Mirror symmetry on the Bruhat-Tits building and the K theory of a p-adic group

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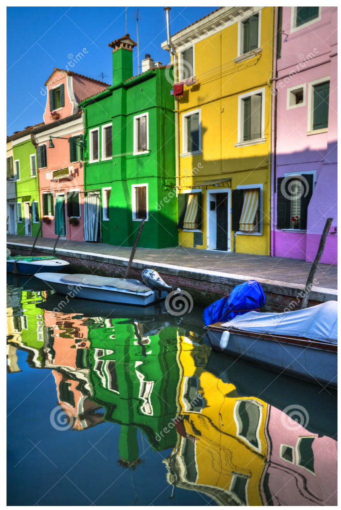

Abstract

Let G be a split, semisimple p-adic group. We construct a derived *localization functor* $\mathrm{Loc} : D^b\overline{\mathrm{Sm}}_{fg} \to D^b\widehat{\mathbf{Sh}}$ from the compactified category of [BK2] associated to G to the category of equivariant sheaves on the building whose stalks have finite-multiplicity isotypic components as representations of the stabilizer. Our construction is motivated by the "coherent-constructible correspondence" functor in toric mirror symmetry and a construction of [CCC]. We show that Loc has a number of useful properties, including the fact that the sections $R\Gamma\,\mathrm{Loc}_c(\overline{V}) = V$ when $\overline{V}$ is an object of $\overline{\mathrm{Sm}}_{fg}$ compactifying the finitely-generated representation V.We also construct a depth-$\leq e$ "truncated" analogue $\mathrm{Loc}_{(e)}$ which has finite-dimensional stalks, and satisfies the property $R\Gamma_c\,\mathrm{Loc}_{(e)}(\overline{V}) = V$ for any V of depth $\leq e$. We deduce that every finitely-generated representation of G has a bounded resolution by representations induced from finite-dimensional representations of compact open subgroups, and use this to compute the K-theory of G in terms of K-theory of its parahoric subgroups.

Contents

To Ren, mom, and dad

Viens avec moi par dessus les buildings
Ça fait WHIN! quand on s'envole et puis KLING!
Après quoi je fais TILT! et ça fait BOING!
SHEBAM! POW! BLOP! WIZZ!

– Serge Gainsbourg, *Comic Strip*

Acknowledgments

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0 Introduction

0.1 K theory of the representation category

Let G be a split, semisimple p-adic group, and let $\mathrm{Sm}_{fg}(G)$ be the category of idempotented finitely-generated representations of the Hecke algebra $\mathcal{H}(G)$ with values in $\mathbb{C}$ (equivalently, smooth finitely-generated representations of G, see e.g. [Ber]). The category $\mathrm{Sm}_{fg}(G)$ is extremely well-behaved: it is a direct sum of countably many Noetherian components, has enough projectives, and has finite homological dimension equal to the rank of the group. In particular, the category has a well-behaved K-theory, with $K^0(\mathrm{Sm}_{fg}(G))$ the Grothendieck group of projectives in $\mathrm{Sm}_{fg}(G)$. Write

$$K^0(G) := K^0(\mathrm{Sm}_{fg}(G)).$$

The group $K^0(G)$ will be a central object of study in this paper. The rational coefficient version $K^0(\mathrm{Sm}_{fg}(G)) \otimes \mathbb{Q}$ was considered in the papers [BDK] and [D], and is related to the character theory of admissible representations of G. Indeed, given any admissible representation A and finitely-generated representation V, the graded spaces $\mathrm{Ext}^i(A, V)$ are finite-dimensional and zero for $i \geq n+1$ (for n the rank of G). The signed sum $\langle A, [V]\rangle := \sum(-1)^i \dim \mathrm{Ext}^i(A, V)$ then defines, for each admissible A, an integral functional on $K^0(G)$. Rationally, the group $K^0(G)$ was computed by Dat [D], who showed that the group $K^0(G) \otimes \mathbb{Q}$ naturally pairs with the vector space of central distributions on compact elements. Namely, we say that an element $\gamma \in G$ is compact if it is contained in some compact subgroup $K \subset G$. Write $G_c \subset G$ for the (open and closed) subset of compact elements. Let $\mathcal{H}_c \subset \mathcal{H}$ be the vector space of (compactly supported locally constant) functions supported on G_c. The group G acts on $\mathcal{H}_c$ by conjugation, and we write HH_0^c for the space of coinvariants $(\mathcal{H}_c)_G$.

Theorem 1 (Dat)**.** *There is an isomorphism $\iota : K^0(G) \otimes \mathbb{C} \to HH_0^c$. In terms of this isomorphism, the pairing $\langle A, [V]\rangle = \langle \chi_A, \iota([V])\rangle$, where χ_A is the Harish-Chandra character of A (here χ_A is viewed as a distribution on G, which acts on $\mathcal{H}_c$ and by conjugation-invariance descends to conjugation coinvariants $(\mathcal{H}_c)_G$).*

One way to get projectives in $\mathrm{Sm}(G)$ is by (compact) induction from compact subgroups. Suppose $J \subset G$ is a compact open subgroup of G. Given a smooth representation V of J, write $\mathrm{Ind}_J^G(V) := V \otimes_{\mathcal{H}_J} \mathcal{H}_G$ for the induced representation. Note that our functor Ind_J^G is left adjoint to the forgetful functor, and is sometimes denoted $\mathrm{Ind}_c |_J^G (V)$ to distinguish it from the right adjoint, which is a non-isomorphic functor (since $[J : G]$ is infinite).

A representation V of a compact group J is finitely-generated and projective if and only if it is finite-dimensional (recall that smooth finite-dimensional representations of a compact group form a semisimple category). As both of these properties are obviously invariant with respect to induction, this gives us a collection of projectives $\mathrm{Ind}_J^G(V) \in \mathrm{Sm}_{fg}(G)$ for pairs (J, V) with $J \subset G$ compact and V finite-dimensional representations of J.

Definition 1. *We say that a representation is* finitely induced *if it is of the form $\mathrm{Ind}_J^G(V)$ for some finite-dimensional representation V of an open compact $J \subset G$.*

Note that it is sufficient to consider maximal compact subgroups J, as if J is compact open and $M \subset J$ is a maximal compact subgroup containing J, then $\mathrm{Ind}_J^G(V) \cong \mathrm{Ind}_M^G(\mathrm{Ind}_J^M V)$, and $\mathrm{Ind}_J^M V$ is a finite-dimensional representation of M. Given a pair (J, V) as above, the class of the corresponding finitely induced representation in Dat's K^0 group is the normalized character $\delta_J \cdot \chi_V$, where χ_V is the function (supported in J) such that $\mathrm{Tr}(h, V) = \langle h, \chi_V \rangle$ for $h \in \mathcal{H}$ supported in J and δ_J is the uniform distribution on J of norm 1. Note that $\delta_J \cdot \chi_V$ is supported on $J \subset G_c$, hence projects to the space of G-coinvariants HH_0^c. Because the characters χ_V form a basis for the vector space of central functions, one sees that these projections of characters span all of HH_0^c over the complex numbers: in particular, the finitely induced representations rationally span the group $K^0(G)$.

In this paper we will consider the *integral* K group $K^0(G)$. Our main result will be the following.

Theorem 2. *The classes $[\mathrm{Ind}_J^G V]$ of finitely induced representations integrally span the group $K^0(G)$.*

An equivalent formulation of this result is as follows.

Theorem'. *For any projective object P of $\mathrm{Sm}_{fg}(G)$, there are finitely induced representations $\mathrm{Ind}_J^G V, \mathrm{Ind}_{J'}^G V'$ such that $P \oplus \mathrm{Ind}_{J'}^G V' \cong \mathrm{Ind}_J^G V$.*

Or, equivalently (as $\mathrm{Sm}_{fg}(G)$ has enough projectives),

Theorem". *Any object of $\mathrm{Sm}_{fg}(G)$ has a (two-sided) resolution by direct sums of finitely induced representations.*

The statement in this last form is a conjecture in Roman Bezrukavnikov's thesis, [BThes].

The functor $\mathrm{Ind}_J^G : \mathrm{Sm}_{fd}(J) \to \mathrm{Sm}_{fg}(G)$ takes direct sums to direct sums, hence induces a linear map on K^0 groups $[\mathrm{Ind}_J^G] : K^0\,\mathrm{Sm}_{fd}(J) \to K^0\,\mathrm{Sm}_{fg}(G)$. The maps $[\mathrm{Ind}_J^G]$ and $[\mathrm{Ind}_{\gamma J \gamma^{-1}}^G]$ are intertwined by the isomorphism $V \mapsto \gamma V : K^0(J) \to K^0(\gamma J \gamma^{-1})$ — hence, in particular, they have the same image in $K^0(G)$. Further, $[\mathrm{Ind}_J^G]$ factors through $\mathrm{Sm}_{fd}(M)$

for M some maximal compact subgroup containing J. If we choose an Iwahori subgroup $I \subset G$, the collection Max_I of maximal compact subgroups containing I is a set of representatives of maximal compact subgroups up to conjugation. With this in mind, we write down the following map.

$$[\mathrm{Ind}_{\max}] := \bigoplus_{M \in \mathrm{Max}_I} [\mathrm{Ind}_I^G] : \bigoplus_{M \in \mathrm{Max}_I} K^0(M) \to K^0(G).$$

Theorem 2 then implies that the map $[\mathrm{Ind}_{\max}]$ is surjective. There are some classes obviously in the kernel of this map: namely, given a subgroup $J \subset M_i \cap M_j$, the two inductions $[\mathrm{Ind}_J^{M_i}(V)]$ and $[\mathrm{Ind}_J^{M_j}(V)]$ have the same image under $[\mathrm{Ind}_{\max}]$ (viewed as elements of the corresponding direct summands). Note that it is enough to take $J = M_i \cap M_j$ above. Write K^0_{cell} for the quotient of $\bigoplus_{M \in \mathrm{Max}_I} K^0(M)$ by relations of the form $[\mathrm{Ind}_{M_i \cap M_j}^{M_i} V] \sim [\mathrm{Ind}_{M_i \cap M_j}^{M_j} V]$.

The map $[\mathrm{Ind}_{\max}]$ induces a map $[\mathrm{Ind}_{\mathrm{cell}}] : K^0_{\mathrm{cell}} \to K^0(G)$. It can be shown from the formula of [D] and basic properties of parahoric subgroups that this map is an isomorphism rationally. Theorem 2 implies that it is a surjection integrally. Hence the map $[\mathrm{Ind}_{\mathrm{cell}}] : K^0_{\mathrm{cell}} \to K^0(G)$ is an isomorphism on torsion-free quotients.

0.2 Compactified category

Our proof will proceed by constructing a resolution for an arbitrary object, in a way that is functorial up to a certain choice of a "normalization" of V. This choice of normalization is provided by the *compactified category* $\overline{\mathrm{Sm}}$ defined in [BK2] and its subcategory $\overline{\mathrm{Sm}}_{fg}$ of locally finitely-generated objects. This category is a powerful tool which in particular allows one to systematically normalize computations with finitely-generated representations of G. Namely, given two objects V, W of $\mathrm{Sm}_{fg}(G)$, the space $\mathrm{Hom}(V, W)$ is in general not finite-dimensional, but has action by the Bernstein center $Z := HH^0(\mathcal{H})$, and is a finitely-generated representation of Z. Equivalently, this Hom space can be considered a sheaf $\underline{\mathrm{Hom}}(V, W)$ over Spec_Z which is coherent and supported over finitely many irreducible components. Similarly the derived Hom space can be written as a finite complex of coherent sheaves $R\underline{\mathrm{Hom}}_{\mathrm{Sm}}(V, W)$ over $\mathrm{Spec}(Z)$. Now components of $\mathrm{Spec}(Z)$ are canonically scheme-theoretic quotients of tori (of dimension between 0 and n) by subgroups of the Weyl group W. Choosing W-equivariant toric compactifications of these tori (something that can be done in a consistent way), we get a canonical compactification $\overline{\mathrm{Spec}(Z)}_{BK}$ of the central spectrum. The idea of [BK2] is to endow the objects V, W of $\mathrm{Sm}(G)$ with some additional data, giving objects $\overline{V}, \overline{W}$ in some upgraded category $\overline{\mathrm{Sm}}_{fg}$, in order to be able to write an inner Hom space $\underline{\mathrm{Hom}}(\overline{V}, \overline{W})$ as a coherent sheaf over $\overline{\mathrm{Spec}(Z)}_{BK}$. One can then reconstruct $\mathrm{Hom}(\overline{V}, \overline{W})$ as $\Gamma\left(\overline{\mathrm{Spec}(Z)}_{BK}, \underline{\mathrm{Hom}}(\overline{V}, \overline{W})\right)$, and $\mathrm{Ext}^*(\overline{V}, \overline{W})$ in $D^b\overline{\mathrm{Sm}}_{fg}$ as the hypercohomology of the double complex $R\Gamma\left(\overline{\mathrm{Spec}(Z)}_{BK}, R\underline{\mathrm{Hom}}(\overline{V}, \overline{W})\right)$. The wonderful advantage of this category and its derived category is that these categories are *proper* (see e.g. [O]), and two objects (under suitable finite generation conditions) have finitely many finite-dimensional Ext spaces. This allows us to define Yoneda functors from $D^b\overline{\mathrm{Sm}}_{fg}$ to $D^b \operatorname{Vect}_{fd}$ given by taking $R\operatorname{Hom}$ with *any* (finitely-generated) object. We will show that any representation V has a resolution by induced representations by choosing a compactification $\overline{V} \in \overline{\mathrm{Sm}}_{fg}$ (something that is relatively easy to construct), and write down a resolution

$$(\jmath_*, d) \xrightarrow{\sim} \jmath$$

of the forgetful functor

$$\jmath : \overline{\mathrm{Sm}} \to \mathrm{Sm}(G)$$

by a finite collection of functors given by direct sum of functors of the form $\jmath_i^J : \overline{V} \mapsto R\operatorname{Hom}(X_i, \overline{V}) \otimes \operatorname{Ind}_J^G V_i$ indexed by J running over corank-i parahoric subgroups of G containing some fixed Iwahori subgroup.

0.3 Localization on the building

The resolution $(\jmath_i, d)$ (as well as a version of this functor depending on depth) will be the focus of this paper, and is interesting independently of its application to K theory. Our construction will be topological in nature, and is motivated by a philosophy of *p-adic localization* introduced in the paper [BThes]. Namely, recall that (for G split and semisimple) the Bruhat-Tits building $\mathbb{B}_G$ is a G-equivariant contractible cell complex with vertices parametrized by maximal compact subgroups and k-dimensional cells parametrized by the collection of all parahoric subgroups of corank k. The space $\mathbb{B}_G$ can be thought of as a p-adic analogue to the equivariant space G/K, either for G a real group and K a compact subgroup, or for G a complex group and K the Borel. The combinatorially constructed topology on $\mathbb{B}$ then takes the place of the smooth or complex structure on the equivariant spaces. In particular, the appropriate analogue to the category of local systems on an equivariant space is the category of constructible sheaves on the building with finite-dimensional fibers, constructible with respect to the cellular stratification. This category can be thought of as having action by something like the Lie algebra of G. To have action by all of G, we consider the category $\boldsymbol{Sh}_{fd}^G$ of G-equivariant constructible sheaves on the building with finite-dimensional stalks. One then is interested in the (compactly supported) global sections functor $\Gamma_c : \boldsymbol{Sh}_{fd}^G \to \mathrm{Sm}_{fg}(G)$ which is analogous to the inverse Beilinson-Bernstein localization functor arising in the theory of category-$\mathbb{O}$ representations of semisimple Lie groups. Unlike the (inverse) localization functor in geometry, the functor Γ_c is far from being an equivalence, and is not a faithful functor; nevertheless, Bezrukavnikov shows in [BThes] that it becomes faithful after factoring out a certain Serre subcategory of $\boldsymbol{Sh}_{fd}^G$ of objects with trivial homology. In fact, the essential image of Bezrukavnikov's functor is precisely the full subcategory of representations in $\mathrm{Sm}_{fg}(G)$ consisting of representations which admit a resolution by direct sums of finitely induced representations. To motivate this, observe the functor Γ_c comes endowed with a resolution (coming from the cellular structure) by functors $\Gamma_c^J : \boldsymbol{Sh}_{fd}^G \to \mathrm{Sm}_{fg}(G)$ with J running over the paraholics and $\Gamma_c^J \cong \operatorname{Ind}_J^G \operatorname{Stalk}_\sigma$ canonically expressed as the induced representation from the stalk functor at a cell σ stabilized by J.

Thus in order to show that any finitely-generated representation has a resolution by finitely induced ones, it would be sufficient to construct a right inverse $\mathrm{Loc}_{\mathrm{Sm}}$ of the functor $\Gamma : \boldsymbol{Sh}^G_{fd} \to \mathrm{Sm}_{fg}(G)$: then the cell complex computing $\Gamma(\mathrm{Loc}_{\mathrm{Sm}}(V)) \cong V$ would give such a resolution. Unfortunately, it is relatively easy to see that such a right inverse does not exist, even in a derived context. Instead, what we do construct is a derived, *compactified* version of the localization functor: a complex of sheaves $\mathrm{Loc}_{\overline{\mathrm{Sm}}}(V)$, which we will call

$$\mathrm{Loc}^{\vee}_{gr} : D^b\overline{\mathrm{Sm}}_{fg} \to \boldsymbol{Sh}^G_{fd},$$

with the property that the following diagram of functors commutes:

$$\begin{tikzcd}
D^b\overline{\mathrm{Sm}}_{fg} \arrow[r, "\mathrm{Loc}^{\vee}_{gr}"] \arrow[d, "\jmath"'] & D^b_{fd}\boldsymbol{Sh}^G \arrow[dl, "R\Gamma_c"] \\
D^b\,\mathrm{Sm}_{fg}(G). &
\end{tikzcd}$$

This commutative diagram, along with the existence for any object $V \in \mathrm{Sm}$ of a (non-unique) compactified object $\overline{V}$ with $\jmath\overline{V} \cong V$, furnishes us with a resolution of every object by finitely induced representations.

Remark 1. *Note here that the category $D^b_{fd}\boldsymbol{Sh}^G$ is the category of (equivariant, cellular constructible) complexes sheaves on $\mathbb{B}$ whose* homology sheaves *have finite-dimensional stalks. By a standard argument, this category is derived equivalent to the category $D^b\boldsymbol{Sh}^G_{fd}$, and in order to get our desired resolution by compactly induced finitely-generated objects, one needs to choose an object-wise inverse lifting the equivalence on homotopy categories of $D^b_{fd}\boldsymbol{Sh}^G \to D^b\boldsymbol{Sh}^G_{fd}$, in a way that does not have to be functorial. Note that in an A_∞ context, functorial such lifts do exist.*

0.3.1 Truncation

The more canonical functor, and the one we will spend the most time studying, is a functor $\mathrm{Loc} : D^b\overline{\mathrm{Sm}}_{fg} \to D^b\,\mathbf{CoSh}^G$ into the category of *cosheaves*, not necessarily with finite-dimensional stalks. In order to get a functor $\mathrm{Loc}^{\vee}$ into sheaves we can use a standard Verdier-type equivalence between derived categories of sheaves and cosheaves (see [Cu]). In order to further project to the category of sheaves with finite-dimensional stalks, we use a procedure of "truncation" and take stalkwise invariants with respect to a coefficient system of congruence subgroups of conductor depending on the depth of $\overline{V}$. It is in fact somewhat surprising that "truncation" does not destroy commutativity of the global sections diagram above, and our proof of this (in section 8) uses extensively ideas of Meyer and Solleveld, [MS].

0.4 Toric mirror symmetry and corridors

The idea behind our construction of the localization functor comes from adapting to the context of buildings and noncommutative geometry a certain functor arising from mirror symmetry of toric varieties: namely, the coherent-constructible correspondence of [FLTZ] (especially in the interpretation of [T] and [CCC]). The basic idea underlying both the point of view of [CCC] and our construction of the localization functor is one of descent: we express (countable) colimit-compatible dg functors $D^b\overline{\mathrm{Sm}} \to D^b\mathcal{C}$ (for arbitrary categories $\mathcal{C}$) as collections of functors from noncommutative affine charts, with certain algebra actions and compatibilities between them. This converts the task of constructing the functor $\mathrm{Loc} : D^b\overline{\mathrm{Sm}} \to D^b\boldsymbol{Sh}^G$ to that of finding several compatible objects of $\boldsymbol{Sh}^G$ with appropriate algebra actions. These objects are constructed using (Verdier duals to) constant sheaves on a new class of contractible geometric subsets of the building which we call *corridors* (analogous to shifts of dual toric cones in the case of toric varieties). Note that both the Beilinson-Bernstein localization functor and the

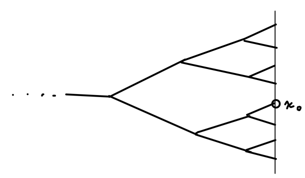

Figure 1: Example of a corridor for $SL_2(\mathbb{Q}_2)$

coherent-constructible correspondence functor are in general fully faithful: not so for the localization functor here. Instead, we have a functor $\mathrm{Col} : D^b\,\mathbf{CoSh}^G \to D^b\overline{\mathrm{Sm}}$ right adjoint to the localization functor with the property that $\mathrm{Col} \circ \mathrm{Loc} : \overline{\mathrm{Sm}} \to \overline{\mathrm{Sm}}$ is close to but not quite the identity functor (as would be the case if Loc were fully faithful). The question of "fixing" Loc to be fully faithful (and thus give an embedding of the compactified category Loc into the category of equivariant sheaves on $\mathbb{B}$) is an interesting one, and one that the author is agnostic about at the moment.

0.5 The Yoneda philosophy and the Morita philosophy

Before continuing, we point out a subtle point about the point of view we adopt in defining functors, which is in a sense dual to the standard one. We will indicate this difference somewhat vaguely in this section, in order to motivate some of the definitional choices we make later in paper. Namely, given two module categories (either Abelian or differential graded), A-Mod and B-Mod there are a few common ways to "represent" dg functors between them. One, which we can call the "Yoneda" philosophy, is to define a functor $F^Y : A\text{-Mod} \to B\text{-Mod}$ by choosing some bimodule $Y \in A\text{-}B\text{-bimod}$, and defining $F^Y(X) := \mathrm{Hom}(Y, X)$. Another, which we call the "Morita" philosophy, is to choose a B-A-bimodule, M, and define

$F_M(X) := M \otimes_A X$. When one has not chosen a generator and starts with two categories $\mathcal{C}, \mathcal{D}$ determined by some sort of algebraic data, it is still often possible to interpret the notion of a $\mathcal{C}$-$\mathcal{D}$-bimodule as an object of another algebraic category, informally "D^{op}-type object in $\mathcal{C}$" (formally, this bimodule category is determined by some universal property, and denoted $\mathcal{C} \boxtimes \mathcal{D}^{op}$ when it exists). In defining the functors in this paper, we will identify the relevant bimodule categories, and almost exclusively use the "Morita" language of tensor product with a "kernel" bimodule M rather than the Yoneda constuction of $\mathrm{Hom}(Y, -)$. Note that our choice is aesthetic: as our categories are smooth, every "Morita-type" object M can (in the dg world) be replaced by a suitably dual "Yoneda-type" object $Y := M^\vee$. However, the relevant duality functors are complicated, and using the Yoneda method of defining functors would make our exposition more cumbersome than it should be.

1 Plan of paper

We begin by gathering together in section 2 some results about the category of representations Sm_G^{fg} that at this point can be considered classical. In section 5 we study homological algebra on the compactified category $\overline{\mathrm{Sm}}$. We begin by recalling basic properties of the compactified category from [BK2], the most important ones being its geometric enrichment over the smooth compact variety $X//W$ for X an n-dimensional toric variety over $\mathbb{C}$ compactifying the spectrum of the spherical center, $\mathrm{Spec}(Z_{sph}) \cong \check{\boldsymbol{T}}//W$. We move between three different points of view of $\overline{\mathrm{Sm}}$ introduced in [BK2]. One point of view is to consider $\overline{\mathrm{Sm}}$ as a collection of compatible representations of the topological algebras $\mathcal{H}_{\mathcal{P}\mathcal{Q}}$, which can be thought of as noncommutative affine charts. A second is a microlocal modification of the first, where we only consider punctured completions $\widehat{\mathcal{H}}_{\mathcal{P}\mathcal{Q}}$ of the $\mathcal{H}_{\mathcal{P}}$ with respect to certain closed strata. The final one is a picture of $\overline{\mathrm{Sm}}$ as sheaves of modules over a sheaf of algebras $\mathcal{A}$ over $X//W$ (which we get after choosing an appropriate generator). The most important results of this section are lemmas 8, giving a formula for higher Hom between compactified representations and 9, which characterizes colimit-compatible (dg) functors $D^b\overline{\mathrm{Sm}} \to \mathcal{C}$ in terms of the data of compatible collections of objects $X_{\mathcal{P}\mathcal{Q}}$ with action by the topological algebras $\mathcal{H}_{\mathcal{P}\mathcal{Q}}$. We call such data $\{X_{\mathcal{P}\mathcal{Q}}\}$ "kernels" for functors. In the next section, 6, we recall some combinatorial models for the category of equivariant cosheaves on the building and its derived category in Section 6, with the main sources being [BThes] and [Cu]. We also introduce a class of sheaves we call constant sheaves on orbifolds, which are orbifold pushforwards of constant sheaves on "étale subsets" of the orbifold $\mathbb{B}/G$. The remainder of the paper defines and studies various functors $D^b\overline{\mathrm{Sm}} \to D^b\,\mathbf{CoSh}^G$ using appropriate kernels $\{X_{\mathcal{P}\mathcal{Q}}\}$. In section 7, we define the "absolute localization" functor $\mathrm{Loc} : D^b\overline{\mathrm{Sm}} \to D^b\,\mathbf{CoSh}^G$ which we glue as a homotopy limit of the functors $\mathrm{Loc}_{\mathcal{P}\mathcal{Q}}$ indexed by pairs of parabolics. The functors $\mathrm{Loc}_{\mathcal{P}\mathcal{Q}}$ are deduced from constant orbifold cosheaves on quotients of certain special contractible subsets of $\mathbb{B}$ which we call *corridors*. We check that Loc satisfies commutativity of the diagram

$$
\begin{tikzcd}
D^b\overline{\mathrm{Sm}} \arrow[r, "\mathrm{Loc}"] \arrow[d, "\jmath"'] & D^b\,\mathbf{CoSh}^G \arrow[dl, "R\Gamma_c"] \\
D^b\,\mathrm{Sm}_G, &
\end{tikzcd}
$$

and show that the stalks of this functor are *profinite-dimensional*, i.e. the stalk over x becomes finite-dimensional upon taking invariants with respect to any open subgroup of the stabilizer $G(x)$.

Now replacing the functors $\mathrm{Loc}_{\mathcal{PQ}}$ by invariants with respect to the "Schneider-Stuhler coefficient system" $G_x^{(e)} \cong G(x,e) \triangleleft G(x)$ over every point x gives a new functor $\mathrm{Loc}^{(e)} : \overline{\mathrm{Sm}} \to \mathbf{Sh}$ with finite-dimensional fibers. We need to show that this functor has the same compatibility

$$
\begin{tikzcd}
D^b\overline{\mathrm{Sm}}^{(\leq e)} \arrow[r, "\mathrm{Loc}^{(e)}"] \arrow[d, "\jmath"'] & D^b\,\mathbf{CoSh}^G \arrow[dl, "R\Gamma_c"] \\
D^b\,\mathrm{Sm}_G, &
\end{tikzcd}
$$

when restricted to compactified representations of depth $\leq e$. It turns out that we in fact have a stronger statement: taking invariants does not change the (compactly supported) sections of any component $\mathrm{Loc}_{\mathcal{PQ}}(\overline{V})$ (provided $\overline{V}$ has depth $\leq e$). In order to prove this, we use the remarkably versatile machinery of "compatible systems of idempotents" of Meyer and Solleveld, [MS]. This is done in section 8. This finally gives us a resolution of the underlying representation V of any object $\overline{V}$ of the compactified category. It remains to observe that any object V of Sm admits compactification to an object $\overline{V}$ of $\overline{\mathrm{Sm}}$ to conclude our proof.

2 Reminders about the representation category and the Bruhat-Tits building

Here we will gather together several known results about the category Sm_G^{fg} of smooth finitely-generated representations. Choosing an integral model for G (easy since G is split), we have a subgroup $G(\mathcal{O}) \subset G(K)$, which is a maximal compact subgroup. A subgroup conjugate to the preimage of a Borel subgroup of $G(k)$ under the residue map $G(\mathcal{O}) \to G(k)$ is called an Iwahori subgroup. A compact open subgroup containing an Iwahori subgroup is called a parahoric. Parahoric subgroups have a geometric incarnation as stabilizers of cells of a contractible G-equivariant cell complex, $\mathbb{B}_G$, called the Bruhat-Tits building. As parahorics are self-normalizing, we have a bijection between cells and parahorics $\sigma \leftrightarrow G(\sigma)$ taking a cell of $\mathbb{B}$ to its stabilizer, or equivalently the stabilizer of any point $x \in \sigma$. Now to every point $x \in \mathbb{B}$ and number $r \in \mathbb{R}_{\geq 0}$, Moy and Prasad

[MP] associate a subgroup $G(x,r) \subset G$, normal in the stabilizer $G(x)$. We say that (for some integer e), a representation $V \in \mathrm{Sm}_G$ has *depth* $\leq e$ if it is generated by the subspaces $V^{G(x,e)}$. It follows from work of Bernstein that the category of all finitely-generated representations of depth $\leq e$ is Noetherian and a direct summand in the category $\mathrm{Sm}_{fg}(G)$. When e is an integer, the groups $G(x,e)$ can be taken to be the *Schneider-Stuhler coefficient system* $G_\sigma^{(e)}$ of [SS], which is constant on cells of $\mathbb{B}$.

Given a parabolic subgroup $P \subset G$, it has a normal unitary radical $U_P \subset P \subset G$, and the quotient P/U_P is a *Levi subgroup*, which we will denote L_P. We have a pair of exact adjoint functors

$$r_P : \mathrm{Sm}_G \rightleftarrows \mathrm{Sm}_{L_P} : i_P,$$

called the Levi restriction and induction, such that $r_P(V) := V_U$ with evident L_P-action. We say that a representation V is *cuspidal* if $i_P(V) = 0$ for any parabolic $P \subsetneq G$, and *admissible* if it has finite depth. Jacquet induction and restriction preserve both the properties of admissibility and of having depth $\leq e$.

We define the Bernstein center $Z := HH^0(\mathrm{Sm}_G)$ to be the center of the category Sm_G. Given any representation $V \in \mathrm{Sm}_G$, it has a central support subvariety $\mathrm{Supp}(V) \subset \mathrm{Spec}(Z)$. The category of representations with central support at a given point $x \in \mathrm{Spec}(Z)$ is not necessarily semisimple, but is always Artinian, with at most $|W|$ irreducibles (for $|W|$ the size of the Weyl group). The depth of a representation depends only on its singular support, and the variety $\mathrm{Spec}(Z)$ is decomposed into a disjoint union by depth. The component $\mathrm{Spec}(Z_{\leq e})$ of bounded depth is a variety of finite type, and it has smooth connected components isomorphic to quotients of tori (of dimension between 0 and n) by subgroups of the Weyl group.

Up to some choices, we can choose a "spherical" central subring $Z_{sph} \subset Z$ such that the $\mathrm{Spec}(Z_{sph}) \cong \check{\boldsymbol{T}}_{\mathbb{C}}//W$ is the scheme-theoretic quotient of the Langlands dual torus to the maximal torus $T \subset G$, taken with coefficients in $\mathbb{C}$ and quotiented by the Weyl group. The resulting map $\mathrm{Spec}(Z) \to \check{\boldsymbol{T}}_{\mathbb{C}}//W$ can be shown to be finite on every central component.

3 Reminders about dg categories

Unless stated otherwise, every instance of the derived category $D^b\mathcal{A}$ of an Abelian category $\mathcal{A}$ will be viewed as a *pretriangulated dg* category (on the category of complexes of objects of $\mathcal{A}$ with bounded cohomology), and not as a triangulated category. Because a formal discussion of dg category formalism would extend this paper unnecessarily, we refer the reader to [**?**, **?**] for a careful introduction. Here we summarize some key properties of the dg language (with some terms borrowed from the closely related ∞-categorical context in a way that by now is standard) that we will be using. A dg category is a category $\mathcal{C}$ fibered in complexes. For two objects $V, W \in \mathcal{C}$ we write the complex

$$\mathrm{Hom}_*(V,W) := \cdots \to \mathrm{Hom}_{-1}(V,W) \to \mathrm{Hom}_0(V,W) \to \mathrm{Hom}_1(V,W) \to \ldots.$$

When we say "a map $f : V \to W$ in $\mathcal{C}$" we will mean (unless otherwise specified) a cocycle

$$f \in Z_0(V, W) := \ker[\mathrm{Hom}_0 \to \mathrm{Hom}_1].$$

Two maps $f, g : V \to W$ are *homotopic* if they differ by a coboundary. The *homotopy category* of $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ with maps $\mathrm{Hom}_{\mathrm{ho}\mathcal{C}}(V, W) := H^0 \mathrm{Hom}_*(V, W)$; in particular, if a map $f \in Z_0(V, W)$ has a homotopy inverse, the objects V, W are isomorphic in the homotopy category. The homotopy category is canonically the 0-graded piece of a graded category $H^*(\mathcal{C})$ with $\mathrm{Hom}_{H^*(\mathcal{C})}(V, W) := H^* \mathrm{Hom}(V, W)$.

A functor $F : \mathcal{C} \to \mathcal{D}$ of dg categories is dg if on Homs it induces maps of complexes (i.e. is linear and commutes with differentials). A dg functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor on homotopy categories, $\mathrm{ho}F : \mathrm{ho}\mathcal{C} \to \mathrm{ho}\mathcal{D}$, as well as a homology functor $H^*F : H^*\mathcal{C} \to H^*\mathcal{D}$. We say that F is a *dg equivalence* if H^*F is an equivalence. Morally, functors which are dg equivalences commute up to homotopy with all suitably derived constructions.

Given an associative algebra A, define the derived category of A, which we denote, $D^-(A)$ to be the category of upper-bounded complexes of *projective* left A-modules, and define $D^+(A)$ to be the category of lower-bounded injective left A-modules. Note that a map $V \to W$ in $D^-(A)$, resp., $D^+(A)$ is a homotopy equivalence if and only if it is a quasiisomorphism (i.e. an isomorphism on the level of the graded A-modules $H^*(V) \to H^*(W)$.

Write $D^b(A)$ for the category of upper-bounded complexes of projective A-modules with bounded cohomology, and write $D^b_{inj}(A)$ for the category of lower-bounded injective A-modules with bounded cohomology. One of the basic results of derived category theory is the following fact:

Fact. *There is a canonical equivalence of graded categories $H^*D^b(A) \cong H^*D^b_{inj}(A)$, which is realized by a chain of dg equivalences.*

3.1 Homotopy limits and colimits

The categories we consider will be pretriangulated, and in fact will have a functorial notion of cone. We write down here the consequences of this property that we will use.

Suppose $\mathcal{C}$ is a pretriangulated dg category, I is a finite partially ordered set and $D : I \to \mathcal{C}$ is a (strict) functor (a "diagram in $\mathcal{C}$ indexed by I"). Then there is a canonical object $\mathrm{hocolim}(D) \in \mathcal{C}$ pronounced "homotopy colimit". When $\mathcal{C}$ is the derived category of some category $\mathcal{A}$ (possibly subject to some conditions on the level of homology groups), this object is a representative of the left derived extension of the ordinary colimit functor $\mathrm{colim} : \mathrm{Fun}(I, \mathcal{A}) \to \mathcal{A}$, and in particular on the level of homotopy categories, it is left adjoint to the identity diagram functor $\mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$. The homotopy limit $\mathrm{holim}(D)$ is defined similarly. If the diagram category I is not finite but has finite depth and $\mathcal{C}$ admits arbitrary direct sums (indexed by sets of cardinality bounded by some ordinal, which also bounds the size of I: by assuming choice of a universe, we can safely ignore set-theoretic issues of this sort) then we can also define $\lim(D)$ and $\mathrm{colim}(D)$ for diagrams indexed by I.

Suppose that either $\mathcal{C}$ admits arbitrary direct sums or if I is finite, or the diagram poset I indexes cell containments of a polyhedral complex $\mathcal{P}$ of bounded dimension $\leq n$. Then a diagram $D : I \to \mathcal{C}$ is equivalent to a polyhedrally constructible sheaf $\mathcal{V}_D$ of objects of $\mathcal{C}$ over $\mathcal{P}$. The homotopy limit of D is then functorially homotopy equivalent to the complex of cellular cochains $C^*(\mathcal{P}, \mathcal{V}_D)$. Similarly, if I^{op} is equivalent to a polyhedrally constructible $\mathcal{P}$ then D induces a *cosheaf* $\mathcal{V}_D^{co}$ on $\mathcal{P}$ and $\mathrm{hocolim}(D)$ is functorially homotopy equivalent to the complex of cellular chains, $C_*(\mathcal{P}, \mathcal{V}_D)$.

3.2 Affine covers and Čech descent

Now suppose X is a scheme of finite type (always assumed separated) and $\mathcal{U} = \{U_i\}$ is an affine open cover of X. We will always assume that open covers are closed under intersections (which preserves the property of being affine for separated schemes of finite type), and the indexing set $i \in I$ is a partially ordered set with $i \preceq j$ iff $U_i \subset U_j$. In this case, for a complex of coherent sheaves $\mathcal{F}$ over X, write $\Gamma_{\mathcal{U}}(\mathcal{F}) := \mathrm{hocolim}_{i \in I^{op}} \Gamma(U_i, \mathcal{F})$, where the functors $\Gamma(U_i, \mathcal{F})$ are the functors induced on the derived category from the exact functors of affine sections.

Fix an affine cover $\mathcal{U}$ of the scheme X. We say that a sheaf $\mathcal{F}$ is $\mathcal{U}$-projective if $\mathcal{F}| \mid U_i$ is projective for any $i \in I$.

Fact.

3.3 Symmetric monoidal dg category and fibered categories

For k a commutative ring, the category of complexes $D(k)$ is canonically a symmetric monoidal category. For complexes $V (= V_*), W (= W_*)$ (assumed projective if k is not a field), we have

$$(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j$$

with differential given by

$$d(v \otimes w) = dv \otimes w + (-1)^{|v|} v \otimes dw.$$

The structural isomorphisms giving the symmetric monoidal structure as as follows. The unit object is k. The associative isomorphism $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ is induced by the one on (graded) vector spaces, while the symmetry isomorphism $\sigma : V \otimes W \cong W \otimes V$ is given by

$$\sigma(v \otimes w) := (-1)^{|v| \cdot |w|} w \otimes v.$$

For a triple of complexes of k-modules have the standard adjunction isomorphism, an isomorphism of complexes of k-complexes,

$$\mathrm{Hom}_k(U \otimes_k V, W) \cong \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)).$$

If $\mathcal{C}, \mathcal{D}$ are dg categories over k and $\mathcal{E}$ is another category, we say that a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is bi-dg if the induced maps on Hom spaces are bilinear and factor through maps of *complexes*

$$\mathrm{Hom}_*^{\mathcal{C}}(X, Y) \otimes \mathrm{Hom}_*^{\mathcal{D}}(X', Y') \to \mathrm{Hom}_*^{\mathcal{E}}(F(X, X'), F(Y, Y')).$$

We say a dg category $\mathcal{C}/k$ is symmetric monoidal if it is endowed with a symmetric monoidal structure (as an ordinary category) such that the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bi-dg and all structure morphisms implicit in symmetric monoidal structure are 0-cocycles.

Given $\mathcal{C}$ a symmetric monoidal dg category over k, we say $\mathcal{D}$ is a *symmetric monoidal dg category over* $\mathcal{C}$ if $\mathcal{D}$ consists of a collection of objects with an assignment $X, Y \mapsto \underline{\mathrm{Hom}}(X, Y)(= \underline{\mathrm{Hom}}_{\mathcal{D}/\mathcal{C}}(X, Y))$, and object of $\mathcal{C}$ together with composition morphisms

$$\underline{\mathrm{Hom}}(Y, Z) \otimes_{\mathcal{C}} \underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(X, Z)$$

which is a 0-cocycle, together with for any $X \in \mathcal{D}$ a unit morphism $\mathbb{I}\mathrm{d} \to \underline{\mathrm{Hom}}(X, X)$, which together admit the evident associativity and unitality natural transformations, which we assume to be 0-cocycles. For any object $\mathcal{F} \in \mathcal{C}$, write $\Gamma(\mathcal{F}) := \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}\mathrm{d}_{\mathcal{C}}, \mathcal{F})$. If the symmetric monoidal structure on $\mathcal{C}$ is closed, i.e., if $\mathcal{C}$ admits a *inner Hom* functor, i.e. a bi-dg functor $\mathcal{C} \times \mathcal{C}^{op} \to \mathcal{C}$ written

$$(X, Y) \mapsto \underline{\mathrm{Hom}}(X, Y) \in \mathcal{C}$$

endowed with an adjunction isomorphism $\alpha : \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z)) \cong \underline{\mathrm{Hom}}(X \otimes Y, Z)$, then any category $\mathcal{D}$ fibered over $\mathcal{C}$ gives rise to a structure of dg category over k on the objects of $\mathcal{D}$ by the assignment $\mathrm{Hom}(X, Y) := \Gamma(\underline{\mathrm{Hom}}(X, Y))$.

4 Lax and strict glueing of categories (maybe put in appendix)

Suppose I is a poset (as in Section 3.1), and suppose $D : I \to \mathcal{C}$ is a diagram of categories and functors. For $i \preceq j$ a morphism in I, write F_{ij} for the corresponding functor $\mathcal{C}_i \to \mathcal{C}_j$. Recall that we interpret the diagram in a two-categorical context: i.e. for $i \preceq j \preceq k$ a morphism in I, we are given functors $F_{ij} : \mathcal{C}_i \to \mathcal{C}_j, F_{ik} : \mathcal{C}_i \to \mathcal{C}_k$ and $F_{jk} : \mathcal{C}_i \to \mathcal{C}_j$ together with a natural equivalence $\phi_{ijk} : F_{jk} \circ F_{ij} \to F_{ik}$, satisfying the compatibility condition that any way of composing the natural transformations of type ϕ_{ijk} produces the same natural transformation $F_{i_{n-1}i_n} \circ F_{i_{n-2}i_{n-1}} \circ \cdots \circ F_{i_1 i_2} \to F_{i_1 i_n}$ for a sequence $i_1 \preceq i_2 \preceq \ldots \preceq i_n$.

In this context we define the oplax limit (a.k.a. the Grothendieck construction) $\lim^{oplax}(D)$ to be the category with objects tuples $\{X_i \in \mathcal{C}_i\}_{i \in I}$ together with compatibile morphisms $x_{ij} : F(X_i) \to X_j$ for each $i \preceq j$, satisfying the evident compatiblity, when $i \preceq j \preceq k$, with the ϕ_{ijk} above. A morphism from $\{X_i, x_{ij}\} \to \{X'_i, x'_{ij}\}$ is a collection of maps $X_i \to X'_i$ compatible with all structure. If all $\mathcal{C}_i$ are dg categories and F_{ij} are dg functors, with ϕ_{ijk} maps of complexes, we say D is a dg diagram, and the oplax limit inherits the structure of a dg category. If all $\mathcal{C}_i$ are pretriangulated, then so is the oplax limit. Diagrams $I \to$ dg-Cat of a fixed poset in derived categories themselves form a two-category. A morphism $D \to D'$ is a collection of functors $T_i : \mathcal{C}_i \to \mathcal{C}_i'$ together with weak equivalences

$$\tau_{ij} : F_{ij}' \circ T_i \cong T_j \circ F_{ij},$$

satisfying the evident compatibilities. A morphism $T := \{T_i\}_i : D \to D'$ induces a functor $T_*^{oplax} : \lim^{oplax} \mathcal{C}_i \to \lim^{oplax} \mathcal{C}_i'$. If all T_i are dg equivalences and the $\mathcal{C}_i$ are pretriangulated, then T_*^{oplax} is also a dg equivalence.

5 Geometry in the compactified category

In order to construct and study our functor Loc, we need a good understanding of the derived category of the Bezrukavnikov-Kazhdan category $\overline{\mathrm{Sm}}$, and more generally, a characterization of dg functors $D^b\overline{\mathrm{Sm}} \to \mathcal{C}$ for all "sufficiently nice" dg categories $\mathcal{C}$, in terms of algebraic data on objects of $\mathcal{C}$. The description we will give will have algebro-geometric flavor. The techniques in this section come directly from ideas of toric mirror symmetry, and in particular from constructions in [CCC] (although the exposition will be self-contained).

5.1 Polarization of G

Here we will introduce some notation and collect some standard results having to do with the combinatorics of roots and polarization of coweight lattices. In particular, to every conjugacy class of parabolic $\mathcal{P} \subset G$ we associate a sublattice $\Lambda_{\mathcal{P}}$ of the coweight lattice of G, and a positive cone $\Lambda_{\mathcal{P}}^+ \subset \Lambda_{\mathcal{P}}$.

Notation. *When making a point to distinguish between a geometric group or space and its set of points, we will use math boldface $\mathbf{G}, \mathbf{X}$ for the geometric object and ordinary symbols $G := \mathbf{G}(K), X := \mathbf{X}(K)$ to denote sets of points. When there is no ambiguity, we reserve the right to abuse notation and use G to denote the geometric group $\boldsymbol{G}$, etc.*

Recall that a *polarized* semisimple algebraic group $\mathbf{G}$ is a pair $\boldsymbol{\mathcal{B}} \subset \mathbf{G}$ with $\boldsymbol{\mathcal{B}}$ a fixed Borel subgroup. Recall that a parabolic subgroup of $\boldsymbol{G}$ is an algebraic group $\boldsymbol{P}$ containing a Borel subgroup. Having chosen a polarization, every parabolic subgroup is conjugate to a unique *standard* parabolic subgroup $\boldsymbol{\mathcal{P}} \supset \boldsymbol{\mathcal{B}}$.

Notation. *We will use calligraphic $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{P}}$ to denote standard Borels or parabolics, and roman $\boldsymbol{B}, \boldsymbol{P}$ to denote their conjugates.*

As any parabolic is its own normalizer, the set of all parabolics conjugate to $\boldsymbol{\mathcal{P}}$ can be canonically identified with points of the space $\boldsymbol{G}/\boldsymbol{\mathcal{P}}$.

Notation. *Abusing notation, we will identify the set of (K-rational) parabolics conjugate to $\boldsymbol{\mathcal{P}}$ with the set $G/\mathcal{P}$, and write $\boldsymbol{P} \in G/\mathcal{P}$ to denote a choice of such a parabolic.*

Definition 2. *For $\boldsymbol{\mathcal{P}}$ a standard parabolic, write $\Lambda_{\mathcal{P}}$ for the unramified quotient of L_P by the minimal open normal subgroup, $\Lambda_{\mathcal{P}} := L_{\mathcal{P}}/L_{\mathcal{P}}^0$.*

For $\boldsymbol{\mathcal{B}}$ the Borel, the lattice $\Lambda_{\mathcal{B}}$ is identified with $X_*(\mathbf{T})$, i.e. the dual lattice to the weight lattice. The lattice $\Lambda_{\mathcal{P}}$ is the sublattice in $\Lambda_{\mathcal{T}}$ of vectors satisfying $\langle \lambda, x_i \rangle = 0$ for any principal roots $x_i \in X_{\text{princ}} \setminus X_{\text{princ}}^{\mathcal{P}} \subset X^*(\mathbf{T})$ *not in* the Lie algebra of $\boldsymbol{\mathcal{P}}$. The choice of polarization endows the lattice $\Lambda_{\mathcal{T}}$ with a distinguished positive cone, $\Lambda_{\mathcal{T}}^+ \subset \Lambda_{\mathcal{T}}$ consisting of $\{\lambda \mid \langle \lambda, x_i \rangle \geq 0 \; \forall \; x_i \in X_{\text{princ}}\}$. Write $\Lambda_{\mathcal{P}}^+ = \Lambda_T^+ \cap \Lambda_{\mathcal{P}}$. There is some ambiguity (depending on convention) on the relationship of polarization on the root lattice (i.e. choice of positive cone) to the polarization data $\mathbf{B} \subset \mathbf{G}$. We choose the convention that guarantees that for any rank-one parabolic $\boldsymbol{\mathcal{P}}_i \supset \boldsymbol{\mathcal{B}}$, the action of $L_{\mathcal{B}}^+$ on the p-adic affine line $U_{\mathcal{B}}/U_{\mathcal{P}_i}$ (viewed as a totally disconnected space with a Haar measure) is *expanding*.

5.2 Definition of $\overline{\mathrm{Sm}}$

Here we will recall the definition and some properties of the compactified category $\overline{\mathrm{Sm}}$ from [BK2]. First, a bit more notation. To a pair of embedded standard parabolics $\mathcal{P} \subset \mathcal{Q}$ we will associate an intermediate cone $\Lambda_{\mathcal{P}}^+ \subset \Lambda_{\mathcal{P}}^{\mathcal{Q}+} \subset \Lambda_P$ as follows.

Definition 3. *Write* $\Lambda_{\mathcal{P}}^{\mathcal{Q}+} := \{\lambda \in \Lambda_{\mathcal{P}} \mid \langle \lambda, x_i \rangle \geq 0 \; \forall \; x_i \in X_{princ}^{\mathcal{Q}}\}$. *Evidently,* $\Lambda_{\mathcal{P}}^{\mathcal{P}+} \cong \Lambda_{\mathcal{P}}^+$.

Definition 4. *For* $\mathcal{P} \subset \mathcal{Q}$ *parabolics in* G*, define* $L_{\mathcal{P}}^+$*, resp.* $L_{\mathcal{P}}^{\mathcal{Q}+}$ *to be the preimage in* $L_{\mathcal{P}}$ *of the semigroups* $\Lambda_{\mathcal{P}}^+$*, resp.* $\Lambda_{\mathcal{P}}^{\mathcal{Q}+}$*, in the unramified quotient* $L_{\mathcal{P}}/L_{\mathcal{P}}^0$.

The category $\overline{\mathrm{Sm}}$ will be "glued" out of smooth representation categories of the semigroups $L_{\mathcal{P}}^{\mathcal{Q}+}$ above.

Definition 5. *Define* $\mathcal{H}_{\mathcal{P}\mathcal{Q}}$ *to be algebra of locally constant, compactly supported functions on the topological semigroup* $L_{\mathcal{P}}^{\mathcal{Q}+}$.

Definition 6. *Define* $\mathrm{Sm}_{\mathcal{P}\mathcal{Q}}$ *to be the category of smooth representations of the algebra* $\mathcal{H}_{\mathcal{P}\mathcal{Q}}$.

One should think of the $\mathrm{Sm}_{\mathcal{P}\mathcal{Q}}$ as a system of "étale" (or more specifically, flat) opens of $\overline{\mathrm{Sm}}$, and the open corresponding to the pair $(\mathcal{P}, \mathcal{Q})$ can be considered to "contain" $(\mathcal{P}', \mathcal{Q}')$ if $\mathcal{P}' \subset \mathcal{P} \subset \mathcal{Q} \subset \mathcal{Q}'$.

Definition 7. *Write* $\mathcal{N}$ *for the poset of pairs* $(\mathcal{P}, \mathcal{Q})$ *of standard parabolics in* G *satisfying* $\mathcal{P} \subset \mathcal{Q}$*, with order* $(\mathcal{P}', \mathcal{Q}') \preceq (\mathcal{P}, \mathcal{Q})$ *when* $\mathcal{P}' \subset \mathcal{P} \subset \mathcal{Q} \subset \mathcal{Q}'$.

Now for any pair $(\mathcal{P}', \mathcal{Q}') \preceq (\mathcal{P}, \mathcal{Q})$ we have a functor $j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} : \mathrm{Sm}_{\mathcal{P}\mathcal{Q}} \to \mathrm{Sm}_{\mathcal{P}'\mathcal{Q}'}$ defined as a composition of the following two functors.

Definition 8. *For any triple* $\mathcal{P}' \subset \mathcal{P} \subset \mathcal{Q}$*, we define the functor* $j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}} : \mathrm{Sm}_{\mathcal{P}\mathcal{Q}} \to \mathrm{Sm}_{\mathcal{P}'\mathcal{Q}}$ *taking* $V_{\mathcal{P}\mathcal{Q}}$ *to the coinvariants* $(\mathcal{V}_{\mathcal{P}\mathcal{Q}})_{U_{\mathcal{P}'}}$*, where we view* $\mathcal{V}_{PQ}$ *as a representation of* $\mathcal{P}^{\mathcal{Q}+}$*, the subsemigroup of* $\mathcal{P}$ *which is the preimage of* $L_{\mathcal{P}}^{\mathcal{Q}+}$*, restrict it to the preimage* $(P')^{\mathcal{Q}+}$ *of* $\mathcal{P}'$ *in* $L_{\mathcal{P}'}^{\mathcal{Q}+}$*, then quotient out by* $\mathcal{U}_{\mathcal{P}}'$ *to obtain a representation of* $L_{\mathcal{P}'}^{\mathcal{Q}+}$.

Definition 9. *For any triple* $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{Q}'$*, we define the functor* $j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}\mathcal{Q}'} : \mathrm{Sm}_{\mathcal{P}\mathcal{Q}} \to \mathrm{Sm}_{\mathcal{P}\mathcal{Q}'}$ *taking a representation* $\mathcal{V}_{\mathcal{P}\mathcal{Q}}$ *of* $\mathcal{H}_{PQ}$ *to its extension of scalars* $\mathcal{V}_{\mathcal{P}\mathcal{Q}} \otimes_{\mathcal{H}_{\mathcal{P}\mathcal{Q}}} \mathcal{H}_{\mathcal{P}\mathcal{Q}'}$.

Definition 10. *Now for a quadruple $\mathcal{P}' \subset \mathcal{P} \subset \mathcal{Q} \subset \mathcal{Q}'$ of parabolics, we write*

$$j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} := j_{\mathcal{P}\mathcal{Q}'}^{\mathcal{P}'\mathcal{Q}'} \circ j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}\mathcal{Q}'} : \mathrm{Sm}_{\mathcal{P}\mathcal{Q}} \to \mathrm{Sm}_{\mathcal{P}'\mathcal{Q}'}.$$

Note that this functor is canonically equivalent to the composition in the opposite order, $j_{\mathcal{P}'\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} \circ j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}}$. More generally, any chain of compositions of functors of this sort with the same range and domain will be canonically equivalent. This is encoded in the following lemma.

Lemma 3 (Bezrukavnikov, Kazhdan)**.** *The categories $\mathrm{Sm}_{\mathcal{P}\mathcal{Q}}$ and the functors $j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'}$ extend to a strict representation of the poset $\mathcal{N}$ in categories. I.e. there are canonical isomorphisms of functors $j_{\mathcal{P}'\mathcal{Q}'}^{\mathcal{P}''\mathcal{Q}''} j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} \cong j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}''\mathcal{Q}''} : \mathrm{Sm}_{\mathcal{P}\mathcal{Q}} \to \mathrm{Sm}_{\mathcal{P}''\mathcal{Q}''}$, and these isomorphisms are compatible in an evident sense.*

We define $\overline{\mathrm{Sm}}$ to be the limit of this diagram of functors parametrized by $\mathcal{N}$ in the category of categories. Namely,

Definition 11 (Bezrukavnikov, Kazhdan)**.** *An object $\overline{V}$ of $\overline{\mathrm{Sm}}$ is a collection of objects $V_{\mathcal{P}\mathcal{Q}}$ of $\mathrm{Sm}_{\mathcal{P}\mathcal{Q}}$ along with compatible isomorphisms $j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} V_{\mathcal{P}\mathcal{Q}} \cong V_{\mathcal{P}'\mathcal{Q}'}$. A morphism $f : \overline{V} \to \overline{V}'$ is a collection of morphisms $f_{PQ} : V_{\mathcal{P}\mathcal{Q}} \to V'_{\mathcal{P}\mathcal{Q}}$ such that $j_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} f_{\mathcal{P}\mathcal{Q}} = f_{\mathcal{P}'\mathcal{Q}'}$.*

5.3 Root toric variety and the geometric center of $\overline{\mathrm{Sm}}$

Let $\check{\boldsymbol{T}}_{\mathbb{C}}$ be the algebraic torus over $\mathbb{C}$ with character lattice Λ (which is dual to the character lattice of the maximal torus $\boldsymbol{T} \subset \boldsymbol{G}$). The collection of dual hyperplanes in $\Lambda_{\mathbb{R}}$ to the roots in Λ form a toric fan. Write $\widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{X}}_{\mathbb{C}}$ for the corresponding toric variety over $\mathbb{C}$, with open orbit $\check{\boldsymbol{T}}_{\mathbb{C}} \subset \widetilde{\boldsymbol{X}}_{\mathbb{C}}$. Then $\widetilde{\boldsymbol{X}}$ is smooth W-equivariant (with action induced from W-action on the fan). The W-action preserves the toric stratification, and induced a stratification on the scheme-theoretic quotient $\widetilde{\boldsymbol{X}}//W$. This stratification on $\widetilde{X}/W$ then has components parametrized by faces of the Weyl chamber (as it is a fundamental domain for the W-action on the fan), which are indexed by standard parabolics $\mathcal{P} \subset G$. For $\mathcal{P}$ a parabolic, let W_P be the intersection $W \cap P$ for $W \subset G$ an embedding of the Weyl group that normalizes some $T \subset \mathcal{B}$. Then W_P acts on the lattice $\Lambda_{\mathcal{P}}$, as well as on the semigroup $\Lambda_{\mathcal{P}}^+$. Write $\widetilde{\boldsymbol{X}}_{\mathcal{P}}$ for the spectrum $\mathrm{Spec}(\Lambda_{\mathcal{P}}^+)$, which is an affine toric subvariety $\widetilde{\boldsymbol{X}}_{\mathcal{P}} \subset \widetilde{\boldsymbol{X}}$, with closed toric stratum isomorphic to the torus $\check{\boldsymbol{T}}_{\mathcal{P}} := \mathrm{Spec}(\Lambda_{\mathcal{P}})$. The $\widetilde{\boldsymbol{X}}_{\mathcal{P}}$ are then an affine cover of $\widetilde{X}$, and the $\check{\boldsymbol{T}}_{\mathcal{P}}$ are an affine stratification of $\widetilde{X}$.

Definition 12. *Write $\boldsymbol{X}_{\mathcal{P}\mathcal{Q}}$ for the quotient $X_{\mathcal{Q}}//W_{\mathcal{P}}$ and $S_{\mathcal{P}\mathcal{Q}}$ for the closed stratum $\check{\boldsymbol{T}}_{\mathcal{Q}}//W_{\mathcal{P}}$ in $X_{\mathcal{P}\mathcal{Q}}$.*

Then the $\boldsymbol{X}_{\mathcal{P}\mathcal{Q}}$ give a finite flat cover of $\boldsymbol{X}$. (This cover has the flat analogue of the Nisnevich property, which we will see in the next section). Now it follows from work of Bernstein that the identity functor in the category $\mathrm{Sm}(G)$ has action by the ring of functions $\mathbb{C}[\Lambda]^W \cong \mathbb{O}(\check{\boldsymbol{T}}//W)$ on the scheme-theoretic quotient by this action. Equivalently, the Hom functor in $\mathrm{Sm}(G)$ is enriched to a functor $\underline{\mathrm{Hom}} : \mathrm{Sm}_G \times \mathrm{Sm}_G \to \mathrm{Qcoh}({}^L\boldsymbol{T}_{\mathbb{C}}^{\mathcal{P}}//W_{\mathcal{P}})$

with composition $\circ : \underline{\mathrm{Hom}}(V,W) \otimes \underline{\mathrm{Hom}}(U,V) \to \underline{\mathrm{Hom}}(U,W)$ fibered over the base ${}^L\boldsymbol{T}^{\mathcal{P}}_{\mathbb{C}}//W_{\mathcal{P}}$, and with canonical isomorphism $\mathrm{Hom}(V,W) \cong \Gamma\left({}^L\boldsymbol{T}^{\mathcal{P}}_{\mathbb{C}}//W_{\mathcal{P}}, \underline{\mathrm{Hom}}(V,W)\right)$. An extension of Bernstein's arguments implies that the category of smooth representations of the semigroup $\mathcal{H}_{\mathcal{PQ}}$ is enriched over the commutative ring $\mathbb{O}(X_{\mathcal{PQ}})$. In the same sense, the paper [BK2] shows that $\overline{\mathrm{Sm}}$ is fibered over the smooth projective scheme X. Namely,

Lemma 4 ([BK2])**.** *1. The category $\overline{\mathrm{Sm}}$ is enriched over $X_{\mathbb{C}}//W$, i.e. there is a functor $\underline{\mathrm{Hom}} : \overline{\mathrm{Sm}} \otimes \overline{\mathrm{Sm}} \to \mathrm{Qcoh}(X_{\mathbb{C}}//W)$ with a composition natural transformation $\circ : \underline{\mathrm{Hom}}(V,W) \otimes \underline{\mathrm{Hom}}(U,V) \to \underline{\mathrm{Hom}}(U,W)$ fibered over the base.*

2. *Ordinary* Hom *in $\overline{\mathrm{Sm}}$ is the composition of $\underline{\mathrm{Hom}}$ with global sections, i.e. $\mathrm{Hom}_{\overline{\mathrm{Sm}}}(\overline{V},\overline{W}) \cong \Gamma(X_{\mathbb{C}}//W, \underline{\mathrm{Hom}}(\overline{V},\overline{W}))$.*

3. *Given an object $\overline{V}$ of $\overline{\mathrm{Sm}}$ and $\mathcal{F}$ of $\mathrm{Qcoh}(X_{\mathbb{C}}//W)$, there is an object $\overline{V} \otimes \mathcal{F} \in \overline{\mathrm{Sm}}$ with a natural adjunction equivalence*

$$\underline{\mathrm{Hom}}_{\overline{\mathrm{Sm}}}(\overline{V} \otimes \mathcal{F}, \overline{W}) \cong \underline{\mathrm{Hom}}_{\mathrm{Qcoh}(X_{\mathbb{C}}//W)}(\mathcal{F}, \underline{\mathrm{Hom}}(\overline{V},\overline{W})).$$

4. *In an étale neighborhood of the boundary stratum $S_{\mathcal{P}}$, the pullback of the inner Hom $\underline{\mathrm{Hom}}(\overline{V},\overline{W})$ to any $X_{\mathcal{PQ}}$ agrees with $\underline{\mathrm{Hom}}_{X_{\mathcal{PQ}}}(V_{\mathcal{PQ}}, W_{\mathcal{PQ}})$ (and in particular, the fiber of $\underline{\mathrm{Hom}}(\overline{V},\overline{W})$ over the open stratum $\check{\boldsymbol{T}}//W$ is $\underline{\mathrm{Hom}}(V,W)$).*

5.3.1 Local projectivity and extension

In order to rightfully call $\overline{\mathrm{Sm}}_{fg}$ a "compactified" category, it would be nice to know that any object $V \in \mathrm{Sm}_{fg}(G)$ can be extended to an object $\overline{V} \in \overline{\mathrm{Sm}}_{fg}$, at least in a dg sense. This is in fact true on the level of abelian categories, but in order to simplify our life a little, we prove it in a simpler setting of *locally projective* objects, which we show to dg span all of $D^b\overline{\mathrm{Sm}}$.

Definition 13. *We say that an object $\overline{V} \in \overline{\mathrm{Sm}}$ is* locally projective *if every V_{PQ} is projective as an object of Sm_{PQ}.*

Lemma 5. *Every object in $\overline{\mathrm{Sm}}$ has a finite locally projective resolution.*

Proof. Every Sm_{PQ} has projective resolution of length $\le n$. Now given any object V such that each $V_{\mathcal{PQ}}$ has projective resolution of length $k \ge 0$ and a map $\overline{F} \to \overline{V}$ from a locally projective object $\overline{F}$ which is surjective on every $\mathcal{PQ}$-component, the kernel $\ker(\overline{P} \to \overline{V})$ locally has projective resolutions of length $\le k-1$. Thus by induction, it is enough to show that any object $\overline{V} \in \overline{\mathrm{Sm}}$ admits a surjective map from a locally projective $\overline{P}$. This is shown in [BK2]. □

Now we prove the following lemma.

Lemma 6. *For any locally projective $V \in \mathrm{Sm}_{fg}(G)$, there is a (not necessarily canonical) object $\overline{V}$ with the underlying representation $\overline{V}_{GG} = V$.*

Proof. We proceed by induction. Suppose we have constructed a collection of compatible (in the sense of Definition 11) objects $V_{\mathcal{P}\mathcal{Q}}$ for all $\mathcal{P}_0 \subsetneq \mathcal{P} \subset \mathcal{Q}$, with $V_{GG} = V$. Then we can automatically extend it to a compatible collection of objects $V_{\mathcal{P}'\mathcal{Q}}$ for all $\mathcal{Q} \supsetneq \mathcal{P}_0$, by taking $V_{\mathcal{P}'\mathcal{Q}} := (\mathcal{V}_{\mathcal{P}_0\mathcal{Q}})_{U_{\mathcal{P}'}}$. Now it suffices to extend this collection of compatible representations by an object of type $V_{\mathcal{P}_0\mathcal{P}_0}$ whose localizations produce compatible $V_{\mathcal{P}_0\mathcal{Q}}$ for $\mathcal{Q} \supsetneq \mathcal{P}_0$. Now note that as (by assumption) the $V_{\mathcal{P}_0\mathcal{Q}}$ are finitely-generated and projective, hence torsion-free, we can choose collections of generators $x_i^{\mathcal{P}_0\mathcal{Q}} \in V_{\mathcal{P}_0\mathcal{Q}}$. Now we define $V_{\mathcal{P}_0\mathcal{P}_0}$ to be subspace of the $L^+_{\mathcal{P}_0}$-span of $x^{\mathcal{P}_0\mathcal{Q}} \in V_{\mathcal{P}_0 G}$ which are contained in all $V_{\mathcal{P}_0\mathcal{Q}} \subset \mathcal{V}_{\mathcal{P}_0 G}$. By Noetherianness of the finitely-generated representation categories, this module is finitely-generated. (In fact, the module is independent of choice of generators when the codimension of $\mathcal{P}_0 \geq 2$ by the S2 property). $\square$

Corollary 7. *These two lemmas imply that the K-theory map $K^0(\overline{\mathrm{Sm}}_{fg}) \to K^0(\mathrm{Sm}_{fg}(G))$ is surjective.* $\square$

5.3.2 Internal and external tensor product

Define the category $\overline{\mathrm{Sm}}^R$ for the category of collections of *right* representations $V_{\mathcal{P}\mathcal{Q}}$ of $\mathcal{H}_{\mathcal{P}\mathcal{Q}}$ with opposite compatibility conditions. Then given a pair of objects $\overline{V} \in \overline{\mathrm{Sm}}, \overline{V}' \in \overline{\mathrm{Sm}}^R$, we can define the complex $\overline{V}\underline{\otimes}\overline{W} \in Q\,\mathrm{coh}(X)$ and $\overline{V} \otimes \overline{W} := \Gamma(X, \overline{V}\underline{\otimes}\overline{W})$. The functor $\underline{\otimes}$ is left exact (as can be seen locally), and we can define its derived functor $\overline{V}\overset{L}{\underline{\otimes}}\overline{W} \in D^b\,\mathrm{coh}(X)$. We can then define the functor $\overline{V}\overset{L}{\otimes}\overline{W} := R\Gamma(\overline{V}\overset{L}{\underline{\otimes}}\overline{W})$ (no longer left or right exact) on the derived category which is a dg functor in each component. When $\overline{V}, \overline{W}$ are locally finitely generated, $\overline{V}\underline{\otimes}\overline{W} \in D^b\,\mathrm{coh}(X)$ is a perfect complex of coherent sheaves, hence $\overline{V}\overset{L}{\otimes}\overline{W}$ is a finite complex of finite vector spaces.

5.4 Formal charts and higher Hom

Lemma 8. 1. *For a pair of objects $\overline{V}, \overline{W} \in \overline{\mathrm{Sm}}$, the derived* Hom *space is computed by the limit in the derived category*

$$\mathrm{rHom}(\overline{V}, \overline{W}) \cong \mathrm{holim}_{(\mathcal{P},\mathcal{Q})\in\mathcal{N}} \left(\mathrm{rHom}_{\mathcal{H}\mathcal{P}\mathcal{Q}}(V_{\mathcal{P}\mathcal{Q}}, W_{\mathcal{P}\mathcal{Q}})\right).$$

Further, this quasiisomorphism is compatible with the fibered structure of $\overline{\mathrm{Sm}}$ over $X_{\mathbb{C}}//W$. Namely,

$$\underline{\mathrm{RHom}}(\overline{V}, \overline{W}) \cong \lim_{\mathcal{N}} j_*^{\mathcal{P}\mathcal{Q}} \left(\underline{\mathrm{RHom}}_{\mathcal{H}_{\mathcal{P}\mathcal{Q}}}(V_{\mathcal{P}\mathcal{Q}}, W_{\mathcal{P}\mathcal{Q}})\right),$$

where we take $j : X_{\mathcal{P}\mathcal{Q}} \to X$ the finite flat map of the previous section.

2. *For a pair of objects $\overline{V} \in \overline{\mathrm{Sm}}$ and $\overline{W} \in \overline{\mathrm{Sm}}^L$, we have*

$$\overline{W} \overset{L}{\otimes} \overline{V} \cong \mathrm{holim}\, W_{\mathcal{P}\mathcal{Q}} \otimes_{\mathcal{H}_{\mathcal{P}\mathcal{Q}}} V_{\mathcal{P}\mathcal{Q}},$$

and, similarly, the inner derived Hom

$$\overline{V}\underline{\overset{L}{\otimes}}\overline{W} \cong \mathrm{holim}_{\mathcal{N}}\, W_{\mathcal{PQ}}\underline{\overset{L}{\otimes}}V_{\mathcal{PQ}},$$

where $W_{\mathcal{PQ}}\underline{\overset{L}{\otimes}}V_{\mathcal{PQ}}$ *are viewed as derived pushforwards from* $\mathrm{coh}(X_{\mathcal{PQ}})$ *to* $\mathrm{coh}(X)$.

In order to prove this lemma we will give an alternative glueing of the compactified category $\overline{\mathrm{Sm}}$ out of formal representation categories $\widehat{\mathrm{Sm}}_{\mathcal{PQ}}$ fibered over punctured formal neighborhoods of closed strata in X. Now write $\widehat{X}_{\mathcal{PQ}} := \widehat{S}_{\mathcal{PP}} \cap X_{\mathcal{PQ}}$ for the formal neighborhood of $S_{\mathcal{PP}}$ in $X_{\mathcal{PP}}$ intersected with $X_{\mathcal{PQ}}$. This is an n-dimensional formal scheme which is a product of one-dimensional tori, disks, and formal disks. Now we observe that the rings $\mathcal{H}_{\mathcal{PQ}}$ are canonically fibered along the $X_{\mathcal{PQ}}$, and we can base change to get rings $\widehat{\mathcal{H}}_{\mathcal{PQ}}$, with representation categories $\widehat{\mathrm{Sm}}_{\mathcal{PQ}}$. It is then straightforward to see that an object $\overline{V}$ of $\overline{\mathrm{Sm}}$ is equivalent to a compatible system of representations $\widehat{V}_{\mathcal{PQ}}$ of $\widehat{\mathcal{H}}_{\mathcal{PQ}}$. Now the spaces $X_{\mathcal{PQ}}$ form a Čech cover of X, so that for two sheaves $\mathcal{F}, \mathcal{F}'$ on X, we can compute $\mathrm{rHom}(\mathcal{F}, \mathcal{F}') \cong \mathrm{holim}_{\mathcal{N}}\left(\mathrm{Hom}_{X_{\mathcal{PQ}}}(\mathcal{F}|_{X_{\mathcal{PQ}}}, \mathcal{F}'|_{X_{\mathcal{PQ}}})\right)$, and similarly for $\mathcal{F}\overset{L}{\otimes}\mathcal{F}'$. This implies from our fibered property that also given two objects $\overline{V}, \overline{W}$ in $\overline{\mathrm{Sm}}$, we have $\mathrm{rHom}(\overline{V}, \overline{W}) \cong \mathrm{holim}_{\mathcal{N}}\left(\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathcal{PQ}}}(\widehat{V}_{\mathcal{PQ}}, \widehat{W}_{\mathcal{PQ}})\right)$, and (for V an object of $\overline{\mathrm{Sm}}^L$) we have $\overline{V}\overset{L}{\otimes}\overline{W} \cong \mathrm{holim}_{\mathcal{N}}\left(\widehat{V}_{\mathcal{PQ}} \overset{L}{\otimes}_{hH_{\mathcal{PQ}}} \widehat{W}_{\mathcal{PQ}}\right)$. Now we observe that, fixing $\mathcal{P}$, both the colimits

$$\mathrm{holim}_{\mathcal{Q}\supset\mathcal{P}}\left(\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathcal{PQ}}}(\widehat{V}_{\mathcal{PQ}}, \widehat{W}_{\mathcal{PQ}})\right) \text{ and}$$

$$\mathrm{holim}_{\mathcal{Q}\supset\mathcal{P}}\left(\mathrm{Hom}_{\mathcal{HPQ}}\right)(V_{\mathcal{PQ}}, W_{\mathcal{PQ}}))$$

compute the same complex which is the complex $H^*_{rel}\left((X_{\mathcal{P}}, X_{\mathcal{P}} \setminus S_{\mathcal{P}}), \underline{\mathrm{Hom}}(V_{\mathcal{PP}}, W_{\mathcal{PP}})\right)$, computing the relative coherent cohomology of the sheaf $\underline{\mathrm{Hom}}_{X_{\mathcal{P}}}(\mathcal{V}_{\mathcal{PP}}, \mathcal{W}_{\mathcal{PP}})$ relative to the complement to the closed stratum. This means that we can introduce filtrations on the complexes

$$\mathrm{holim}_{\mathcal{N}}\, \mathrm{rHom}_{\mathcal{H}_{\mathcal{PQ}}}(V_{\mathcal{PQ}}, W_{\mathcal{PQ}}) \text{ and}$$

$$\mathcal{R}\,\mathrm{Hom}(\overline{V}, \overline{W}) \cong \mathrm{holim}_{\mathcal{N}}\, \mathrm{rHom}_{\widehat{\mathcal{H}}_{\mathcal{PQ}}}(\widehat{V}_{\mathcal{PQ}}, \widehat{W}_{\mathcal{PQ}})$$

compatible relative to the obvious map $\mathcal{R}\,\mathrm{Hom}(\overline{V}, \overline{W}) \to \mathrm{holim}_{\mathcal{N}}\, \mathrm{Hom}_{\mathcal{H}_{\mathcal{PQ}}}(V_{\mathcal{PQ}}, W_{\mathcal{PQ}})$, which induce isomorphisms on associated graded components. The arguments for $\overline{V} \overset{L}{\otimes} \overline{V}'$ are analogous. This proves the lemma.

5.5 Noncommutative pushforwards

We've defined the forgetful functors $\jmath^*_{\mathcal{PQ}} : \overline{\mathrm{Sm}} \to \mathrm{Sm}_{\mathcal{PQ}}$; these are exact, hence have obvious derived analogues $\jmath^*_{\mathcal{PQ}} : D^b\overline{\mathrm{Sm}} \to \mathrm{Sm}_{\mathcal{PQ}}$. These functors are noncommutative analogues of affine pull-back and so it makes sense to look for a right adjoint functor $R\jmath^{\mathcal{PQ}}_*$. It is proved in [BK2] that it is possible to define a sheaf of algebras $\mathcal{A}$ over $X_{\mathbb{C}}//W$ such that $\overline{\mathrm{Sm}}$ is equivalent to the category of sheaves of modules $\mathrm{Sm}(\mathcal{A})$. Now $\mathcal{H}_{\mathcal{PQ}}$

induces a sheaf of algebras $\mathcal{A}_{\mathcal{PQ}}$ over $X_{\mathcal{PQ}}$ such that the category of sheaves of representations of $\mathcal{A}_{\mathcal{PQ}}$ is equivalent to $\mathrm{Sm}_{\mathcal{PQ}}$. Write $j_*^{\mathcal{PQ}}\mathcal{A}_{\mathcal{PQ}}$ for the algebra $\mathcal{H}$ considered as an algebra over X via the map of algebraic varieties $j_{\mathcal{PQ}} : X_{\mathcal{PQ}} \to X$. Then it follows from formal arguments that the functor $j_{\mathcal{PQ}}^*$ interpreted as a functor $\mathcal{A} - \mathrm{Mod} \to j_*^{\mathcal{PQ}}\mathcal{A}_{\mathcal{PQ}} - \mathrm{Mod}$ is given by tensor product with some bimodule $M_{\mathcal{PQ}}$ flat over both $\mathcal{A}$ and $j_*^{\mathcal{PQ}}\mathcal{A}_{\mathcal{PQ}}$. From this it follows formally that we have a well-defined right adjoint functor $j_*^{\mathcal{PQ}}$ to $j_{\mathcal{PQ}}^*$, whose derived functor will give a left adjoint in the derived category, $Rj_*^{\mathcal{PQ}}$. We will not write down a formula for the affine components of $(Rj_*^{\mathcal{PQ}}(V_{\mathcal{PQ}}))_{\mathcal{P}'\mathcal{Q}'}$ here (as it is rather involved), but rather just use the existence of this adjoint functor.

5.6 Dg functors out of $\overline{\mathrm{Sm}}$

Now we are ready to characterize DG functors from $\overline{\mathrm{Sm}}$ to an arbitrary dg category $\mathcal{C}$.

Lemma 9. *Suppose that $\mathcal{C}$ is a dg category with all colimits. Then a colimit-preserving functor $D^b\overline{\mathrm{Sm}} \to \mathcal{C}$ is equivalent to a collection of objects $A_{\mathcal{PQ}}$ of $\mathcal{C}$ with right actions by $\mathcal{H}_{\mathcal{PQ}}$, together with compatible identifications $j_{\mathcal{PQ}}^{\mathcal{P}'\mathcal{Q}'}A_{\mathcal{PQ}} \cong A_{\mathcal{P}'\mathcal{Q}'}$, where the functor $j_{\mathcal{PQ}}^{\mathcal{P}'\mathcal{Q}'}(A) := (A \otimes_{H_{\mathcal{PQ}}} H_{\mathcal{PQ}'})_{U_{\mathcal{P}'}}$ is defined as a colimit in the category $\mathcal{C}$.*

Proof. Define $\mathrm{Fun}_{\mathrm{local}}$ for the category of collections $A_{\mathcal{PQ}}$ as above and $\mathrm{Fun}(D^b\overline{\mathrm{Sm}}, \mathcal{C})$ for the category of (dg) colimit-compatible functors. Then we have a functor $\alpha : \mathrm{Fun}_{\mathrm{local}} \to \mathrm{Fun}(D^b\overline{\mathrm{Sm}}, \mathcal{C})$ given by $\alpha(\{X_{\mathcal{PQ}}\}) : \overline{V} \mapsto \mathrm{holim}\, X_{\mathcal{PQ}} \otimes_{\mathcal{H}_{\mathcal{PQ}}} \overline{V}$ and $\beta : \mathrm{Fun}(D^b\overline{\mathrm{Sm}}, \mathcal{C}) \to \mathrm{Fun}_{\mathrm{local}}$ given by $\beta(F)_{\mathcal{PQ}} := F(j(\mathcal{H}_{\mathcal{PQ}}))$ (which have obvious right $\mathcal{H}_{\mathcal{PQ}}$-action as $\mathcal{H}_{\mathcal{PQ}}$-modules). It follows from the previous two subsections that α, β are inverse to each other. $\square$

Notation. *We call the data $X_{\mathcal{PQ}}$ like in Lemma 9 the* kernel *of the functor $F := \alpha(\{X_{\mathcal{PQ}}\})$. The notation comes from the theory of Fourier-Mukai kernels, since in fact, the data of $\{X_{\mathcal{PQ}}\}$ above is most naturally an object of the tensor product category $\overline{\mathrm{Sm}} \boxtimes \mathcal{C}$.*

In Section 7, we will write down a kernel $\{\mathfrak{Loc}_{\mathcal{PQ}}\}$ which we will use to define the localization functor Loc.

6 Algebra on the building

In this section we write down some standard results about the derived category of G-equivariant cosheaves on the Bruhat-Tits building. Our main sources are [BThes] and [Cu].

6.1 Models for sheaves and cosheaves

Definition 14. *Write* $\mathbf{CoSh}^G$ *for the category of cosheaves on the building $\mathbb{B}$ which are constructible with respect to the cellular stratification and equivariant with respect to the G-action on $\mathbb{B}$. Equivalently, this is the opposite category of the category of sheaves with values in* Vect^{op}.

Given any point $x \in \sigma \subset \mathbb{B}$ and cosheaf $\mathcal{V} \in \mathbf{CoSh}^G$, the costalk $\mathcal{V}_x$ has action by the parahoric subgroup G_σ. Because the strata of the cellular stratification are contractible, the data of these stalks together with the specialization morphisms on costalks is sufficient to reconstruct $\mathcal{V}$. More precisely, choose a top-dimensional cell $\Sigma \subset \mathbb{B}$. Note that as we are studying sheaves which are constant on cells and the stabilizer $G(\sigma)$ coincides with the stabilizer $G(x)$ for any $x \in \sigma$, we can unambiguously write $\mathcal{V}_\sigma$ for the costalk of $\mathcal{V}$ at an arbitrary $x \in \sigma$.

Definition 15. *Write $P\Sigma$ for the (non-additive) category with objects cells $\sigma \subset \Sigma$ and morphisms*

$$\mathrm{Hom}_{P\Sigma}(\sigma, \sigma') := \begin{cases} G(\sigma), & \sigma' \subset \sigma \\ \emptyset, & \sigma' \not\subseteq \sigma \end{cases}$$

with compositions given by embeddings of subgroups and the group structure.

This category is generated by the automorphisms $G(\sigma) = \mathrm{Aut}_{P\Sigma}(\sigma)$ together with "specialization" morphisms $\iota_{\sigma\sigma'} : \sigma \to \sigma'$ for $\sigma' \subset \sigma$ (in fact, it's enough to take the two cells to be of consecutive dimension). Then we have

Lemma 10. *There is an equivalence of categories between $\mathbf{CoSh}^G$ and the category of left modules $P\Sigma - Mod$ taking a cosheaf $\mathcal{V}$ to the representation $R\mathcal{V}(\sigma) := \mathcal{V}_\sigma$, with $\mathrm{Aut}_{P\Sigma}(\sigma)$ action induced by equivariance and action of $\iota_{\sigma\sigma'}$ given by cospecialization morphisms of stalks of cosheaves.*

We will abuse notation and go between these two interpretations freely. The most important category for us will be the category $\mathbf{CoSh}^G$ of equivariant cosheaves above. However, it will also be useful for us to have similar "representation-theoretic" models for the categories $\mathbf{CoSh}$ (non-equivariant cosheaves) as well as the categories $\mathbf{Sh}^G, \mathbf{Sh}$ of equivariant and non-equivariant sheaves on $\mathbb{B}$. We define another poset category.

Definition 16. *Define the category $P\mathbb{B}$ to be the poset of closed cells of $\mathbb{B}$, ordered by reverse containment.*

Now the same arguments as above give us the following equivalences.

Lemma 11. *With this definition, we have*

1. *The category of nonequivariant cosheaves $\mathbf{CoSh}_{\mathbb{B}}$ is equivalent to the category of representations of the category $P\mathbb{B}$*
2. *The category of equivariant sheaves $\boldsymbol{Sh}^G_{\mathbb{B}}$ is equivalent to the category of representations of the opposite category $P\Sigma^{op}$.*
3. *The category of nonequivariant sheaves $\boldsymbol{Sh}_{\mathbb{B}}$ is equivalent to the category of representations of $P\mathbb{B}^{op}$.* □

In particular, as the pairs of categories $\mathbf{Sh}, \mathbf{CoSh}$ and $\mathbf{Sh}^G, \mathbf{CoSh}^G$ can be interpreted as representation categories of opposite rings, we obtain tensor product functors $\otimes : \mathbf{Sh} \times \mathbf{CoSh} \to \mathrm{Vect}$ and $\otimes : \mathbf{Sh}^G \times \mathbf{CoSh}^G \to \mathrm{Vect}$, as well as left derived versions $\overset{L}{\otimes}$.

6.2 Projective and injective objects

We will be interested in the derived category $D^b\,\mathbf{CoSh}^G$. It will be convenient for us to have a notion of derived tensor product between sheaves and cosheaves. Namely, for a sheaf $\mathcal{V} \in \mathbf{Sh}_{\mathbb{B}}$ and a cosheaf $\mathcal{V}' \in \mathbf{CoSh}_{\mathbb{B}}$, write $\mathcal{V}\otimes_{P\mathbb{B}}\mathcal{V}'$ for the tensor product of $\mathcal{V}, \mathcal{V}'$ as right and left $P\mathbb{B}$-modules. We define tensor product $\mathcal{V}\otimes_{P\Sigma}$ similarly for $\mathcal{V} \in \mathbf{Sh}^G, \mathcal{V}' \in \mathbf{CoSh}^G$, and write $\mathcal{V} \overset{L}{\otimes} \mathcal{V}'$ for the derived functor. By standard homological-algebraic arguments, this derived tensor product can be computed in terms of a projective resolution of either side. We will be especially interested in the case $\mathcal{V} = \underline{\mathbb{C}}$ the constant sheaf, in which case as we will see $\underline{\mathbb{C}} \overset{L}{\otimes}_{P\mathbb{B}} \mathcal{V}'$ returns the homology of the cosheaf $\mathcal{V}'$. First, we recall from [BThes] a classification of projective objects in $\mathbf{Sh}^G$.

Definition 17. *Given a cell $\sigma \subset \mathbb{B}$ and a vector space V, write $\bigstar_\sigma(V)$ for the constant sheaf on the stellar neighborhood of σ.*

This definition has an equivariant analogue,

Definition 18. *Given a cell σ and a representation V_σ of $G(\sigma)$, write $\bigstar_\sigma(V_\sigma) \in P\Sigma - Mod$ for the sheaf with $\bigstar_\sigma(V_\sigma)_{\sigma'} = V|_{G(\sigma')}$ if $\sigma' \supset \sigma$ and 0 otherwise, where for any pair of cells $\sigma' \supset \sigma$ the cospecialization morphism $\bigstar_\sigma(V_\sigma)_\sigma \xrightarrow{\sim} \bigstar_\sigma(V_\sigma)_{\sigma'}$ is the identity map.*

Lemma 12 ([BThes])**.** *The sheaves $\bigstar_\sigma(V)$ for V irreducible are a complete collection of indecomposable projectives in $\mathbf{Sh}^G$. Their dual cosheaves, $\bigstar_\sigma(V)$ for V irreducible form a complete collection of indecomposable injectives in $\mathbf{CoSh}^G$.* □

Lemma 13 ([BThes])**.** *Any sheaf $\mathcal{V} \in \mathbf{Sh}$ has a projective resolution*

$$\begin{array}{l}\bigoplus_{\sigma\subset\Sigma} \bigstar_\sigma \mathcal{V}_\sigma \leftarrow \bigoplus_{\sigma\supset\tau,\ |\sigma|-|\tau|=1} \bigstar_\sigma \mathcal{V}_\tau \leftarrow \ldots \leftarrow \bigoplus_{\sigma\supset\tau,\ |\sigma|-|\tau|=n} \bigstar_\sigma \mathcal{V}_\tau \to 0 \\ \quad\downarrow \\ \quad\mathcal{V}\end{array}$$

and, analogously, every sheaf $\mathcal{V} \in \mathbf{Sh}^G$ has a projective resolution

$$\begin{array}{l}\bigoplus_{\sigma\subset\Sigma} \bigstar_\sigma \mathcal{V}_\sigma \leftarrow \bigoplus_{\sigma\supset\tau,\ |\sigma|-|\tau|=1} \bigstar_\sigma \mathcal{V}_\tau \leftarrow \ldots \leftarrow \bigoplus_{\sigma\supset\tau,\ |\sigma|-|\tau|=n} \bigstar_\sigma \mathcal{V}_\tau \to 0 \\ \quad\downarrow \\ \quad\mathcal{V}\end{array}$$

This in particular tells us that $\mathbf{Sh}$ and $\mathbf{Sh}^G$ have projective dimension n. Additionally, it gives us a formula for a complex $\mathcal{V} \overset{L}{\otimes} \mathcal{W}$ as follows indexed by pairs $\sigma \supset \tau$:

$$\begin{array}{l}\bigoplus_{\sigma\subset\Sigma} \mathcal{V}_\sigma \otimes_{G_\sigma} \mathcal{W}_\sigma \leftarrow \bigoplus_{|\sigma|-|\tau|=1} \mathcal{V}_\tau \otimes_{G(\sigma)} \mathcal{W}_\sigma \leftarrow \ldots \leftarrow \bigoplus_{|\sigma|-|\tau|=n} \mathrm{Hom}_{G(\sigma)} \mathcal{V}_\sigma \otimes_{G(\sigma)} \mathcal{W}_\sigma \\ \quad\Big\downarrow{\scriptstyle\text{quasi-iso}} \\ \quad\mathcal{V} \overset{L}{\otimes} \mathcal{W},\end{array}$$

both in the equivariant and the non-equivariant settings. Putting in $\mathcal{V} = \underline{\mathbb{C}}_{\mathbb{B}}$ the constant sheaf, we recover the standard complex $R\Gamma_c(\mathcal{V})$ computing the homology of $\mathbb{B}$ with coefficients in $\mathcal{W}$, with respect to the baricentric subdivision of our cellular decomposition. Putting in $\mathcal{V} = \underline{\mathbb{C}}_{\mathbb{B}/G}$, the constant sheaf viewed as an object of $\mathbf{Sh}^G$, we recover a complex computing $R\Gamma(\mathcal{V})_{hG}$, the derived G-coinvariants in the homology of $\mathcal{W}$.

6.3 Constant sheaves on orbifold subsets

Here we will introduce a class of sheaves corresponding to "étale suborbifolds" $\mathbb{S}/H$ of $\mathbb{B}/G$. We will use the notations $\mathbb{B}$, G for the building and the group G, although the same analysis will apply to an arbitrary polyhedrally stratified locally finite CW complex $\mathbb{B}$ with smooth action by a totally disconnected topological group G with compact open stabilizers. Suppose that $\mathbb{S} \subset \mathbb{B}$ is a (closed, cellular) subset and $H \subset G$ a closed subgroup fixing $\mathbb{S}$. Then we define the G-equivariant topological space $G \times_H \mathbb{S} = \frac{G \times \mathbb{S}}{H}$, where H acts diagonally. We define the "action map" $\beta : G \times_U \mathbb{S} \to \mathbb{B}$ via $(g, x) \mapsto gx$, and define

$$\underline{\mathbb{C}}_{\mathbb{S}/H} := \beta_!(\underline{\mathbb{C}}_{G \times_U \mathbb{S}}),$$

the "constant cosheaf on the orbifold $\mathbb{S}/H$", to be the !-pushforward of the constant cosheaf on $G \times_U \mathbb{S}$ via β. This is the sheaf whose stalk over a point $x \subset \mathbb{B}$ is the vector space of compactly supported functions on $G/G(x) \cap U$. (Here $G(x)$ is the stabilizer of x in G, equivalently the stabilizer of a small symmetric open neighborhood of x). We have the following important observation.

Proposition 14. $R\Gamma_c(\underline{\mathbb{C}}_{\mathbb{S}/U}) \cong \mathcal{H}_G \times_{\mathcal{H}_H} R\Gamma_c(\mathbb{S})$.

This follows from the fact that $R\Gamma_c(\mathcal{V}) := R\mathrm{pt}_!(\mathcal{V})$, for $\mathrm{pt} : \mathbb{B} \to *$ the map to a point, hence

$$R\Gamma_!(R\beta_!(\underline{\mathbb{C}}_{\mathbb{S} \times_H G})) \cong R\mathrm{pt}_!(\underline{\mathbb{C}}(\mathbb{S} \times_H G)) \cong C_*(\mathbb{S} \times_H G).$$

(Here we write C_* to denote the complex of chains.) The terminology of constant cosheaf is motivated by the fact that $\underline{\mathbb{C}}_{\mathbb{S}/H} \in \mathbf{CoSh}^G$ corepresents the functor of invariants in cochains,

$$\mathrm{rHom}(\underline{\mathbb{C}}_{\mathbb{S}/H}, \mathcal{V}) \cong C^*(\mathbb{S}, \mathcal{V})^{hH}$$

for arbitrary $\mathcal{V} \in \mathbf{CoSh}$. In particular, if we have $\mathbb{S}_1 \subset \mathbb{S}_2$ and $H_1 \subset H_2$, then we have a canonical map $\iota : \underline{\mathbb{C}}^!_{\mathbb{S}_1/H_1} \to \underline{\mathbb{C}}^!_{\mathbb{S}_2/H_2}$ corresponding to the constant section on $\mathbb{S}_1$ of the constant sheaf on $\mathbb{S}_2$. In fact, this construction can be extended. Let $G_{(\mathbb{S}_1, \mathbb{S}_2)} \subset G$ be the collection of all $\gamma \in G$ with $\gamma(\mathbb{S}_1) \subset \mathbb{S}_2$.

Definition 19. *Given two subsets $\mathbb{S}_1, \mathbb{S}_2 \subset \mathbb{B}$ invariant with respect to $H_1, H_2 \subset G$, respectively, define the "geometric Hom"*

$$\mathrm{Hom}_{geom}(\mathbb{S}_1/H_1, \mathbb{S}_2/H_2) := (H_2 \backslash G_{(\mathbb{S}_1, S_2)})^{H_1}$$

to be the set of right H_1-invariant points in the quotient $H_2 \backslash G_{\mathbb{S}_1/H_1, \mathbb{S}_2/H_2}$ (with evident right H_1-action).

Then the space $C^\infty_{cpt}\operatorname{Hom}_{geom}(\mathbb{S}_1/H_1, \mathbb{S}_2/H_2)$ of compactly supported locally constant functions on $\operatorname{Hom}_{geom}(\mathbb{S}_1/H_1, \mathbb{S}_2/H_2)$ (viewed as a complex concentrated in degree 0) maps to $\operatorname{Hom}_{\mathbf{CoSh}^G}(\mathbb{C}_{\mathbb{S}_1/H_1}, \mathbb{C}_{\mathbb{S}_2/H_2}) \cong H^0\operatorname{rHom}_{D^b\,\mathbf{CoSh}^G}(\mathbb{C}_{\mathbb{S}_1/H_1}, \mathbb{C}_{\mathbb{S}_2/H_2})$. Note that $\operatorname{Hom}_{geom}(\mathbb{S}_1/H_1, \mathbb{S}_2/H_2)$ defines a category structure on pairs $\mathbb{S}/H$ (with $H \subset G$ acting on $\mathbb{S} \subset \mathbb{B}$), and the map

$$C^\infty_{cpt}\operatorname{Hom}_{geom}(\mathbb{S}_1/H_1, \mathbb{S}_2/H_2) \to \operatorname{Hom}_{\mathbf{CoSh}^G}(\mathbb{C}_{\mathbb{S}_1/H_1}, \mathbb{C}_{\mathbb{S}_1/H_2})$$

is compatible with this category structure.

7 Definition of the localization functor

7.1 Corridors

There is a tradition of making papers on Bruhat-Tits theory read like manuals on real estate. Buildings contain apartments that consist of alcoves. There is however a problem with buildings that until now has not been resolved: there is no a priori way of getting from one apartment to another. Here we will finally propose a solution for the long-suffering tenants. We will introduce a notion of corridors, parametrized by standard parabolic subgroups $\mathcal{Q}$, each of which connects together all apartments along a Weyl chamber corresponding to $\mathcal{Q}$.

Fix a basepoint of the building, $x_0 \in \mathbb{B}$, fixed by a maximal compact $K \subset G$. Write $\pi_{\boldsymbol{\mathcal{B}}} : \mathbb{B} \to \mathbb{A}$ for the projection to the quotient $\mathbb{A} \cong \mathbb{B}/U_{\mathcal{B}}$. For a standard parabolic $\mathcal{Q} \supseteq \mathcal{B}$, write $\Lambda^{\perp}_{\mathcal{Q}}$ for the kernel of the composition $\Lambda \subset T \to \mathcal{Q} \to L \to X^*(L/[L,L])$. Then $\Lambda^{\perp}_{\mathcal{Q}} \otimes \mathbb{R}$ acts on $\mathbb{A}$. Write $\mathbb{A}_{\mathcal{Q}} := \mathbb{A}/\Lambda^{\perp}_{\mathcal{Q}} \otimes \mathbb{R}$ and $\pi_{\mathcal{Q}} : \mathbb{B} \to \mathbb{A}_{\mathcal{Q}}$ for the evident composed projection. Write $\mathbb{A}^-_{\mathcal{Q}} \subset \mathbb{A}_{\mathcal{Q}}$ for the cone of all points strictly *smaller* than $\pi_{\mathcal{Q}}(x_0)$ in the usual poset structure on the coweight lattice of G.

Definition 20. *The standard corridor of type $\mathcal{Q}$ is the preimage $\mathbb{D}_{\mathcal{Q}} := \pi^{-1}_{\mathcal{Q}}(\mathbb{A}^-_{\mathcal{Q}}) \subset \mathbb{B}$.*

Example. *Let $G = SL_2(\mathbb{Q}_2)$. Then*

- *$\mathbb{D}_G$ is the whole building $\mathbb{B}$.*
- *$\mathbb{D}_{\mathcal{B}}$ is the contractible graph that looks like this.*

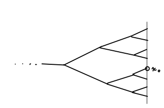

Figure 2: $\mathbb{D}_{\mathcal{B}}$ for SL_2

From the "hyperbolic" point of view, corridors should be thought of as cylinders in the parabolic geometry, tangent to the boundary (in the polyhedral compactification, see [La] or section 8.1) at a parabolic subspace (which will always be a building for a Levi subgroup). For example, $\mathbb{D}_{\mathcal{B}}$ in the example above is a disk that meets the boundary at a point:

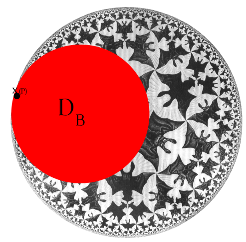

Figure 3: Hyperbolic corridor

In particular, we have the following theorem about the geometry of corridors.

Theorem 15.

1. *$\mathbb{D}_P$ is convex and contractible.*
2. *The normalizer of the standard corridor $\mathbb{D}_P$ is L^0U_P.*
3. *For two corridors $\mathbb{D}_P, \mathbb{D}_Q$ and $\gamma \in G$ we have $\gamma\mathbb{D}_P \subset \mathbb{D}_Q$ if and only if $P \subset Q$ and $\gamma \in Q^+$.*
4. *Suppose $\gamma \in L_Q^{P+}$ such that it is not in any $L_Q^{P'+}$ for $P' \subsetneq P$. Then the union $\cup_{n\geq 0}\gamma^{-n}\mathbb{D}_Q \cong \mathbb{D}_P$.*

We will not give a proof of these relatively straightforward results here.

7.2 The localization kernel

Note that each $\mathbb{D}_\mathcal{Q}$ (being a preimage of a subset of $\mathbb{B}/U_\mathcal{B}$) is invariant with respect to the unitary group $U_\mathcal{B}$, hence also invariant with respect to all $U_\mathcal{P}$ (which are contained in $U_\mathcal{B}$). Now our localization kernel $\mathfrak{Loc}$ will be constructed out of the equivariant cosheaves

$$\mathfrak{Loc}_{\mathcal{P}\mathcal{Q}} := \underline{\mathbb{C}}_{\mathbb{D}_\mathcal{Q}/U_\mathcal{P}}$$

in the terminology of section 6.3.

Namely, observe that for two arbitrary parabolics $\mathcal{Q}, \mathcal{Q}'$, Theorem 15 implies that the set of elements sending $\mathbb{D}_\mathcal{Q}$ to $\mathbb{D}_{\mathcal{Q}'}$ is

$$G_{(\mathbb{D}_\mathcal{Q},\mathbb{D}_{\mathcal{Q}'})} = \begin{cases} L^+_{\mathcal{Q}'}U_{\mathcal{Q}'}\ , & \mathcal{Q} \subseteq \mathcal{Q}' \\ \emptyset\ , & \mathcal{Q} \nsubseteq \mathcal{Q}'. \end{cases}$$

This means that

$$\mathrm{Hom}_{geom}(\mathbb{D}_\mathcal{Q}/U_\mathcal{P}, \mathbb{D}_{\mathcal{Q}'}/U_{\mathcal{P}'}) = (U_{\mathcal{P}'}\backslash L^+_{\mathcal{Q}'}U_{\mathcal{Q}'})^{U_\mathcal{P}} = \left((U_{\mathcal{P}'}\cap L_{\mathcal{Q}'})\backslash L_{\mathcal{Q}'}^{U_\mathcal{Q}}\right)^{L'_\mathcal{Q}\cap U_\mathcal{P}},$$

so long as $\mathcal{P}' \subset \mathcal{P} \subset \mathcal{Q} \subset \mathcal{Q}'$. If $\mathcal{P} = \mathcal{Q} = \mathcal{P}' = \mathcal{Q}'$, then we have $L_\mathcal{P}^{\mathcal{Q}+}U_\mathcal{P} \subset L_\mathcal{Q}^+U_\mathcal{Q}$, which is bi-invariant with respect to $U_\mathcal{P}$, and so $L_\mathcal{P}^{\mathcal{Q}+} \subset \mathrm{Hom}_{geom}(\mathbb{D}_\mathcal{Q}/U_\mathcal{P}, \mathbb{D}_{\mathcal{Q}'}/U_{\mathcal{P}'})$. This gives us the desired $L_\mathcal{P}$-action. Further, we have tautologically for $\mathcal{P} \subset \mathcal{Q}$ that $\mathrm{Loc}_{\mathcal{P}'\mathcal{Q}} \cong (\mathrm{Loc}_{\mathcal{P}\mathcal{Q}})_{U_{\mathcal{P}'}}$. The identity class $1\cdot U_{\mathcal{P}'} \subset L^+_{\mathcal{Q}'}U_{\mathcal{Q}'}$ is right $U_\mathcal{P}$-invariant (since $U_{\mathcal{P}'} \supset U_\mathcal{P}$), hence gives a class $\iota_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} \in \mathrm{Hom}_{geom}(\mathbb{D}_{\mathcal{P}}/\mathcal{Q}, \mathbb{D}_{\mathcal{P}'}/\mathcal{Q}')$, inducing a map $\iota_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}'\mathcal{Q}'} : (\mathfrak{Loc}_{\mathcal{P}\mathcal{Q}})_{U_{\mathcal{P}'}} \to \mathfrak{Loc}_{\mathcal{P}\mathcal{Q}}$, which is visibly $L_{\mathcal{P}}^{\mathcal{Q}+}$-equivariant. In order to check that the $\mathfrak{Loc}_{\mathcal{P}\mathcal{Q}}$ indeed define a kernel, we need to check that the map $\mathfrak{Loc}_{\mathcal{P}\mathcal{Q}} \otimes_{L_{\mathcal{P}}^{\mathcal{Q}+}} \mathcal{L}_{\mathcal{P}}^{\mathcal{Q}'+} \to \mathfrak{Loc}_{\mathcal{P}\mathcal{Q}'}$ adjoint to $\iota_{\mathcal{P}\mathcal{Q}}^{\mathcal{P}\mathcal{Q}'}$ is an isomorphism. This follows from part 4 of Theorem 15. Having checked conditions of Lemma 9, we get a functor $\mathrm{Loc} : \overline{\mathrm{Sm}} \to \mathbf{CoSh}^G$ defined as follows.

Definition 21. *Define*

$$\mathrm{Loc}(\overline{V}) := \mathrm{holim}_{\mathcal{P}\mathcal{Q}}\, \mathfrak{Loc}_{\mathcal{P}\mathcal{Q}} \otimes_{\mathcal{H}_{\mathcal{P}\mathcal{Q}}} V_{\mathcal{P}\mathcal{Q}} = \mathrm{holim}_{\mathcal{P}\mathcal{Q}}\, \underline{\mathbb{C}}_{\mathbb{D}_{\mathcal{P}}/U_{\mathcal{Q}}} \otimes_{\mathcal{H}_{\mathcal{P}\mathcal{Q}}} V_{\mathcal{P}\mathcal{Q}} \in D^b\, \mathbf{CoSh}^G.$$

We introduce also some notation for the "affine components" of Loc, namely

Definition 22. *We define the functor* $\mathrm{Loc}_{\mathcal{P}\mathcal{Q}} : \mathrm{Sm}_{\mathcal{P}\mathcal{Q}} \to \mathbf{CoSh}^G$ *by*

$$\mathrm{Loc}_{\mathcal{P}\mathcal{Q}} : V_{\mathcal{P}\mathcal{Q}} \mapsto V_{\mathcal{P}\mathcal{Q}} \otimes_{\mathcal{H}_{\mathcal{P}\mathcal{Q}}} \underline{\mathbb{C}}_{\mathbb{D}_{\mathcal{Q}}/U_{\mathcal{P}}}$$

for $V_{\mathcal{P}\mathcal{Q}}$ *a reprsentation of* $\mathcal{H}_{\mathcal{P}\mathcal{Q}}$.

Recall here that $\underline{\mathbb{C}}_{\mathbb{D}_{\mathcal{Q}}/U_{\mathcal{P}}}$ is defined as the !-pushforward of the constant sheaf on $\mathbb{D}_{\mathcal{Q}} \times_{U_{\mathcal{P}}} G$ under the action map $\beta : G \times_{U_{\mathcal{P}}} \mathbb{D}_{\mathcal{Q}} \to \mathbb{B}$. This means that we can describe the stalks of $\mathbb{D}_{\mathcal{Q}}/U_{\mathcal{P}}$ alternatively via the following definition.

Definition 23. *Define*

$$H_{\mathcal{P}\mathcal{Q}}^{\sigma} := \frac{\{\gamma \in G \mid \gamma \cdot \mathbb{D}_{\mathcal{Q}} \supset \sigma\}}{U_{\mathcal{P}}} \subset U_{\mathcal{P}} \diagdown G.$$

This is an open subset of $G/U_{\mathcal{P}}$ *left invariant with respect to* $G(\sigma)$ *and right invariant with respect to* $L_{\mathcal{P}}^{+\mathcal{Q}} \subset \mathrm{Hom}_{geom}(\mathbb{D}_{\mathcal{Q}}/U_{\mathcal{P}}, \mathbb{D}_{\mathcal{Q}}/U_{\mathcal{P}})$.

Proposition 16. *We can then have*

$$\mathfrak{Loc}_{\mathcal{P}\mathcal{Q}}^{\sigma} := C_c^{\infty}(H_{\mathcal{P}\mathcal{Q}}^{\sigma}).$$

Definition 24. *Define (abusing notation somewhat)* $\mathfrak{Loc}^{\sigma}$ *for the* $G(\sigma)$*-equivariant object of the right compactified category* $(\overline{\mathrm{Sm}}^R)^{G(\sigma)}$ *to be the object with affine components* $\mathfrak{Loc}_{\mathcal{P}\mathcal{Q}}^{\sigma}$*, and evident componentwise* $G(\sigma)$*-action.*

From our definition of kernels, we now have:

Proposition 17. *With this notation, have stalks* $\mathrm{Loc}(\overline{V})_{\sigma} \cong \mathfrak{Loc}^{\sigma} \overset{L}{\otimes} \overline{V}$.

Having defined the functor, we begin verifying its properties.

Lemma 18. *For any sheaf* $\overline{V} \in \overline{\mathrm{Sm}}$*, we have canonically* $R\Gamma_c \,\mathrm{Loc}(\overline{V}) \cong V$.

Proof. We will prove a corresponding statement independently for each component $\mathrm{Loc}_{\mathcal{P}\mathcal{Q}}$.

Proposition 19. $R\Gamma_c \,\mathrm{Loc}_{\mathcal{P}\mathcal{Q}}\, V_{\mathcal{P}\mathcal{Q}} \cong V_{\mathcal{P}\mathcal{Q}} \otimes_{L_{\mathcal{P}}^{\mathcal{Q}+}} \mathcal{H}_{L_{\mathcal{P}}} \otimes_{L_{\mathcal{P}}} C_c^{\infty}(U_{\mathcal{P}} \backslash G)$.

Proof. It suffices to check this in the universal case, with $V_{\mathcal{P}\mathcal{Q}} = \mathcal{H}_{\mathcal{P}\mathcal{Q}}$, in which case it follows from Proposition 14 and the contractibility of $\mathbb{D}_{\mathcal{Q}}$, (Theorem 15, part 1). □

Note that this proposition in particular implies that, given an object $\overline{V}$ of the compactified category, $R\Gamma_c(\mathrm{Loc}_{\mathcal{PQ}}(\overline{V})) \cong R\Gamma_c(\mathrm{Loc}_{\mathcal{PQ}'}(\overline{V}))$ for any $Q' \supset Q$ (as $V_{\mathcal{PQ}} \otimes_{L_{\mathcal{P}}^{\mathcal{Q}+}} \mathcal{H}_{L_{\mathcal{P}}}$ is independent of $\mathcal{Q}$). Now since all our functors are dg functors and commute with finite homotopy limits, we can compute $R\Gamma_c(\mathrm{Loc}(\overline{V}))$ as the limit of $R\Gamma_c(\mathrm{Loc}_{\mathcal{PQ}}(V_{\mathcal{PQ}}))$, giving

$$R\Gamma_c\left(\mathrm{Loc}(\overline{V})\right) \cong \mathrm{holim}_{\mathcal{P},\mathcal{Q}}\, V_{\mathcal{PQ}} \otimes_{L_{\mathcal{P}}^{\mathcal{Q}+}} C_c^{\infty}(G/U_{\mathcal{P}}).$$

Thus decomposing the partially ordered set $\mathcal{N}$ of pairs $(\mathcal{P},\mathcal{Q})$ into subcategories $(\mathcal{P},-) \subset \mathcal{N}$, we are taking the homotopy limit along a diagram which is constant along each $(\mathcal{P},-)$. Since each of these categories has a terminal object: namely, $(\mathcal{P},G)$, the nerve of the corresponding subcategories is contractible, and the limit computation can be simplified to

$$\mathrm{holim}_{\mathcal{P}}\, V_{(\mathcal{P},G)} \otimes_{L_{\mathcal{P}}} C_c^{\infty}(G/U_{\mathcal{P}}).$$

But the subcategory $(\mathcal{P},G) \subset \mathcal{N}$, in turn, has a terminal object, namely, (G,G) leaving us with $R\Gamma_c(\mathrm{Loc}(\overline{V})) \cong V_{(G,G)} \otimes_{\mathcal{H}} \mathcal{H} \cong V$.

Remark 2. *For SL_2, the category $\mathcal{N}$ has three objects, and looks like this:*

$$(\mathcal{B},\mathcal{B}) \succeq (\mathcal{B},G) \preceq (G,G),$$

and the colimit computation above then identifies the homotopy limit of the fibered product diagram

$$\begin{tikzcd} & V \arrow[d] \\ V_{\mathcal{B}} \otimes_T C_c^{\infty}(G/U_{\mathcal{B}}) \arrow[r, "\sim"] & V_{\mathcal{B}} \otimes_T C_c^{\infty}(G/U_{\mathcal{B}}) \end{tikzcd}$$

with V (note that the colimit of any diagram of the form $A \xrightarrow{\sim} A \leftarrow B$ is B).

This concludes the proof. $\square$

The next lemma establishes that $\mathrm{Loc}(\overline{V})$ *almost* has finite-dimensional stalks when $\overline{V} \in \overline{\mathrm{Sm}}_{fg}$. In the next chapter, we will see how to get rid of the "almost".

Lemma 20. *So long as $\overline{V}$ is locally finitely generated,* $\mathrm{Loc}(\overline{V})$ *has stalks that have finite-dimensional invariants with respect to open compact subgroups.*

Proof. Since $G(\sigma)$ is compact, taking invariants with respect to an open subgroup is an exact functor, and hence for $J \subset G(\sigma)$, we have ${}^J\,\mathrm{Loc}(\overline{V}) \cong \overline{V} \otimes {}^J(\mathfrak{Loc}^{\sigma})$. Now it is sufficient to show (see Section 5.3.2) that ${}^J(\mathfrak{Loc}^{\sigma})$ is locally finitely-generated. In order to see this, we observe that ${}^J\mathfrak{Loc}^{\sigma}_{\mathcal{PQ}} \cong C_c^{\infty}(J\backslash H_{\mathcal{PQ}})$ is the space of compactly supported functions on the subset $J\backslash H_{\mathcal{PQ}}$ of the discrete double quotient $J\backslash G/U_{\mathcal{P}}$. Now in the special case $J = G(\sigma)$, the double quotient $J\backslash H_{\mathcal{PQ}} \subset G(\sigma)\backslash G/U$ can be identified with the collection of cells of $\sigma' \subset \mathbb{A}$ of the apartment which are

$\widetilde{W}$-conjugate to σ and satisfy $\sigma \succeq x_0$ (in the partial order induced by polarization on $\mathbb{A} = \mathbb{B}/U_{\mathcal{B}}$). Hence it is generated over $\Lambda_{\mathbb{Q}}^{+}$ by finitely many classes (corresponding to the minimal cells in each Λ-conjugacy class of $\widetilde{W} \cdot \sigma$). Let these generators be $\{x_i\} \in G(\sigma)\backslash G/U$. Then their finitely many preimages in $J\backslash G/U$ will give a generating set for ${}^{J}(\mathfrak{Loc}^{\sigma})$. This gives us finite generation of $(\mathfrak{Loc}_{\mathcal{PQ}}^{\sigma})^{J}$, and completes our proof. $\square$

8 The truncated localization functor

Fix an integer $e \geq 1$, which we will assume to be chosen larger than the depth of our compactified representation $\overline{V}$. The paper [SS] defines a conjugation invariant system of open normal subgroups $G_\sigma^{(e)} \triangleleft G_\sigma$ indexed by cells $\sigma \subset \mathbb{B}$, with the property that $G_\tau^{(e)} \subset G_\sigma^{(e)}$ for $\tau \subset \sigma$. This allows us to define a "truncation" functor $I^{(e)} : \mathbf{CoSh}^{(e)} \to \mathbf{CoSh}^{(e)}$ defined as follows:

Definition 25. *For $\mathcal{V} \in \mathbf{CoSh}^{G}$ define $I^{(e)}\mathcal{V} \in \mathbf{CoSh}^{(e)}$ to be the cosheaf whose stalks are invariants,*

$$(I^{(e)}\mathcal{V})_\sigma := (\mathcal{V}_\sigma)^{G_\sigma^{(e)}}$$

with respect to the ("Schneider-Stuhler") system of subgroups above.

Definition 26. *Define* $\mathrm{Loc}^{(e)}$ *to be the composition* $I^{(e)} \circ \mathrm{Loc} : \overline{\mathrm{Sm}} \to \mathbf{Sh}$.

By Lemma 20, the functors $\mathrm{Loc}^{(e)}$ have finite-dimensional cohomology of stalks. This section will be devoted to proving the following theorem.

Theorem 21. *Suppose that $\overline{V} \in \overline{\mathrm{Sm}}$ has depth $\leq e$. Then the compactly supported global sections,* $R\Gamma_c(\mathrm{Loc}(\mathcal{V})^{(e)})$

In fact, we will prove a stronger result.

Definition 27. *Define* $\mathrm{Loc}_{\mathcal{PQ}}^{(e)} := I^{(e)}\,\mathrm{Loc}_{\mathcal{PQ}} : \overline{\mathrm{Sm}} \to \mathbf{CoSh}$.

Then we have

Theorem 22. *Suppose $V_{\mathcal{PQ}} \in \mathrm{Sm}_{\mathcal{PQ}}$ has depth $\leq e$. Then the compactly supported global sections,*

$$R\Gamma_c\left(\mathrm{Loc}_{\mathcal{PQ}}^{(e)}(V_{\mathcal{PQ}})\right) \cong R\Gamma_c\left(\mathrm{Loc}_{\mathcal{PQ}}(V_{\mathcal{PQ}})\right).$$

8.1 Building combinatorics

We will give here some reminders about the theory of buildings and the polyhedral compactification of [La]. This subsection and the next will be inspired by constructions and notation in the paper [MS]. We will take a combinatorial point of view based on the Weyl partial order on the coweight lattice. Namely, for an algebraic group $\mathbf{G}$, write $\mathbf{T}$ for its torus, with lattice of characters (the weight lattice) $X^*(\mathbf{T})$ and lattice of coweights $X_*(\mathbf{T})$. We will choose a uniformizer $\varpi \in \mathbb{G}_m(K)$, and write $\Lambda \subset T$ for the lattice $\Lambda \cong X_*(\mathbf{T})$ of coweights embedded in the K-point group $T := \mathbf{T}(K)$ via multipowers of the uniformizer. Write $\Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R}$ and choose a polarization on G. Let $x_1, \ldots, x_n \in \Lambda^\vee$ be the collection of simple roots. Let $\Lambda^+ \subset \Lambda$ be the sublattice of all element which pair positively with the x_i. This is naturally identified (via the metric) with a Weyl chamber. We define a partial order on Λ with $\alpha \preceq \beta$ if $\beta - \alpha \in \Lambda_{\mathbb{R}}^+$. Now to any torus $T \subset G$ there corresponds an apartment $\mathbb{A}_T \subset G$. If we choose a containment $T \subset B$ in a Borel (equivalently, a polarization), then we get a partial order on $\mathbb{A}$ with $a \succeq b$ when we have a containment of stabilizers $U(a) \supset U(b)$ in the unipotent radical $U \subset B$. This partial order satisfies $\varpi^\lambda a \geq a$ if and only if $\lambda \in \Lambda^+$ (where the parametrized embedding $\Lambda \subset T$ is determined by the polarization as above). In particular, if we choose in addition a $x \in \mathbb{A}$, we get a canonical identification $\mathbb{A} \cong \Lambda_{\mathbb{R}}$ compatible with partial order.

We will use the shorthand notation *polarized apartment* to denote an apartment with choice of partial order corresponding to a pair $T \subset B$ as above.

Definition 28 (Meyer and Solleveld)**.** *The* convex hull *of two cells $\sigma, \tau \subset \mathbb{B}$, denoted $\overline{\sigma, \tau}$, is the intersection of all apartments containing both σ and τ. More generally, the convex hull of a subset $\Gamma \subset \mathbb{B}$ is the union of all convex hulls of pairs of points of Γ. This notion generalizes in an obvious way to a subset $\Gamma \subset \overline{\mathbb{B}}$*

Remark 3. *The idea behind this terminology is to replace the notion of a geodesic line segment, which is the collection of points on a shortest path between a, b, by a "partially ordered geodesic line segment", which is the collection of points x in a parametrized apartment between $a, b \in \mathbb{B}$ which satisfy $a \preceq x \preceq b$ in the Weyl partial order corresponding to the parametrization.*

Definition 29. *Given a point $x \in \mathbb{A}$ in a polarized apartment, we say that a subset $R \subset \mathbb{A}$ with the data of a partial order is a* geodesic ray *out of x in the given polarization if R is cofinal with minimal point x in the partial order, i.e. if any $y \in R$ satisfies $y \succeq x$ and for any pair $y, y' \in R$, there is $z \in R$ with $z \succeq y, z \succeq y'$.*

Remark 4. *We are not using the usual notion of metric on the Bruhat-Tits building, and a geodesic ray is in general* **not one-dimensional***.*

For an arbitrary pair of points x, y, there is then a unique geodesic ray $\overrightarrow{x, y}$ with minimal point x and maximal point y. If σ, σ' are a pair of closed simplices, then there is a unique vertex $x \in \sigma, y \in \sigma'$ such that $\overrightarrow{x, y}$ contains both σ and σ'. We define $\overrightarrow{\sigma, \sigma'}$ to be this subset (with induced partial order).

Remark 5. *The notion of a geodesic ray allows us to define the polyhedral compactification of $\overline{\mathbb{B}}$ (see [La]) as follows. Define $\overline{\mathbb{B}}$ to be the quotient of the collection of closed geodesic rays $R \subset \mathbb{B}$ by the equivalence relation that $R \sim R'$ if $R \cap R'$ is cofinal in both R and R'. In particular, the convex hulls $\overline{x, y}$ and $\overline{x', y}$ (with partial order such that y is maximal) are equivalent as their intersection contains y. This gives the embedding $\mathbb{B} \subset \overline{\mathbb{B}}$.*

8.2 Consistent systems of idempotents

Definition 30 (Meyer and Solleveld)**.** *Let V be a vector space with action of G. We say that a system of idempotents $E_\sigma \in \operatorname{End}_{\mathbb{C}} V$ indexed by cells* *of the building is* consistent *if it satisfies the following three properties.*

(a) (local commutativity) E_σ, E_τ commute if σ, τ are in the closure of the same face

(b) (local multiplicativity) The idempotent corresponding to a cell is the product of those corresponding to its vertices, i.e. $E_\sigma = \prod_{x\in\sigma^0} E_x$.

(c) (convexity) For any triple of cells τ, σ, σ' with $\tau \subset \sigma$ and σ in the convex hull of τ, σ', we have the identity $E_\sigma E_{\sigma'} = E_\tau E_{\sigma'}$.

Lemma 23 (Meyer and Solleveld)**.** *For any depth $e \geq 1$, the idempotents $E_\sigma^{(e)} := \delta G_\sigma^{(e)}$ form a consistent system of idempotents.* □

We mention that in the proof of [MS], property b above is reduced to the following group identity, which will be useful to us as well:

Proposition 24 (Meyer and Solleveld)**.** $G_\tau^{(e)} \cdot G_{\tau'}^{(e)} \supset G_\sigma^{(e)}$. □

Given a system $E = \{E_\sigma\}$ of consistent idempotents, write $\mathcal{V}^E$ for the coefficient system with the vector space V^{E_σ} over cell σ. Note that this is admits a map to the constant coefficient system $\underline{V}$.

Lemma 25 (Meyer and Solleveld)**.** *For any consistent system of idempotents $\{E_\sigma\}$ and any closed convex subset $\Gamma \subset \mathbb{B}$, the derived global sections $R\Gamma(V^E)$ are quasiisomorphic to the vector space $\sum V^{E_\sigma} \subset V$ concentrated in degree 0. This identification is consistent with the embedding of coefficient systems $R\Gamma(V^E) \subset R\Gamma(\underline{V}) = V[0]$.* □

Choose a parabolic $P \supset B$. Now let $V_P = i_P r_P \mathcal{H}_{L_P} = C_c^\infty(G/U_P)$. We have two algebras acting on V_P. First, $\mathcal{H}_G$ acts via the usual representation structure. Secondly, the commutative algebra

$$R := C^\infty(G/U)$$

of *all* locally constant functions acts on compactly supported functions by multiplication. The two actions combine into an action of the crossed product algebra

$$A_P := R\#G.$$

Now choose another parabolic $Q \supset P$. Observe that the subsets $H_{PQ}^{+\sigma}$ giving local action of the localization functor correspond to idempotent functions δ_σ^{PQ+} on R, and that these are preserved by the subgroups G_σ, hence commute with the idempotents $E_\sigma^{(e)} = \delta(G_\sigma^{(e)})$ (hence their products are idempotent). To unburden notation, write

$$\delta_\sigma := \delta_\sigma^{PQ+},$$

$$\mathcal{E}_\sigma := E_\sigma^{(e)} \text{ and}$$

$$\Phi_\sigma := \delta_\sigma \mathcal{E}_\sigma.$$

The idempotents Φ_σ act on the space $V_P \cong i_P \mathcal{H}_{L_P}$ of compactly supported functions, and have image $\mathrm{Loc}_{PQ}^{\sigma(e)}(\mathcal{H}_{PQ})$ (compactly supported, $G_\sigma^{(e)}$-equivariant functions on $H_\sigma^{PQ} \subset G/U_P$). Up to a universality argument, it suffices for us to prove the following.

Proposition 26. *The idempotents $E_{PQ}^{\sigma(e)}$ form a consistent system of idempotents.*

Proof. It will be convenient for us to give a formula for products of functions of the form δ_σ. Namely, given an element $\gamma \in G/U_P$, write $\mathbb{D}_\gamma^Q := \gamma \cdot \mathbb{D}_0^Q$ for the corresponding corridor (of type Q). Recall that the space H_σ^{PQ} is the collection of all $\gamma \in G/U$ such that $\sigma \subset \mathbb{D}_\gamma^Q$. Because corridors are convex and closed, we can multiply idempotents of the form δ_σ in the following way. Suppose that $\sigma_1, \ldots, \sigma_k$ are a collection of cells (of arbitrary dimension). Write Σ for the convex hull of the closed cells $\overline{\sigma}_i$. Write $H_\Sigma^+ = \{\gamma \in G/U \mid \mathbb{D}_\gamma^Q \supset \Sigma\}$. By convexity of corridors, we have H_Σ^{PQ} in this case is the intersection of all $H_{\sigma_i}^{PQ} \subset G/U_P$. Write δ_Σ for the corresponding characteristic function. We deduce that we have the following formula.

Lemma 27. *We have*

$$\prod_{i=1}^{k} \delta_{\sigma_i} = \delta_\Sigma$$

with Σ the convex hull of the closed cells $\overline{\sigma}_i$ as above. ☐

In order to prove proposition 26, we need to check the three properties of Definition 30 for the Φ_σ. By Meyer and Solleveld's Lemma 23, we have consistency of the $\mathcal{E}_\sigma$ and the lemma 27 applied to the vertices of a single cell cell gives us the local multiplicativity property for the system of idempotents δ_σ. The other two properties are obvious from commutativity of the δ_σ, giving us consistency of the system $\{\delta_\sigma\}$ as well. Note that this is not yet good enough to give us the desired consistency of the $\{\Phi_\sigma\} = \{\mathcal{E}_\sigma \delta_\sigma\}$ as $\mathcal{E}_\sigma$ may not commute with $\delta_{\sigma'}$ for σ, σ' far apart.

First, we observe that local commutativity and multiplicativity follows from the corresponding properties of the systems $\{\mathcal{E}_\sigma\}, \{\delta_\sigma\}$ by checking that idempotents of the two systems mutually commute at nearby vertices.

Claim. *the idempotents $\mathcal{E}_\tau, \mathcal{E}_{\tau'}, \delta_\tau$ and $\delta_{\tau'}$ pairwise commute for τ, τ' both in the closure of a cell σ.*

The only pairs for which we still need to check this are $(\mathcal{E}_{\tau'}, \delta_\tau)$ and $(\mathcal{E}_\tau, \delta_{\tau'})$. Now we have by construction that $G_{\tau'}^{(e)} \subset G_\sigma^{(e)} \subset G_\tau$ (see [SS], I.2, where these groups are called $U_F^{(e)}$). Since G_τ normalizes H_τ^{PQ}, the idempotents $\mathcal{E}_{\tau'}$ and δ_τ commute and we are done WLOG. ☐

It remains for us to check convexity. Note that, fixing a Haar measure, the twisted product $A = R\#G$ can be identified with locally constant functions on $G/U \times G$ which are supported over a bounded subset of G. Product is computed via the multiplication kernel

$$x\#\gamma d(G/U)dG \cdot x'\#\gamma' d(G/U)dG := \delta_{x,\gamma x'} \cdot x\#\gamma\gamma' d(G/U)dG.$$

Suppose we have a triple $\tau \subset \sigma$, and τ' such that $\sigma \subset \overline{\tau\tau'}$ is in the convex hull (of the open cells). Write $\Sigma := \overline{\overline{\tau}, \overline{\tau}'}$ for the convex hull of the closures of the cells, which coincides with $\overline{\overline{\sigma\tau'}}$. We compute

$$\Phi_\tau\Phi_{\tau'} = \int_{\gamma\in G_\tau^{(e)},\eta\in G/U|\mathbb{D}_\eta\ni\tau} \frac{d\gamma d\eta}{|G_\tau^{(e)}|}\eta\#\gamma\cdot\int_{\gamma'\in G_{\tau'}^{(e)},\eta'\in G/U|\mathbb{D}_{\eta'}\ni\tau'}\frac{d\gamma' d\eta'}{|G_{\tau'}^{(e)}|}\eta'\#\gamma' \tag{1}$$

$$= \int_{\gamma\in G_\tau^{(e)},\mathbb{D}_\eta\ni\tau,\gamma'\in G_{\tau'}^{(e)},\mathbb{D}_\eta\ni\tau'}\frac{d\gamma d\eta d\gamma' d\eta'}{|G_\tau^{(e)}|\cdot|G_{\tau'}^{(e)}|}\delta(\eta,(\eta')^\gamma)\#\gamma\gamma' \tag{2}$$

Where we are using the notation $\delta(\eta,\eta')$ for the delta measure on the diagonal $\eta=(\eta')^\gamma$. Now note that for $\gamma\in G_\tau$, the conditions $\mathbb{D}_\eta\ni\tau,\mathbb{D}_{\eta'}\ni\tau'$ and $\eta=(\eta')^\gamma$ together are equivalent to $\mathbb{D}_\eta\ni\tau,\mathbb{D}_\eta\ni\gamma\tau'$ and $\eta'=\eta^{\gamma^{-1}}$, which can further be reduced to $\mathbb{D}_\eta\supset\gamma\Sigma$ (as $\tau=\gamma\tau$). This lets us rewrite

$$\Phi_\tau\Phi_{\tau'} = \int_{\gamma\in G_\tau^{(e)},\gamma'\in G_{\tau'}^{(e)},\eta\in G/U,\mathbb{D}_\eta\supset\gamma\Sigma}\frac{d\gamma d\gamma' d\eta}{|G_\tau^{(e)}|\cdot|G_{\tau'}^{(e)}|}\eta\#\gamma\gamma' \tag{3}$$

Now note that multiplying γ in the above expression on the right by any element $\gamma_0\in G$ that fixes Σ and is contained in the product $G_\tau^{(e)}G_{\tau'}^{(e)}$ will not change the result. In particular, this is true for any $\gamma_0\in G_\sigma^{(e)}\cap G_\Sigma$. Write $G_0:=G_\sigma^{(e)}\cap G_\Sigma$ and averaging over $\gamma_0\in G_0$ as above, we can safely introduce a new variable γ_0 in the integral above:

$$\Phi_\tau\Phi_{\tau'} = \int_{\gamma\in G_\tau^{(e)},\gamma_0\in G_0,\gamma'\in G_{\tau'}^{(e)},\eta\in G/U,\mathbb{D}_\eta\supset(\gamma)\Sigma}\frac{d\gamma d\gamma_0 d\gamma' d\eta}{|G_\tau^{(e)}|\cdot|G_0|\cdot|G_{\tau'}^{(e)}|}\eta\#\gamma\gamma_0\gamma'. \tag{4}$$

Now we observe that $G_\tau^{(e)}\cdot G_0=G_\sigma^{(e)}$, as we know from Proposition 24 that $G_\tau^{(e)}\cdot G_{\tau'}^{(e)}\supset G_\sigma^{(e)}$, so it follows that $G_\tau^{(e)}\cdot\left(G_{\tau'}^{(e)}\cap G_\sigma^{(e)}\right)=G_\sigma^{(e)}$. Using this identity (and some obvious homogeneity considerations), the expression above can be rewritten as

$$\Phi_\tau\Phi_{\tau'} = \int_{\gamma\in G_\sigma^{(e)},\gamma'\in G_{\tau'},\eta\in G/U,\mathbb{D}_\eta}\frac{d\gamma d\gamma' d\eta}{|G_\sigma|\cdot|G_{\tau'}|}\eta\#\gamma\gamma' \tag{5}$$

which, by the arguments above, $$=\Phi_\sigma\Phi_{\tau'}. \tag{6}$$

This concludes the proof that the Φ_σ are coherent. $\square$

From this we deduce by Lemma 25 that the derived global sections of the cosheaf $\sigma\mapsto\Phi_\sigma\cdot C_c^\infty(G/U)=\mathrm{Loc}_{PQ}^{(e)}(\mathcal{H}_{PQ})$ have no higher cohomology, and in degree zero give the subspace of $C_c^\infty(G/U)$ spanned by all $\Phi_\sigma C_c^\infty(G/U)$, which evidently are equivalent to the depth-e component of $C_c^\infty(G/U)$ (see this by moving σ towards the boundary in the direction of the polarization on $\mathbb{A}$). Thus the natural transformation $\mathrm{Loc}_{PQ}^{(e)}(\mathcal{H}_{PQ})\to(\mathrm{Loc}_{PQ}(\mathcal{H}_{PQ}))^{(e)}$ is an equivalence on global sections. As both functors, as well as $R\Gamma_c$ are derived exact and commute with all colimits, and since $\mathcal{H}_{PQ}$ has right action by $\mathcal{H}_{PQ}$ that commutes with the left module structure, this implies that for any module V_{PQ} over $\mathcal{H}_{PQ}$ of depth $\leq e$, we have

$$\Gamma_c \operatorname{Loc}^{(e)}_{PQ} V_{\mathcal{PQ}} \cong \Gamma_c \operatorname{Loc}^{(e)}_{PQ}(\mathcal{H}_{PQ} \otimes_{\mathcal{H}_{PQ}} (V_{PQ}))$$
$$\cong (\Gamma_c \operatorname{Loc}_{PQ}(\mathcal{H}_{PQ}))^{(e)} \otimes_{\mathcal{H}_{PQ}} (V_{PQ}) \cong \Gamma_c \operatorname{Loc}_{PQ}(V),$$

where for a representation $V \in \mathrm{Sm}(G)$, we define $V^{(e)}$ to be its depth-$\leq e$ component. From this we deduce that $R\Gamma_c \left(\operatorname{Loc}^{(e)}(\overline{V})\right) \cong R\Gamma_c \operatorname{Loc}(\mathcal{V})$, which by Lemma 18 is just V.

With Theorem 21 in hand, our main result, Theorem 2 easily follows. Namely, given an arbitrary $V \in \mathrm{Sm}$ of depth $\leq e$, write $[\overline{V}] \in K^0(\overline{\mathrm{Sm}}_{fg})$ a preimage of the class $[V] \in \mathrm{Sm}_{fg}(G)$, which exists by Corollary 7. Write $\operatorname{Loc}^{(e)}_\sigma$ for the fiber over σ of $\operatorname{Loc}^{(e)}$. As these are dg functors, they define maps on K-theory $[\operatorname{Loc}^{(e)}_\sigma]$. Now from the identity $V \cong R\Gamma_c \operatorname{Loc}(\overline{V}) \cong R\Gamma_c \operatorname{Loc}(\overline{V})$ (and using Lemma 20) we deduce $[V] = [\mathrm{Ind}] \sum_{\sigma \in \Sigma} (-1)^{|\sigma|} [\operatorname{Loc}^{(e)}_\sigma]([\overline{V}])$. $\square$

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