Math 113 Homework 4, selected solutions

1, Book exercises: 5.1-5.6: determine whether the following groups are subgroups of the group $\mathbb{C}$ of complex numbers under addition.

5.1, $\mathbb{R}$ Yes: $\mathbb{R}$ is closed under addition, $0 \in \mathbb{R}$, and if $r \in \mathbb{R}$ then $-r \in \mathbb{R}$ so closed under additive inverse.

5.2, $^+$ **(positive rational numbers)** No. Closure under $+$ holds but not identity or inverse axioms ($0 \notin \mathbb{Q}^+$ and $-2 \notin \mathbb{Q}^+$ though $2 \in \mathbb{Q}^+$).

5.3, $7\mathbb{Z}$ Yes. Closed under addition, $0 \in 7\mathbb{Z}$ and closed under additive inverse.

5.4, $i\mathbb{R}$ **the set of purely imaginary numbers including** 0 Yes. Closed under addition, $0 = 0 \cdot i \in i\mathbb{R}$ and closed under inverse.

5.5, the set $\pi\mathbb{Q}$ **of rational multiples of** π Yes. Closed under addition, $0 = 0 \cdot \pi \in \pi\mathbb{Q}$ and closed under inverse.

5.6, the set $\{\pi^n \mid n \in \mathbb{Z}\}$. No. 0 is not a power of π, and not closed under addition $\pi + \pi$ is not a power of π).

5.7 Which of these are subgroups of the group $\mathbb{C}^*$ of nonzero complex numbers under multiplication?

(1) $\mathbb{R}$ is not even a subset of $\mathbb{C}^*$ (since $0 \in \mathbb{R}$ but $0 \notin \mathbb{C}^*$), so the question does not make sense.

(2) $\mathbb{Q}^+$, surprisingly, is a mutiplicative subgroup. Indeed, $q \cdot r \in \mathbb{Q}^+$ (is positive and rational) if both q, r are positive and rational. $1 \in \mathbb{Q}^+$ and the inverse of a positive rational number is positive and rational.

(3) $7\mathbb{Z}$ no. It is not a subset of $\mathbb{C}^*$, as $0 \in 7\mathbb{Z}$. (If one were to remove 0 to turn it into $7\mathbb{Z}^*$, it would be closed but not have the unit element, $1 \notin 7\mathbb{Z}^*$.)

(4) $i\mathbb{R}$ no. Not a subset of $\mathbb{C}^*$. (If we were to remove 0 to turn it into $i\mathbb{R}^*$ then it still wouldn't be: for example $i \cdot i = -1 \notin i\mathbb{R}^*$.)

(5) The set $\pi\mathbb{Q}$ of rational multiples of π is not a subset of $\mathbb{C}^*$.

(6) $\pi^n \mid n \in \mathbb{Z}$ is a subgroup! It is contained in $\mathbb{C}^*$ (no power of π is equal to 0), it is closed under $\cdot$, contains the identity elemetn $1 \in \mathbb{C}^*$ and is closed under inverse. In fact, it is a cyclic subgroup generated by π: so $\{\pi^n \mid n \in \mathbb{Z}\}\langle\pi\rangle \subseteq \mathbb{C}^*$

5. (a) *Write down an addition table for the Klein 4-group V (look it up in the book!). Write down an addition table for the Gaussian numbers modulo 2, i.e. $\mathbb{G}/2 \cdot \mathbb{G}$. Give a function $V \to \mathbb{G}/2 \cdot \mathbb{G}$ which takes one table to the other (i.e. is an isomorphism).*[1]

	e	a	a^2	b	c	d
e	e	a	a^2	b	c	d
a	a	a^2	e	c	d	b
a^2	a^2	e	a	d	b	c
b	b	d	c	e	a^2	a
c	c	b	d	a	e	a^2
d	d	c	b	a^2	a	e

(b) Recall that the direct product $\mathbb{Z}_n \times \mathbb{Z}_n$ is the group of pairs $([a], [b])$ of residues modulo n with *componentwise* addition $([a], [b]) + ([a'], [b']) = ([a + a'], [b + b'])$. Construct an isomorphism from $\mathbb{Z}_n \times \mathbb{Z}_n$ to $(\mathbb{G}/n \cdot \mathbb{G}, +)$.

6. Extra credit, worth either 1/2 a problem or alternatively write “doing this problem instead of problem x” to replace one of problems 1.-4. but not 5: **book exercise 4.29**