisms between* $\operatorname{Shv}(\Theta_A)$ *and* $\operatorname{Shv}(\Theta_\mathbb{R})$.

Proof. Since A is dense, the subtopology Θ_A generates $\Theta_\mathbb{R}$. In order to see that they define equivalent categories, it is sufficient to check that every cover of a divisor $d \in \operatorname{QDiv}_\mathbb{R}$ can be refined by a cover with all objects in QDiv_A and that any two such covers have a common refinement (also only involving objects in A). This is once again clear from the density of A. $\square$

It follows that $(\operatorname{Qcoh}^a X_\Sigma^{Nov-A})^{T_{Nov-A}} \cong \operatorname{Shv}(M_\mathbb{R})$ and hence, by the Pontrjagin duality theorem in section 11, we get the following theorem, which implies the main theorem 5.

Theorem 40. *In the setting above, we have an equivalence of derived categories*

$$(\operatorname{Qcoh}^a X_\Sigma^{Nov-A})^{(A/M)^\sim} \cong \operatorname{Shv}(S).$$

8.2 Log-perfectoid sheaves

Inspired by Theorem 40, we write down here a general procedure for obtaining a new quasicoherent category associated to a pair of spaces (U, X) with normal complex complement, and make some conjectures about independence on choice of X. Assume for simplicity that the base field k has characteristic zero. Suppose that U is a variety and X is a compactification. Suppose that $\partial := X \setminus U$ is a normal crossings divisor. Assume that the algebraic fundamental group of U maps surjectively to the local fundamental group (spanned by ramification) of the neighborhood of any point of ∂. (If this is not the case, we can remove some extra divisors from the interior of U to enlarge the fundamental group, then glue in an invariant procedure, but we will not worry about this level of generality here.) Let Γ_U be the geometric fundamental group of U. Let Γ_N be a cofiltering family of finite quotient groups with inverse limit Γ_U. Let $U^{(N)}$ be the corresponding étale covers of U. For each boundary component $\partial_i \subset \partial$, let $I_N^i \subset \Gamma_N$ be a ramification group associated to ∂_i (note that it depends on the choice of a branch of the cover near ∂_i). Now there is a unique finite flat extension of the family $U^{(N)}$ over U to a flat Γ_N-equivariant family over $U \cup \mathring{\partial}_i \subset X$. The preimage of $\mathring{\partial}_i$ will be $|\Gamma_N / I_N^i|$ copies of ∂_i

with multiplicity $|I_N^i|$. Glueing such extensions, we extend to a cover of the codimension-two subvariety

$$U \cup \bigcup \mathring{\partial}_i \subset X.$$

There is a unique extension of this family to an affine family over X: étale locally over any component of the preimage of some (possibly high-codimension) boundary component ∂_0, the map will look isomorphic to a product of power maps $(x_1, \ldots, x_n) \mapsto (x_1^{N_1}, x_2^{N_2}, \ldots, x_n^{N_n}) : \mathbb{A}^n \to \mathbb{A}^n$.

Now the pro-sequence of affine maps $X^{(N)} \to X$ will have a well-defined projective limit scheme $X^{(\infty)}$ which is affine over X, and has action by Γ, where (abusing notation), we view Γ also as an affine projective limit of affine (group) schemes. Let $\partial^{(\infty)} \subset X^{(\infty)}$ be the boundary. Our condition on ramification behavior of Γ implies that the category of sheaves on $X^{(\infty)}$ pushed forward from $\partial^{(\infty)}$ is a Serre subcategory. We define $\mathrm{Qcoh}^a(X^{(\infty)})$ to be the quotient of $\mathrm{Qcoh}(X^{(\infty)})$ by the Serre subcategory of sheaves pushed forward from $\partial^{(\infty)}$. This category has natural action by the group scheme Γ. Define $\mathrm{Qcoh}_{\text{log-perf}}(X)$ to be the category of Γ-equivariant quasicoherent sheaves in $\mathrm{Qcoh}^a(X^{(\infty)})$. Note that while our definition uses the (possibly very large and non-commutative) Galois group Γ, the stabilizers of the stacky points of $X^{(\infty)}/\Gamma_\infty$ will be tame. Specifically, they will all be products of ramification groups of several ∂_i, and in particular commutative. Thus we could define the category $\mathrm{Qcoh}_{\text{log-perf}}(X)$ starting not with the full Galois group Γ but with any continuous subgroup Γ' of Γ which maps surjectively to local ramification groups (in fact, it should be possibly to replace surjectivity by a density requirement).

If $U = T$ and X is a toric variety, the Galois group of T (viewed as a group scheme) is $\Gamma \cong \big((\mathbb{Q}/\mathbb{Z})^\sim\big)^n$. A canonical choice of tower $X^{(N)}$ is provided by the N-frobenius maps, induced from the maps of lattices $N_\Sigma \to N \cdot N_\Sigma$ without changing the fan (with apologies for the awkward notation: N is an integer and N_Σ is the lattice). Theorem 40 then implies that $\mathrm{Qcoh}_{\text{log-perf}}(U, X)$ is equivalent to the category of quasicoherent sheaves on S. In particular, it is independent of choice of X. This leads us to make the following conjecture.

Conjecture 1. *Assume the base field $k = \mathbb{C}$. Then for any U, the derived category $D^b\,\mathrm{Qcoh}_{\textit{log-perf}}(U, X)$ is independent of choice of (n.c.) compactification X.*

If this is true, we would have a well-defined category $D^b\,\mathrm{Qcoh}_{\text{log-perf}}(U)$ depending only on an open variety, and this would perhaps be the right category to study for some wider class of mirror symmetry statements for open varieties. Further, whether of not this conjecture is true, SYZ mirror symmetry provides us with an abundance of pairs of varieties (U, X), where X is an algebraic Calabi-Yau manifold admitting an SYZ torus fibration with singularities, and $U \subset X$ is the complement to the space of singular fibers (which has codimension 2 and is in fact algebraic). Suppose U, X is such a pair and let M be the symplectic manifold obtained from the dual torus filtration on the nonsingular fibers. Then we ask the following question.

Question. *Is there a way to define a "large" Fukaya category $Fuk^?(M)$ on M (which for T^*S^1 would be equivalent to the derived category of all topological sheaves on S^1) such that there is an equivalence of categories $Fuk^?(M) \cong D^b\,\mathrm{Qcoh}_{log\text{-}perf}(U, X)$?*