Math 113 Homework 3, selected solutions

1. Book exercise 3.3: is $\varphi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ given by $\varphi(n) = 2n$ an isomorphism? No. It is a homomorphism, since $\varphi(a+b) = 2(a+b) = \varphi(a) + \varphi(b)$. It is also injective, since $2a = 2b$ implies $a = b$ for $a, b \in$. But it is not surjective: 1 is not $2a$ for any integer a!

BTW, it is a little surprising that there is a function from $\mathbb{Z}$ to $\mathbb{Z}$ that is injective but not surjective: wouldn't this imply that $\mathbb{Z}$ is "bigger" than itself? Turns out this is not the case for infinite sets: you can map them to themselves injectively with "room left over". If you're curious, you can look up "infinite hotel" online (or if you want to learn in a little more depth, look up "cardinality" on wikipedia).

3. (a) Show that "two-sided cancellation" holds in groups. I.e., if $a, b, \tilde{b}, c$ are elements of a group G with operation $\cdot$, and $a \cdot b \cdot c = a \cdot \tilde{b} \cdot c$ then $b = \tilde{b}$.

Say $abc = a\tilde{b}c$. Take both sides of the identity, multiply on the left by a^{-1} (which exists since we're working in a group) and on the right by c^{-1}. We obtain:

$$b = a^{-1} \cdot a \cdot b \cdot c \cdot c^{-1} = a^{-1} \cdot a \cdot \tilde{b} \cdot c \cdot c^{-1} = \tilde{b},$$

so $b = \tilde{b}$, as desired. (Notice that when applying this argument it is important that I multiplied by a^{-1} on the left and by b^{-1} on the right – if I was careless about order of factors, they would not cancel, since groups are not in general abelian!)

(b) If G is a group and $g \in G$ is a fixed element, show that the function i_g with $i_g(x) = g \cdot x \cdot g^{-1}$ is an isomorphism of G with itself. Three things to show. First, homomorphism property:

$$i_g(x) \cdot i_g(y) = (g \cdot x \cdot g^{-1})(\cdot g \cdot y \cdot g).$$

By associativity, we can first multiply the $g^{-1} \cdot g$ in the middle *(as this involves changing parenthesization, but not order of factors in the expression)*. Giving us:

$$= g \cdot x \cdot e \cdot y \cdot g^{-1} = g \cdot x \cdot y \cdot g^{-1} = i_g(xy).$$

This concludes checking the homomorphism property. Now it remains to check bijectivity. As usual, this is done in two steps: *Surjectivity*: the value x is $i_g(g^{-1}xg)$ (since $\varphi_g(g^{-1}xg) = g(g^{-1}xg)g^{-1} = e \cdot x \cdot e = x$.) And *injectivity:* say $i_g(x) = i_g(\tilde{x})$. Then $gxg^{-1} = g\tilde{x}g^{-1}$, and $x = \tilde{x}$ by part (a).