Formality of little disks and algebraic geometry

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Abstract

We construct a canonical chain of formality quasiisomorphisms for the operad of chains on framed little disks and the operad of chains on little disks. The construction is done in terms of logarithmic algebraic geometry and is remarkable for being rational (and indeed definable integrally) in de Rham cohomology.

Contents

1 Introduction

Let FLD be the operad of framed little disks. The author showed in [55] that FLD is equivalent (via a chain of quasiisomorphisms of topological operads) to the topological realization of an operad FLD^{log} of operads in logarithmic algebraic geometry (in the sense of Kato). In this note we write down explicitly the log spaces and maps involved, and show that these spaces have an incredibly well-behaved Hodge splitting property (i.e. quasiisomorphism between cohomology of forms and de Rham cohomology), which we call the *proper acyclicity* property, something that is only possible in the log geometry context. From this property we deduce an explicit formality splitting quasiisomorphism between dg operads $H_*(FLD^{log}, \mathbb{C})$ and $C_*(FLD^{log}, \mathbb{C})$. We deduce similar splittings for the operad LD of little disks as a consequence. Our construction is canonical and explicit. It does not depend on a choice of an associator or a Frobenius element in the Grothendieck-Teichmüller group, or a choice of obstruction-theoretic splitting. Its algebro-geometric provenance also gives it automatic compatibility with a number of structures. In particular this splitting is rational (and indeed definable integrally) for the "de Rham" rational structure on the operad FLD (the de Rham structure on the complex $C^*(FLD)$ as well as on $C^*(LD)$ is part of a larger known derived mixed Hodge structure on these dg operads: see for exampe [12] for a definition in terms of the Grothendieck-Teichmüller group, or [55] for a log algebro-geometric definition.) No explicit splitting that has previously been constructed was known to be rational in any lattice, though rational splittings have been known to exist by certain standard lifting theorems. Our rationality implies compatibility of this formality quasiisomorphism with the theory of derived vertex algebras with rational coefficients, and (via results of Vallette and Drummond-Cole) with rational structure on the genus 0 Deligne-Mumford-Knutsen operad. In the upcoming paper [53], the author shows that this formality isomorphism has an interesting and new deformation in the presence of monoidal structure determined by an associator.

1.1 Relation to previous work

A formality splitting for the operad of little disks was first proven to exist by Dmitry Tamarkin using path integral ideas of Kontsevich in [51], then using Drinfeld associators in [50]. Tamarkin's proof was extended to the operad of framed little disks independently by Giansiracusa and Salvatore in [20] and by Severa in [45]. A differently flavored proof, using Grothendieck-Teichmüller action, was given in [11]. A splitting equivalent to the one constructed here was sketched out in an unpublished short letter of Beilinson to Kontsevich, [3], and it would not be wrong to say that the present paper is a formalization of an idea of Beilinson.

1.2 Idea of proof and structure of paper

For convenience we construct the formality quasiisomorphism in a dual context, i.e., for the cooperad $C^*(FLD, \mathbb{C})$ of Čech[1] cochains instead of chains. Formality occurs most naturally in the log de Rham context,

$$H^*(FLD, \mathbb{C}) \to C^*_{dR}(FLD^{log})$$

for FLD^{log} our log algebro-geometric model for framed little disks. Here the formality follows from the following two observations, each of which follow from the remarkable *proper acyclicity* property of the log spaces FLD^{log}_n:

1. For each n, the log space of operations FLD^{log}_n satisfies

$$H^p\Omega^q(FLD_n) \cong \begin{cases} H^q(FLD_n, \mathbb{C}), & p = 0 \\ 0, & p \geq 1. \end{cases}.$$

2. The de Rham differentials $d : H^0\Omega^q \to H^0\Omega^{q+1}$ are zero.

In other words, the Hodge to de Rham spectral sequence degenerates at the $E1$ term and only has one row. This immediately implies formality of the cooperad $C^*_{dR}(FLD^{log})$. We then compare $C^*_{dR}(FLD^{log})$ with $C^*_{\text{Čech}}(FLD, \mathbb{C})$ (de Rham cohomology makes sense since FLD is an operad of smooth manifolds with boundary) by a topological argument: the so-called "Kato-Nakayama" topological realization of FLD^{log} is equivalent to FLD via a standard chain of quasiisomorphisms.

1.3 Acknowledgments

Much of the log geometry in this paper was learned in discussions with Clemens Koppensteiner and Mattia Talpo. Some of the weight filtration techniques in Section 3.3 were explained to the author by Joana Cirici. The idea for giving the explicit definition of the log composition morphisms in section 2.2 was suggested an anonymous reviewer of the paper [55], to whom the author is indebted. A number of discussions were instrumental for the gestation of this paper, and the author would like to in particular thank Pavel Etingof, Akhil Mathew, Agustí Roig and Dmitry Tamarkin.

2 Logarithmic geometry and the operad of framed little disks

2.1 The category of logarithmic schemes

See [39] for an in-depth development of Kato's logarithmic geometry, and [47] for an informal introduction; we give a minimalistic picture of the

[1] We use Čech cochains for the standard topology to work with cohomology on complex level. This gives better functoriality properties and can be compared more clearly to the de Rham theory.

relevant theory here. We work over a characteristic-zero field k. Given a scheme X, a log structure on X is an étale sheaf of monoids $\mathcal{M}/X$ with a certain multiplicative relationship to the sheaf of functions $\mathbb{O}_X$. Schemes with log structure form a category LogSch with behavior analogous to the category of schemes. In particular, if X is an algebraic variety then one can turn it into a log scheme by taking "trivial log structure" on X (which we will abusively still denote $X \in$ LogSch); this realizes the category Sch of schemes as a full subcategory of the category of log schemes. In the other direction, we can define a functor taking a log scheme $\mathcal{X}$ corresponding to a log structure on the scheme X to the underlying space,

$$\mathcal{X} \mapsto \mathring{\mathcal{X}} := X.$$

The forgetful functor $\mathcal{X} \mapsto \mathring{\mathcal{X}}$: LogSch $\to$ Sch is left adjoint to the canonical embedding of Sch in LogSch, with unit of the adjunction given by natural transformation

$$\pi_{\mathcal{X}} : \mathcal{X} \to \mathring{\mathcal{X}}$$

for $\mathcal{X} \in$ LogSch, which we call projection to the underlying scheme.

Remark 1. *All log schemes we work with will be fine and saturated. Moreover, they will be of* normal-crossings type*.*

Remark 2. *There is an additional enrichment of the category of log schemes called the category of* idealized log schemes*, see e.g. [39] I.1.3, consisting of log schemes with a choice of sheaf of ideal monoids $I \subset \mathcal{M}$ (closed under multiplication by $\mathcal{M}$). We will treat every log scheme $\mathcal{X}$ we work with as an idealized log scheme with a* tautological *idealized structure,*

$$\mathcal{X}_{id} = (\mathcal{X}, \mathcal{M}, I_{taut}).$$

Namely to a log scheme $\mathcal{X}$ with sheaf of log monoids $\mathcal{M}$ over an underlying scheme X, we associate tautologically the maximal ideal

$$I_{taut} := \mathcal{M} \setminus \mathbb{O}_X^\times.$$

We say that a log scheme $\mathcal{X}$ is idealized smooth *if $\mathcal{X}_{id}$ is smooth (in the sense of [39]) as an idealized log scheme.*

We now give a suite of definitions and results which will provide sufficient background on log schemes of normal-crossings type to explicitly define and work with our operad FLD^{log}. Proofs of all of these statements can be found in [39].

1. Given a smooth scheme X together with a normal-crossings divisor $D \subset X$, there is a log scheme $(X, D)_{log}$ with underlying space X. This log scheme is idealized smooth (indeed, it is also smooth in a non-idealized sense).
2. To a log algebraic variety $\mathcal{X}$ with underlying scheme X (resp., a map of log schemes $\mathcal{X} \to \mathcal{Y}$), we can associate a bundle

$$\Omega_{\mathcal{X}}$$

over X of log differentials (resp., $\Omega_{\mathcal{X}/\mathcal{Y}}$ of relative log differentials). We define

$$\Omega^k_{\mathcal{X}} := \Lambda^k_{/\mathbb{O}_X}\Omega,$$

in particular $\Omega^0_{\mathcal{X}} = \mathbb{O}_X$. For $\mathcal{X} = X$ (trivial log structure), $\Omega_{\mathcal{X}} = \Omega_X$. For a log scheme of type $\mathcal{X} = (X, D)_{log}$, we have $\Omega_{\mathcal{X}} = \Omega_X(D, \log)$, the bundle of rational differentials generated locally by $\operatorname{dlog} f$ for f functions with no zeroes or poles outside D.

3. We define the log tangent bundle $T_{\mathcal{X}} := \Omega^\vee_{\mathcal{X}}$ to be dual to the cotangent bundle.

4. There is a differential $d : \Omega^k_{\mathcal{X}} \to \Omega^{k+1}_{\mathcal{X}}$ which is a map of sheaves of vector spaces over X (similarly to the non-logarithmic context). The de Rham cohomology

$$H^*_{dR}(\mathcal{X})$$

of a log scheme $\mathcal{X}$ is defined as the hypercohomology of the complex of sheaves $(\Omega^*_{\mathcal{X}}, d)$ on X.

5. To a scheme X and a line bundle L on X one associates a log scheme $(X, L)_{log}$ with underlying scheme X, with monoid $\mathcal{M}_{(X,L)}$ given by all homogeneous functions on the total space of L. For $X = \text{pt}, L = \mathbb{O}_{pt}$ the trivial line bundle, we define the "log point"

$$\text{pt}_{log} := (\text{pt}, \mathbb{O})_{log}.$$

Schemes of the form $(X, L)_{log}$ (for X smooth) are smooth *in an idealized sense* (see remark **??**). Moreover the projection to the underlying scheme $(X, L)_{log} \to X$ is a log smooth map in an idealized sense.

6. $\Omega(X, L)_{log}$ is the sheaf whose sections over $U \subset X$ are $\mathbb{G}_m$-equivariant differentials of the $\mathbb{G}_m$-torsor $\mathbb{G}L$ on U given by removing the zero section of L.

7. There is a useful intuition for the log scheme $(X, L)_{log}$, which has a geometric correlate in terms of Kato-Nakayama spaces (introduced in the next section). Namely, $(X, L)_{log}$ can be thought of as a "zeroth order logarithmic neighborhood" of the zero section in the total space of L. Indeed, functions on $(X, L)_{log}$ are simply functions on X, while one-forms on $(X, L)_{log}$ are locally generated by restrictions to the zero-section $X \subset L$ of regular one-forms on L (i.e., one-forms on X) and "residue terms" of singular one-forms on L with first order singularity along the zero section.

8. Log schemes have a notion of base change which is compatible with base change of underlying schemes, and base change with respect to an idealized smooth map preserves idealized smoothness and induces a pullback diagram of sheaves of (pulled back) log tangent bundles in a standard sense.

9. In particular, given a smooth scheme X with a collection of line bundles $L_1, L_2, \ldots, L_d$, write $\underline{L} := (L_1, L_2, \ldots, L_d)$ for the tuple of bundles (this is given by the same data as a $\mathbb{G}_m^d$-principal bundle

on X). We define $(X, \underline{L})_{log}$ for the iterated fiber product of the $(X, L_i)_{log}$ over X. This is an idealized smooth log scheme, and its projection to the underlying space X is idealized smooth with fibers pt^d_{log}.

10. More generally given a smooth scheme X with a tuple $\underline{L} = (L_1, \dots, L_d)$ of line bundles as above and a normal-crossings divisor $D \subset X$, write

$$(X, D, \underline{L}) := (X, \underline{L})_{log} \times_X (X, D)_{log}.$$

This is the base change of the smooth log scheme $(X, D)_{log}$ under an idealized smooth map, hence is idealized smooth.

11. Schemes étale locally of the type $(X, D, \underline{L})_{log}$ as above are called log schemes of normal-crossings type. Schemes *Zariski* locally of the type $(X, D, \underline{L})_{log}$ with $D \subset X$ a strict normal-crossings divisor are called of strict normal crossings type. All log schemes we deal with will be of strict normal-crossings type.

12. To an idealized smooth log scheme $\mathcal{X}$, one associates two numbers: the log dimension, defined as the rank of the log tangent bundle $\Omega_{\mathcal{X}}$ and the geometric dimension, defined as the dimension of the underlying scheme X. There is also a log fiber dimension, defined as the difference of the two. The scheme $(X, D, \underline{L})_{log}$ has geometric dimension n and log dimension $n + d$, where $n = \dim(X)$ and d is the number of line bundles in the tuple $\underline{L}$.

13. Given two log schemes $\mathcal{X}, \mathcal{X}'$ both with underlying scheme X, we say that a *map of log structures* is a map $\mathcal{X} \to \mathcal{X}'$ over the identity map of underlying schemes.

14. Given a map of schemes $f : X \to Y$ and a log structure $\mathcal{Y}$ on Y, there is a "pullback log structure" $f^*(\mathcal{Y})$ with underlying scheme X and canonical map $f^*(\mathcal{Y}) \to \mathcal{Y}$. Any map $\mathcal{X} \to \mathcal{Y}$ with map of underlying schemes f factors uniquely through this map as $\mathcal{X} \to f^*\mathcal{Y} \to \mathcal{Y}$, with the map $\mathcal{X} \to f^*\mathcal{Y}$ a map of log structures.

15. Assume $\mathcal{X} = (X, D, \underline{L})$ and $\mathcal{X}' = (X, D', \underline{L}')$ are two normal-crossings log structure with the same underlying scheme X. Then a map of log structures

$$\mathcal{X} \to \mathcal{X}'$$

is classified by the following data.

$$\mathrm{Maps}_X(\mathcal{X}, \mathcal{X}') = \begin{cases} \emptyset, & D' \not\subseteq D \\ \{\alpha : \mathbb{A}\underline{L} \to \mathbb{A}\underline{L}' \mid \alpha \text{ homogeneous}\}, & D' \subseteq D. \end{cases}$$

Here $\mathbb{A}\underline{L}$ is the total space of the vector bundle $L_1 \oplus \cdots \oplus L_d$ and a map between two such bundles is homogeneous if elements of pure degree $(m_1, m_2, \dots, m_d)$ map to elements of pure degree $(m'_1, m'_2 \dots, m'_{d'})$, equivalently if it is torus-equivariant with respect to a map of tori $\mathbb{G}_m^d \to \mathbb{G}_m^{d'}$. For example the set of maps of log structures between pt^d_{log} and $\mathrm{pt}^{d'}_{log}$ is in bijection with $d \times d'$ matrices with positive integer coefficients (here matrices with integer coefficients classify torus maps $\mathbb{G}_m^d \to \mathbb{G}_m^{d'}$ and the positivity condition ensures that they extend to the partial compactification $\mathbb{A}^d \to \mathbb{A}^{d'}$).

16. Suppose X is a scheme with D a strict normal-crossings divisor and $\underline{L} = (L_1, \ldots, L_d)$ a tuple of line bundles. Suppose that $Y := \overline{D}^k_\alpha$ is a closed normal-crossings stratification component of dimension k and codimension $c = n - k$. Let

$$\iota : Y \to X$$

be the closed embedding of this stratum. Then then there are c distinct codimension-one closed strata $\overline{D}_1^{n-1}, \ldots, \overline{D}_c^{n-1}$ which contain Y, and we have a canonical identity relating normal bundles

$$N_X Y \cong \bigoplus_{i=1}^{c} N_i,$$

for

$$N_i := \iota_Y^* N_X \overline{D}_i^{n-1}.$$

Let D_Y be the union of all normal-crossings strata contained in Y of higher codimension. Then we have the following canonical isomorphism:

$$\iota_Y^* \big((X, D, \underline{L})_{log}\big) \cong \big(Y, D_Y, (L_1, L_2, \ldots, L_d, N_1, N_2, \ldots, N_d)\big).$$

Note that this immediately permits us to classify maps between strict normal-crossings schemes, so long as they lie over the embedding of a normal-crossing stratum. This will be the case for the maps defining our operad structure on FLD^{log}.

2.2 The operad of log framed little disks

We can now define the operad of log framed little disks in terms of the minimalistic sketch of log geometry of the previous section. Note that our definition here is equivalent to the less explicit moduli-theoretic definition of the (reduced) operad FLD^{log} in [55]. Recall that an operad O in a category $\mathcal{C}$ with symmetric monoidal structure $\times$ is a collection of "space of operations" objects $O_n \in \mathcal{C}$ together with composition maps $\mathrm{comp}_i : O_n \times O_m \to O_{m+n-1}$ for $1 \leq i \leq n$, and symmetric group actions $\Sigma_n \curvearrowright O_n$, satisfying certain compatibilities. In the presence of a covariant functor of points $S : \mathcal{C} \to \mathrm{Sets}$, the sets $S(O_n)$ represent $n \to 1$ operations in an algebra structure, and the composition map $\mathrm{comp}_i : (o_m, o_n) \mapsto o_n \,\mathrm{comp}_i\, o_m$ represents the operation obtained by plugging in the output of o_m as the ith input of o_n (keeping all inputs of o_m and the $n-1$ remaining inputs of o_n free), and the Σ_n action produces new operations from old ones by permuting the inputs.

Remark 3 (A note about identity operations)**.** *In this text we will prove formality for non-unital operads, as the formality property for a unital operad follows from formality for the corresponding non-unital operad: see for example [43]. In particular our model FLD^{log} will be non-unital (though it can be made unital, see [55]). Nevertheless after taking chains, all our quasiisomorphisms are compatible with appropriate unital structure, and we leave it as an exercise for the interested reader to check this compatibility.*

In particular, in order to define a log operad FLD^{log} in the category of log schemes, we need to define log spaces FLD_n of n-ary operations, the composition maps $\mathrm{comp}_i : FLD_m \times FLD_n \to FLD_{m+n-1}$ and permutation actions $\Sigma_n \curvearrowright FLD_n^{log}$.

We begin by reminding the reader about a closely related operad in the category of (ordinary) algebraic varieties, which we call the Deligne-Mumford-Knudson (DMK) operad. Namely, define for $n \geq 2$,

$$DMK_n := \overline{\mathcal{M}}_{0,n+1},$$

the moduli space of stable genus zero nodal curves with $n+1$ marked points labeled $x_0, x_1, \ldots, x_n$. We think of the point x_0 as representing the output, and all $x_{\geq 1}$ as inputs. There is an obvious action of the symmetric group Σ_n permuting inputs and we define composition morphisms

$$\mathrm{comp}_i : \overline{\mathcal{M}}_{0,m+1} \times \overline{\mathcal{M}}_{0,n+1}$$

as maps of moduli representing the geometric gluing construction

$$(X, x_0, \ldots, x_{m+1}) \, \mathrm{comp}_i (X', x_0', \ldots, x_{n+1}) :\cong$$
$$\left(\frac{X \sqcup X'}{(x_0' \sim x_i)}, x_0, x_1 \ldots, x_{i-1}, x_1', \ldots, x_m', x_{i+1}, \ldots, x_n \right)$$

(here as stable genus zero curves have no automorphisms, the isomorphism assignment uniquely defines the map).

Checking these conditions on $DMK_{\geq 2}$ satisfy the operad axioms is equivalent to checking that (up to isomorphism), the composite glueing $\frac{X \sqcup Y \sqcup Z}{x \sim y, y' \sim z}$ along disjoint pairs of points is independent of the order of spaces glued, a tautology. We formally extend the operad structure on $DMK_{\geq 2}$ above to a unital operad by defining

$$DMK_1 := \mathrm{pt}\,.$$

It is helpful to think of the point in DMK_1 as classifying a "fully collapsed" genus 0 curve with a single "node" point and no 1-dimensional components, and with marked input and output (which happen to coincide in this case).

Now we define FLD^{log} to be a log operad on top of the operad DMK. Namely, let $D_n \subset DMK_n$ be the normal-crossings divisor classifying all strictly nodal curves in $\overline{\mathcal{M}}_{0,n+1}$. Let

$$D_{m,m'} := \mathrm{comp}_1(DMK_m \times DMK_m') \subset DMK_n$$

be the image of the first composition map from $DMK_m \times DMK_{m'}$ for $m + m' = n + 1$. Let $L_k = T_k^*$ (for $k = 0, \ldots, n$) be the line bundle over DMK classifying the cotangent line to the marked point x_k. We have the following crucial lemma.

Lemma 1. *1.* $\mathrm{comp}_1 : DMK_m \times DMK_{m'} \to DMK_n$ *is an embedding of a closed normal-crossings component.*

2. $D = \cup \sigma(D_{m,m'})$ *for* $\sigma \in \Sigma_n$ *ranging over* $(m', m-1)$ *shuffles.*

3. $\mathrm{comp}_i^* N_{DMK_n}(D)$ *is canonically isomorphic to the tensor product line bundle* $L_i^* \boxtimes L_0^*$ *on* $DMK_m \times DMK_{m'}$.

Proof. Parts 1 and 2 are equivalent to the well-known fact that the normal-crossings boundary $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ consists of a union of moduli spaces classifying genus-zero curves that have a node that splits the set of marked points via a fixed bipartite partition. See for example Section 2.5.1 of [10] for 3. □

Now we are ready to define the operad FLD^{log}.

Definition 1. *Define*

$$FLD_n^{log} := (DMK_n, D_n, \underline{L})$$

for $\underline{L} = (L_0, L_1, \ldots, L_n)$ *the tuple of cotangent bundles at all the marked points. Define* Σ_n*-action by permuting the marked input points* $x_1, \ldots, x_n$ *and the corresponding line bundles* $L_1, \ldots, L_n$*. Define gluing maps* comp_i^{FLD} : $FLD_m^{log} \times FLD_n^{log} \to FLD_{m+n-1}^{log}$ *as the composition*

$$FLD_m^{log} \times FLD_n^{log} \xrightarrow{\alpha_{m,n}} (\mathrm{comp}_i^{DMK})^* DMK_{m+n-1} \xrightarrow{\iota_{m,n}} DMK_{m+n-1},$$

where $\iota_{m,n}$ *above is the universal map from the pullback log structure, and* α *is a map between log structures to be defined below. Observe that we have via the canonical identification (see* **??***)*

$$(\mathrm{comp}_i^{DMK})^* DMK_{m+n-1} \cong$$
$$\Big(DMK_m \times DMK_n, D, \big(\mathrm{comp}_i^*(L_1), \ldots, \mathrm{comp}_i^*(L_{m+n-1}), N_{DMK_{m+n-1}} \mathrm{comp}_i\big)_{log} \Big).$$

Now observe that each $(\mathrm{comp}_i^{DMK})^*(L_k)$ is a bundle on $\overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1}$ which for each pair of curves (X, X') classifies the tangent line at a marked point of the glued curve $\frac{X \times X'}{x_i' \sim x_0}$, equivalently the tangent line at either some x_j or x_j', depending on k. The explicit matching doesn't matter very much, but explicitly we have

$$\mathrm{comp}_i^*(L_k) \cong \begin{cases} L_k \boxtimes \mathbb{O}_{\overline{\mathcal{M}}0,m+1}, & k < i \\ \mathbb{O}_{\overline{\mathcal{M}}n+1,0} \boxtimes L_{k-i+1}, & i \leq k \leq i+m-1 \\ L_{k-m+1}, & i \geq i+m. \end{cases}$$

Observe also that the pulled back normal line bundle to the "i-composition divisor" $N_{DMK_{m+n-1}} \mathrm{comp}_i$ on $\overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1}$ is $T_i \boxtimes T_0 \cong L_i \boxtimes L_0$, the tensor product of the tangent bundles at the two glued points. Therefore if $j \neq i$ indexes a marked points of the curve X and $k \neq 0$ a marked point of X', we can define the (homogeneous) map α from each $L_i \boxtimes \mathbb{O}$ to $\mathrm{comp}_i^*(\underline{L}_{0,m+n-1})$ by sending the corresponding summand isomorphically to the corresponding tangent line of the glued curve. It remains to define a map on the total space of $L_i \boxtimes \mathbb{O} \oplus \mathbb{O} \boxtimes L_0$ to the pulled back normal bundle $N_{DMK_{m+n-1}} \mathrm{comp}_i \cong L_i \boxtimes L_0$. We define this to be the quadratic multiplication map from the two-dimensional affine bundle $\mathbb{A} L_i \boxtimes \mathbb{O} \oplus \mathbb{O} \boxtimes L_0$ to the tensor product of the two coordinates $L_i \boxtimes L_0$.

Note that we assumed that $n \geq 2$ in the above. We define

$$FLD_1^{log} := \mathrm{pt}_{log}.$$

(The log point, equivalently $(\mathrm{pt}, k)_{log}$.) We define the $1 \to 1$ operad compositions structure (equivalently, monoid structure) on FLD_1^{log} as that induced from multiplicative monoid structure on k (this is the standard monoidal structure on pt_{log}). To define action maps $\mathrm{comp}_i : FLD_n \times \mathrm{pt}_{\mathrm{log}} \to FLD_n$ and $\mathrm{comp}_0 : \mathrm{pt}_{\mathrm{log}} \times FLD_n \to FLD_n$ we need to specity an equivariant action of $(\mathbb{A}^1, \cdot)$ on the affine space $L_0 \times \cdots \times L_n$; we do this by having $\mathbb{A}^1$ act linearly on the corresponding factor L_i (with $i \in \{1, 2, \ldots, n\}$ for right action and $i = 0$ for left action of FLD_n).

To check that this defines an operad structure, one must verify several standard relations, chief among them the associativity relation on a pair of operations of type comp_i. This is a somewhat laborious definition check; alternatively, it can be deduced by observing that our explicit construction coincides with the moduli-theoretic operad composition operations in [55].

2.3 Forgetful maps and FLD_1^{log}

If X is a stable curve with marked points indexed by a finite set Γ and $\Gamma' \subset \Gamma$ is a subset of order ≥ 3, the curve $X_{\Gamma'}$ is defined as the curve obtained from X by forgetting all points not in Γ' and contracting all unstable components. This induces "forgetful" maps $\mathtt{forg}_{\Gamma,\Gamma'} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n'}$ for $n \geq n' \geq 3$. This map is compatible with normal-crossings structure, and it is straightforward to lift this map to a canonical map $FLD_{n-1}^{log} \to FLD_{n'-1}^{log}$ for $n \geq n' \geq 3$. Though we will not use this until section 4, it will be convenient for us to extend this to $n \geq 2$, i.e., define maps

$$\mathtt{forg}_{[n],\{i\}} : FLD_n^{log} \to FLD_1^{log} (= \mathrm{pt}_{log})$$

(which we will also call θ_i^{log}) corresponding to "forgetting all inputs except the ith input". A convenient way to do so is as follows: let $j : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,2(n+1)}$ be the map that takes a curve and glues a curve with three labels to each point. Then (by comparing normal bundles) we see that FLD_n^{log} for $n \geq 2$ canonically fits into the following pullback diagram, with $\mathcal{M}_{0,n+1}^{log}$ the space $(\overline{\mathcal{M}}_{0,n+1}, D_{0,n+1})_{log}$:

$$\begin{array}{ccc} FLD_n^{log} & \xrightarrow{j} & \mathcal{M}_{0,2(n+1)}^{log} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \overline{\mathcal{M}}_{0,n+1} & \xrightarrow{j} & \overline{\mathcal{M}}_{0,2(n+1)}. \end{array}$$

Here in the moduli problem of $\overline{\mathcal{M}}_{0,2(n+1)}$ one should label the marked points $x_0, \ldots, x_n, y_0, \ldots, y_n$ and the image of $\overline{\mathcal{M}}_{0,n+1}$ under j is the set of curves where each pair x_i, y_i are on their own nodal component with only one node.

Now the forgetful maps $\mathtt{forg} : FLD_n^{log} \to FLD_{n'}^{log}$ are the induced maps on pullback log schemes for the following map of triples of forgetful maps

$$\begin{tikzcd}
 & \mathcal{M}^{log}_{0,2(n+1)} \arrow[d, "\pi"] \arrow[dr] & \\
\overline{\mathcal{M}}_{0,n+1} \arrow[r, "j"] \arrow[dr] & \overline{\mathcal{M}}_{0,2(n+1)} \arrow[dr] & \mathcal{M}^{log}_{0,2(n'+1)} \arrow[d, "\pi"] \\
 & \overline{\mathcal{M}}_{0,n'+1} \arrow[r, "j"] & \overline{\mathcal{M}}_{0,2(n'+1)}.
\end{tikzcd}$$

We note that the space $FLD^{log}_1 \cong \mathrm{pt}_{log}$ fits into the diagram

$$\begin{tikzcd}
FLDlog_1 \arrow[r, "j"] \arrow[d, "\pi"] & \mathcal{M}^{log}_{0,4} \arrow[d, "\pi"] \\
\mathrm{pt} \arrow[r, "j"] & \overline{\mathcal{M}}_{0,4},
\end{tikzcd}$$

Here we label the points of $\mathcal{M}_{0,4}$ by x_i, y_i for $0 \leq i \leq 1$, and the point $j(\mathrm{pt})$ is the unique nodal curve with x_0, y_0 on one nodal component and x_1, y_1 on the other. We can now define forgetful maps

$$\theta^{log}_i : FLD^{log}_n \to FLD^{log}_1$$

as the induced maps on pullbacks for the diagram

$$\begin{tikzcd}
 & \mathcal{M}^{log}_{0,2(n+1)} \arrow[d, "\pi"] \arrow[dr] & \\
\overline{\mathcal{M}}_{0,n+1} \arrow[r, "j"] \arrow[dr] & \overline{\mathcal{M}}_{0,2(n+1)} \arrow[dr] & \mathcal{M}^{log}_{0,4} \arrow[d, "\pi"] \\
 & \mathrm{pt} \arrow[r, "j"] & \overline{\mathcal{M}}_{0,4},
\end{tikzcd}$$

where the forgetful map $\overline{\mathcal{M}}_{0,2(n+1)} \to \overline{\mathcal{M}}_{0,4}$ forgets all indices x_k, y_k for k not in $\{0, i\}$. Note that this gives additional meaning to the notion that one should consider $\overline{\mathcal{M}}_{0,2}$ to be a single point, classifying the "single node" curve (with dual graph the "open edge" graph).

Remark 4. *In fact, the structure of FLD^{log}_n together with all forgetful maps is equivalent to an extension of the operad FLD^{log} to an operad $FLD^{log,+}$ with $0 \to 1$ operations classified by* pt; *"forgetting" an index is then the operation of "gluing" the point of $FLD^{log,+}_0$ at that index. We will not use this fact here.*

3 De Rham cohomology and Hodge theory

We review here the topological and analytic aspects of logarithmic geometry over $\mathbb{C}$, focusing on de Rham cohomology and Hodge-de Rham comparison. We take in this section $k = \mathbb{C}$ and study the category LogSch_k of $\mathbb{C}$-log schemes.

3.1 Kato-Nakayama spaces, the de Rham comparison and acyclicity

There is a category AnLog of complex analytic spaces with log structure, and an *analytification functor*

$$\mathcal{X} \mapsto \mathcal{X}^{an}$$

from log schemes over $\mathbb{C}$ to log analytic schemes, extending the functor $X \mapsto X^{an}$ from schemes with trivial log structure to analytic schemes with trivial log structure. Log schemes also have a wonderfully behaved *topological* realization (discovered by Kato and Nakayama), but unlike the case of ordinary schemes, this is not equivalent to the analytic realizqation, i.e., the "underlying topological space" of a log scheme is not provided by the $\mathbb{C}$-points with analytic topology: rather, this has to be modified to take into account the logarithmic structure.

The "correct" topological realization functor

$$\mathcal{X} \mapsto \mathcal{X}^{top} : \mathrm{LogSch} \to \mathrm{Top}$$

is defined as

$$\mathcal{X}^{top} := \mathcal{X}(\mathrm{pt}_{KN})$$

with pt_{KN} the "Kato-Nakayama point", a certain nontrivial log structure on $\mathrm{Spec}(\mathbb{C})$. (Note that this structure is very different from the "log point" structure, pt_{log}, which is defined over any base field.) The Kato-Nakayama point carries a certain analytic structure which gives $\mathcal{X}^{top}$ the structure of, first, a topological space and second, a locally ringed space. When $\mathcal{X} = X$ has trivial log structure, we have naturally

$$\mathcal{X}^{top} \cong \mathcal{X}^{an}$$

as locally ringed topological spaces, but for general log schemes $\mathcal{X}^{an}$ might not be analytic, and indeed might have odd (real) dimension. For a scheme of type $(X, D)_{log}$, its analytification is the real blow-up

$$\mathrm{Bl}_D(X(\mathbb{C}))$$

of the divisor $D(\mathbb{C}) \subset X(\mathbb{C})$. This is a real $2n$-dimensional manifold with corners (for $n = \dim(X)$) whose interior is canonically an n-dimensional complex analytic manifold, namely $(X \setminus D)(\mathbb{C})$. Note that in this case the map of topological spaces $U^{top} \to \mathcal{X}^{top}$ is a homotopy equivalence, for $U = X \setminus D$ the locus of trivial log structure on $\mathcal{X}$. The topological realization of the log point is the unit circle,

$$(\mathrm{pt}_{log})^{top} \cong S^1.$$

There is a theory of GAGA (Géométrie Algébrique-Géométrie Analytique, after Serre [44]) comparisons of invariants for a log scheme $\mathcal{X}$ which is richer than for non-logarithmic schemes, as it involves comparisons of invariants for not two but three objects $\mathcal{X}$, $\mathcal{X}^{an}$ and $\mathcal{X}^{top}$ with their accompanying local ringed and logarithmic structures. The relevant comparison in our case is the pair of isomorphisms

$$H^*_{dR}(\mathcal{X}) \cong H^*_{dR}(\mathcal{X}^{an})$$

and

$$H^*_{dR}(\mathcal{X}^{an}) \cong H^*_{\text{Čech}}(\mathcal{X}^{top}, \mathbb{C})$$

which hold for any idealized smooth log scheme $\mathcal{X}$ over $\mathbb{C}$. We will in fact use a refinement of these homology isomorphisms to chain level, which is easier to state for schemes with a nice acyclicity property.

Definition 2. *We say that a log scheme $\mathcal{X}$ is* acyclic *if the sheaves $\Omega^k(\mathcal{X})$ are acyclic for all k, i.e., $H^{\geq 1}(\Omega^k(\mathcal{X}), X) = 0$.*

For example any log scheme with affine underlying scheme is acyclic (since coherent sheaves have no higher cohomology), meaning that any log scheme has an acyclic Zariski cover. The complex of de Rham chains is easiest to write down for $\mathcal{X}$ acyclic, where we simply write

$$C^*_{dR}(\mathcal{X}) := \left(H^0\Omega^*(\mathcal{X}), d\right).$$

This definition then extends to a DG functor on non-acyclic sheaves by gluing (it is canonical in an ∞-categorical sense, though can be made directly functorial by choosing covers in a universal way). We will only need the acyclic statement here.

Theorem 2. *Let $\mathcal{X}$ be an acyclic idealized smooth log scheme with underlying space X. Then there is a sequence of symmetric monoidal functors from with symmetric monoidal quasiisomorphisms between the functor of Čech cochains $C^*(\mathcal{X}^{top})$ and $C^*_{dR}(\mathcal{X})$, viewed as functors from acyclic log spaces.*

Proof. This essentially follows from results in [39]. See the Appendix **??** for more details. Note that the acyclicity condition is not needed if the functor C^*_{dR} is defined in an appropriate hypercohomology sense. □

3.2 Hodge to de Rham and proper acyclicity

Like in the classical case of smooth schemes, the hypercohomology interpretation of H^*_{dR} implies a Hodge to de Rham spectral sequence with $E1$ term

$$H^p(X, \Omega^q_{\mathcal{X}/X}) \implies H^*_{dR}(\mathcal{X}).$$

Also analogously to the classical case, this spectral sequence degenerates when $\mathcal{X}$ is smooth and proper [27] (see also [23])[2] This spectral sequence is associated to a chain-level *Hodge* filtration on $C^*(\mathcal{X})$. In the previous section we defined a log scheme to be acyclic if $H^{\geq 1}\Omega^k_{\mathcal{X}} = 0$ for all k. The main new input into our formality splitting is the acyclicity property. Note that an n-dimensional smooth and proper variety with trivial log structure has a dualizing class in $H^n\Omega^n_X$, and hence cannot be acyclic unless $n = 0$. However a nontrivial log variety can be both acyclic and

[2]In fact this degeneration also holds when $\mathcal{X}$ is idealized smooth with underlying scheme X proper. As the author could not find a reference for this more general fact in the literature, we will deduce a statement of this type by converting an idealized smooth scheme into a log smooth scheme with the same Hodge structure (something that is always possible using toric geometry).

proper (we say that a log scheme is proper if its underlying scheme is proper). A key example is the smooth and proper log scheme

$$\mathcal{X} = (\mathbb{P}^1, D)_{log}$$

for D a divisor with $d \geq 1$ points. Here acyclicity is elementary: indeed, the sheaf of functions Ω^0 does not depend on log structure giving $\Omega^0_{\mathcal{X}/X} = \mathbb{O}_{\mathbb{P}^1}$, which is acyclic, and $\Omega^1_{\mathcal{X}/X} = \Omega^1_{\mathbb{P}^1}(D) \cong \mathbb{O}(d-2)$ is also acyclic. Note that log smoothness is not required for proper acyclicity: for example, the log point pt_{log} is trivially proper acyclic.

Let

$$\mathrm{PALog}$$

be the category of proper acyclic log schemes (a full subcategory of LogSch). Hodge to de Rham degeneration implies the following theorem (see also **??**.)

Theorem 3. *Over a field k of characteristic 0, there is a canonical formality natural quasiisomorphism $\Omega : H^* \to C^*_{dR}$ between the two functors PALog $\to$ dgVect (for dgVect the category of complexes). Moreover, Ω is compatible with (lax) symmetric monoidal structure.*

Proof. By our definition, C^*_{dR} is defined for acyclic sheaves as the complex $(H^0\Omega^k_{\mathcal{X}}, d_{dR})$. Hodge-to-de Rham degeneration for proper schemes implies that $d_{dR} = 0$ for each $\mathcal{X} \in$ PALog. Thus the functor C^*_{dR} as a (lax) symmetric monoidal functor from PALog to the category of complexes factors through the subcategory of complexes with zero differential (and is in fact strict symmetric monoidal). □

Functoriality and symmetric monoidicity of this natural transformation implies that any "algebraic co-structure" on de Rham cochains associated to an algebraic structure on proper acyclic spaces is itself canonically proper. In particular we have the following corollary.

Corollary 4. *Suppose O is an operad in PALog. Then the co-operad $C^*_{\check{C}ech}(O^{KN}, \mathbb{C})$ is related by a canonical sequence of natural quasiisomorphisms of cooperads to $H^*(O^{KN}, \mathbb{C})$.*

Proof. We combine Theorem 3 with Theorem 2. □

3.3 Proper acyclicity of FLD^{log}

In order to prove formality of framed little disks it remains to prove the following theorem.

Theorem 5. *The spaces FLD^{log}_n are proper acyclic.*

Proof. As the underlying scheme of FLD^{log}_n is $\overline{\mathcal{M}}_{0,n+1}$ (or pt for $n = 1$), properness is automatic. It remains to demonstrate acyclicity of Ω^k_{log}. We give a proof using Hodge theory, using interactions between the Hodge and weight filtrations (we note that more explicit proofs of this computation in the cohomology of coherent sheaves are possible. See also section 3 of [18] for a related computation). We say that a (smooth but not necessarily proper) algebraic variety X (with trivial log structure) is 2-*pure* if the associated graded $Gr_W^k(H^*(X,\mathbb{C}))$ is purely in cohomological degree $\frac{k}{2}$ (and in particular, is trivial for k odd). The quintessential example is $X = \mathbb{G}_m$, which has $H^1(\mathbb{G}_m) = \mathbb{C}(1)$ the Tate Hodge module, pure of weight 2. In fact all 2-pure varieties can be seen to be of Tate type. (Note that a smooth variety X is "1-pure", i.e., one can drop the factor $\frac{1}{2}$ in the comparison between the homological and weight gradings, if and only if X is proper.)

We begin by proving that a (non-idealized) proper smooth log variety that compactifies a 2-pure variety is proper acyclic.

Remark 5. *In fact, it is possible to extend this statement to idealized smooth log varieties, though we do not need the full generality here. Namely, there is a way to associate motives more generally to log varieties (explained for example in upcoming work of Vologodsky et al, [56]), and it is possible to show that an idealized log variety is proper acyclic if and only it is proper and its motive is 2-pure.*

Lemma 6. *Suppose that U is a 2-pure smooth algebraic variety over $\mathbb{C}$ and X is a normal-crossings compactification of U with $D = X \setminus U$. Then*

$$\mathcal{X} := (X, D)_{log}$$

is acyclic.

Proof. There is a weight spectral sequence for $H^*(\mathcal{X})$ built out of the cohomology groups of X and the closed strata of D, with $E1$ term as follows:

$$\begin{array}{cccc}
\ddots & \vdots & \vdots & \vdots \\
\dots & H^0(D^2) \xrightarrow{d_1} & H^2(D^1) \xrightarrow{d_1} & H^4(X) \\
\dots & 0 & H^1(D^1) \xrightarrow{d_1} & H^3(X) \\
\dots & 0 & H^0(D^1) \xrightarrow{d_1} & H^2(X) \\
\dots & 0 & 0 & H^1(X) \\
\dots & 0 & 0 & H^0(X),
\end{array}$$

for $D^k := \sqcup \widetilde{D}_i^{n-k}$ is the disjoint union of normalizations of irreducible components of closed codimension-k strata of D, and with differential d_1 a Gisin differential. The weight filtration splits on the $E1$ term, with $E_1^{p,q} = H^{p-2q}(D^p)$ in weight p; so 2-purity is equivalent to vanishing of all terms except on the $E\infty$ page except $E_\infty^{2p,p}$ (note that the spectral sequence degenerates at the E_2 page, hence the vanishing is also true on E_2). The weight filtration is compatible with the Hodge filtration on columns, with $F^p_{Hodge}(D^q)$ concentrated in $H^{\geq p}$. Thus 2-purity implies that $F^{\geq 1}_{Hodge} = 0$, i.e., the Hodge filtration is concentrated in degree 0. Now the associated graded $Gr^p_{Hodge}(U)$ is computed by the complex $(H^p(\Omega^*_U), d_{dR})$, which can be computed on in terms of log forms in a normal-crossings compactification. Hodge-to-de Rham degneration then guarantees that $H^{\geq 1}\Omega^*(X, D)_{log} = 0$. $\square$

The relevant advantage of the weight filtration over the Hodge filtration in our case is its good behavior in families. In particular the weight filtration behaves "flatly" in families, and we have the following lemma.

Lemma 7. *Suppose $\pi : E \to B$ is a smooth family of schemes over a smooth base B (over $\mathbb{C}$). Suppose that B is 2-pure and a fiber $F \subset E$ over some point of B is 2-pure. Then E is 2-pure.*

Proof. This follows from appropriate compatibility of weight filtrations with the Serre spectral sequence. $\square$

Corollary 8. *The spaces $\mathcal{M}_{0,n}$ are 2-pure.*

Proof. Indeed, $\mathcal{M}_{0,3} = \mathrm{pt}$ is pure acyclic and there is a smooth fibration $\mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n}$ with fibers isomorphic to n times punctured $\mathbb{P}^1$ (which is 2-pure as $H^1(\mathbb{P}^1 \setminus D) \cong \mathbb{C}^{d-1}(1)$, for D a collection of $d \geq 3$ distinct points), whose H^1 motive is isomorphic to $\mathbb{C}(1)^{d-1}$, of weight 2. $\square$

This implies that the moduli space of marked genus zero log curves $(\overline{\mathcal{M}}_{0,n}, D)_{log}$ is proper acyclic. To deduce a result for $FLD_n^{log} = (\overline{\mathcal{M}}_{0,n+1}, D, \underline{L})_{log}$ we need to deal with the additional line bundles $L_1, \ldots, L_n$. Define the smooth log variety

$$\mathbb{P}(\overline{\mathcal{M}}_{0,n+1}, \underline{L})$$

to be the (total space of the) $(\mathbb{P}^1)^{n+1}$-bundle $\mathbb{P}(L_1) \times \cdots \times \mathbb{P}(L_n)$ which compactifies the total space of $L_0 \oplus \cdots \oplus L_n$. We introduce a log structure

$$\mathbb{P}(\overline{\mathcal{M}}_{0,n+1}, \underline{L})_{log} = \big(\mathbb{P}(\overline{\mathcal{M}}_{0,n+1}, \underline{L})_{log}, \mathbb{P}D\big),$$

where $\mathbb{P}D$ is the union of the preimage of $D \subset \overline{\mathcal{M}}_{0,n+1}$ and the union of the 0-section and the ∞-section of each $\mathbb{P}^1$-bundle. We have

$$\iota : (\overline{\mathcal{M}}_{0,n+1}, D, \underline{L})_{log} \hookrightarrow \mathbb{P}(\overline{\mathcal{M}}_{0,n+1}, \underline{L})_{log}$$

embedded as the induced log structure on the simultaneous zero section of all $\mathbb{P}^1$-bundles. Now note that the embedding $\mathrm{pt}_{log} \hookrightarrow (\mathbb{P}^1, 0 \sqcup 1)_{log}$ embedded as the log structure at 0 induces an isomorphism on global sections Ω^* (with both spaces proper acyclic). Taking a tensor power, we deduce the same for the embedding $\mathrm{pt}^{n+1}_{log} \subset (\mathbb{P}^1, 0 \sqcup \infty)^{n+1}_{log}$, hence arguing fiberwise we see that the map ι above induces an isomorphism on each $H^p\Omega^q$, and in particular preserves the acyclicity property. Thus it remains to prove that $\mathbb{P}(\overline{\mathcal{M}}_{0,n+1}, \underline{L})_{log}$ is pure acyclic. Since the open scheme $\mathbb{P}(\overline{\mathcal{M}}_{0,n+1}, \underline{L}) \setminus \mathbb{P}D$ is a $\mathbb{G}_1^{n+1}$-bundle over $\mathcal{M}_{g,n}$ we see that it is 2-pure by applying Lemma 7 to 8, hence done with Theorem 5. $\square$

4 From framed little disks to little disks

The formality problem was first posed and first proven for the E_2 operad, equivalent to the operad LD of ordinary (i.e., not framed) little disks. The operads LD and FLD are closely related: indeed, FLD is a semidirect product of LD and S^1. One aspect of this relationship is a fiber square of operads of operads (defined below), as follows.

$$\begin{array}{ccc} LD & \longrightarrow & FLD \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \theta} \\ \mathrm{Comm} & \xrightarrow{i} & \mathrm{Comm}^{S^1}. \end{array} \tag{1}$$

Here Comm *commutativity operad* (the terminal object in the category of operads), uniquely defined by $\mathrm{Comm}_n = \mathrm{pt}$ for all n. It is so named because the category of algebras over this operad (viewed as a unital operad) is the category of commutative monoids. For any (not necessariy commutative or unital) topological monoid G, there is a "G-equivariant commutativity" operad Comm^G, with category of algebras equal to the category of G-equivariant monoids. This operad is defined by $\mathrm{Comm}^G_n := G^n$, with composition

$$(g_1, \ldots, g_n) \circ_i (g'_1, \ldots, g'_k) := (g_1, \ldots, g_{i-1}, g_i g'_1, g_i g'_2, \ldots, g_i g'_k, g_{i+1}, \ldots, g_n).$$

The map $i : \mathrm{Comm} \to \mathrm{Comm}^{S^1}$ is the map on operads induced by the unique map of groups $1 : \{e\} \to S^1$. Explicitly, it is defined by $i_n(*) = (1, 1, \ldots, 1) \in \mathrm{Comm}^{S^1}_n$.

Recall that FLD_n classifies the data of a collection of n nonintersectiong maps $\iota_1, \ldots, \iota_n$ from closed disks $\mathbb{D}^2$ to a single $\mathbb{D}^2$ which are complex homotheties (i.e., compositions of a translation, scaling and rotation). Given such a configuration, each map ι_n rotates the disk by an angle θ^{log}_n. Let $\theta : FLD_n \to \mathrm{Comm}^{S^1}_n$ be the map recording the angles, $(\iota_1, \ldots, \iota_n) \mapsto (\theta^{log}_1, \ldots, \theta^{log}_n) \in (S^1)^n$. This is a map of operads. This defines the three lower right objects and maps of the square 1. Now the operad $LD \subset FLD$ has spaces of operations $LD_n \subset FLD_n$ given by tuples of nonoverlapping *real* homotheties $\iota_n : \mathbb{D}^2 \to \mathbb{D}^2$, equivalently, elements of FLD with "framing" angles $\theta_1, \ldots, \theta_n = 0$. For each n we thus have $LD_n = \theta^{-1}(1, 1, \ldots, 1) \subset FLD_n$, verifying the commutativity and the pullback property of the diagram 1.

Note that each map $FLD_n \to (S^1)^n$ is a Hurewicz (therefore also a Serre) fibration, and so (since fibrancy for operads is inherited from spaces, see [6]), the diagram 1 is a homotopy basechange diagram. We would like to get Hodge splitting properties for the diagram LD by re-interpreting this diagram in a logarithmic context, though there are some new complications here as we shall see. First we observe that the right column $FLD \to \mathrm{Comm}^{S^1}$ has a logarithmic analog. Namely, recall (section 2.3) that for $i = 1, \ldots, n$, we have maps $\theta^{log}_i : FLD_n \to FLD_1$. The space $FLD_1 = \mathrm{pt}_{log}$ has the structure of a (non-unital) monoid in the category of log schemes, hence induces an "equivariance" operad and the maps

$$\theta^{log} := (\theta_1, \ldots, \theta_n) : FLD^{log}_n \to \mathrm{pt}^n_{log}$$

combine to a map of operads $FLD^{log} \to \mathrm{Comm}^{\mathrm{pt}_{log}}$. A direct comparison (see Appendix) shows that after K-N realization, this map of operads is related (by a pair of quasiisomorphisms of maps of operads) to the map of topological operads $\theta : FLD^{log} \to \mathrm{Comm}^{S^1}$. We would like to draw a diagram of log operads analogous to the lower right three entries of 1, and define an operad "FLD^{log}" as the pullback, but we run into a problem. Namely, pt_{log} is non-unital and indeed, there is no map $\mathrm{pt} \to \mathrm{pt}_{log}$, hence the bottom row of the diagram cannot be interpreted in the log category. In the paper [55], this problem is resolved by moving to a more flexible motivic category; however in this paper we use a more concrete solution, involving the de Rham complex of sheaves in the Kato-Nakayama realization.

We begin by replacing this diagram with a quasiisomorphic one: namely, let $\mathbb{R}(1)$ be the group isomorphic to $\mathbb{R}$, but understood as the group of purely imaginary complex numbers. The map $\exp : \mathbb{R} \to S^1$ induces a map $\mathrm{Comm}^{\mathbb{R}} \to \mathrm{Comm}^{S^1}$. Define the operad $\widetilde{LD}$ to be the pullback of the diagram

$$\begin{tikzcd}
\widetilde{LD} \arrow[r] \arrow[d] & FLD \arrow[d, "\theta"] \\
\mathrm{Comm}^{\mathbb{R}} \arrow[r, "\iota"] & \mathrm{Comm}^{S^1}.
\end{tikzcd}$$

(Geometrically, a point of $\widetilde{LD}_n$ classifies a point of FLD together with a homotopy class of paths in S^1 from each of the angles θ_i to $1 \in S^1$.) The map $\mathrm{pt} \to \mathbb{R}$ induces the homotopy equivalence of operads $LD \to \widetilde{LD}$.

Now on the level of Čech cochains, the last diagram can be compared to a diagram of logarithmic origin. Namely, recall that for $\mathcal{X}$ a logarithmic variety, $\mathcal{X}^{top}$ is a locally ringed space with ring of "log holomorphic" functions $\mathbb{O}^{top}$. The sheaf $\mathbb{O}^{top} =: \Omega^0_{top}$ is part of a complex

$$(\Omega^*_{top}(\mathcal{X}), d) := \Omega^0_{top} \to \Omega^1_{top} \to \cdots \to \Omega^{n+d}_{top}$$

(here $n+d$ is the "log dimension"), which (assuming $\mathcal{X}$ is idealized smooth) resolves the sheaf $\mathbb{C}^{top}$ of locally constant functions on $\mathcal{X}^{top}$. For $\mathcal{X} = \mathrm{pt}_{log}$, let us choose a basepoint $\overrightarrow{1} \in \mathrm{pt}^{top}_{log}$ (the point of the exceptional fiber of the real blowup of $\mathbb{C}$ in the direction of $1 \in \mathbb{C}$), and let $\widetilde{\mathrm{pt}}^{top}_{log}$ be the universal cover with respect to this basepoint, canonically isomorphic to $\mathbb{R}(1)$. Then the complex $\Omega^0(\mathrm{pt}^{top}_{log}) \to \Omega^1(\mathrm{pt}^{top}_{log})$ has a lift to $\widetilde{\mathrm{pt}}^{top}_{log}$, which we denote

$$\Omega^0(\widetilde{\mathrm{pt}}^{top}_{log}) \to \Omega^1(\widetilde{\mathrm{pt}}^{top}_{log}).$$

Since this complex resolves $\mathbb{C}_{\mathbb{R}(1)}$, its global sections are one-dimensional, spanned by the unit global section $1 \in \Omega^0(\widetilde{\mathrm{pt}}^{top}_{log})$, and there is no higher cohomology.

Remark 6. *In fact, the complex of sheaves*

$$\Omega^0(\widetilde{\mathrm{pt}}^{top}_{log}) \to \Omega^1(\widetilde{\mathrm{pt}}^{top}_{log})$$

is quite simple. Ω^0 is the constant sheaf with fiber the polynomial algebra $\mathbb{C}[\log]$ for $\log$ a variable (corresponding to the log function on the universal cover of $\mathbb{C}^*$) and $\Omega^1 = d\log \cdot \mathbb{C}[\log]$.

Let

$$\widetilde{LD}^{log,top}$$

be the covering of the Kato-Nakayama operad $FLD^{log,top}$ which fits into the pullback diagram

$$\begin{tikzcd}
\widetilde{LD}^{log,top} \arrow[r] \arrow[d] & FLD^{log,top} \arrow[d, "\theta^{log,top}"] \\
\mathrm{Comm}^{\widetilde{\mathrm{pt}}^{top}_{log}} \arrow[r, "\iota"] & \mathrm{Comm}^{\mathrm{pt}^{top}_{log}}.
\end{tikzcd}$$

Define the complex of sheaves $\Omega^*(\widetilde{LD}_n^{log,top})$ to be the pullback of the complex of Kato-Nakayama log differential forms $\Omega^*(FLD_n^{log,top})$ on the topological space $FLD_n^{log,top}$ to its cover. Note that this complex once again resolves the constant sheaf $\mathbb{C}$ on the topological space $\widetilde{LD}_n^{log,top}$.

Remark 7. *The operad $\widetilde{LD}^{log,top}$ with the complex of logarithmic forms $\Omega^*(\widetilde{LD}^{log,top})$ is a natural "analytic geometry" home for the mixed Hodge structure on the operad of chains C_*LD given by Tamarkin's construction [49] (see also [12]): in particular, both the Hodge and the weight filtration are clearly visible in this picture. Moreover, the complex of sheaves $\Omega^*(\widetilde{LD}^{log,top})$ has naturally a rational lattice (lifting the rational structure on $\Omega^*(FLD^{log})$), giving naturally a de Rham lattice $C^*_{dR}(LD)$. It can be checked that this is the same lattice as the one given in [55].*

Applying the Čech cochains functor $C^* =: C^*_{\text{Čech}}$, we obtain the following diagram of cooperads in the category of cdga's:

$$\begin{tikzcd}
C^*(\widetilde{LD}^{log,top}, \Omega^*_{log,top}) & C^*(FLD^{log,top}, \Omega^*_{log,top}) \arrow[l] \\
C^*(\mathrm{Comm}^{\widetilde{\mathrm{pt}}^{top}_{log}}, \Omega^*_{log,top}) \arrow[u] & C^*(\mathrm{Comm}^{\mathrm{pt}^{top}_{log}}, \Omega^*_{log,top}). \arrow[l] \arrow[u]
\end{tikzcd}$$

Since each $\Omega^*_{log,top}$ is a resolution of a constant sheaf, and the diagram of spaces is a fibration, we see that this is a homotopy pushforward diagram in the category of cdga's for each space of operations (therefore also a homotopy pullback diagram in the category of co-operads of connective cdga's). Now consider the following diagram of triples of complexes.

$$\begin{tikzcd}
C^*(\mathrm{Comm}^{\widetilde{\mathrm{pt}}^{top}_{log}}, \mathbb{C}) \arrow[d] & C^*(\mathrm{Comm}^{\mathrm{pt}^{top}_{log}}, \mathbb{C}) \arrow[l] \arrow[r] \arrow[d] & C^*(FLD^{log,top}, \mathbb{C}) \arrow[d] \\
C^*(\mathrm{Comm}^{\widetilde{\mathrm{pt}}^{top}_{log}}, \Omega^*_{log,top}) & C^*(\mathrm{Comm}^{\mathrm{pt}^{top}_{log}}, \Omega^*_{log,top}) \arrow[l] \arrow[r] & C^*(FLD^{log,top}, \Omega^*_{log,top}) \\
\mathbb{C} \arrow[u, "\iota"] & H^*(\mathrm{Comm}^{S^1}, \mathbb{C}) \arrow[l] \arrow[r] \arrow[u, "\alpha"] & H^*(FLD, \mathbb{C}). \arrow[u, "\beta"]
\end{tikzcd}$$

The maps between the first and second row are given by the map of complexes of sheaves $\mathbb{C} \to \Omega^*_{log,top}$ which is a quasiisomorphism (by the de Rham resolution property stated above), the maps α, β between the bottom two rows are the formality maps of Theorem **??** and ι is the embedding of constant functions. This is easily seen to be a commutative diagram with all vertical maps quasiisomorphisms. Now note that a triple of connective cdga's $R' \leftarrow R \rightarrow R''$ is "pullback-exact", i.e. satisfies the property that

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R'' & \longrightarrow & R' \otimes_R R'' \end{array}$$

is a homotopy pushforward diagram, if either $R \to R'$ or $R \to R''$ is cofibrant in the Quillen model structure. For the top two rows, the map on the left is cofibrant (since it is induced by pullback of sections for a covering space), and for the bottom diagram, the map on the right is cofibrant since the map of (ordinary) topological operads $FLD_n \to (S^1)^n$ is a fibration. Thus taking cdga pushout of each row gives a new triple of quasiisomorphisms of operads of cdga's, with the bottom pushout isomorphic to $H^*(LD)$ and the top pushout mapping quasiisomorphically to $C^*(\widetilde{LD}^{log,top})$ which is related by a pair of quasiisomorphisms of topological operads to $C^*(FLD)$.

5 Applications and extensions

Here we sketch out very briefly several applications and extensions of the results of this paper and make some conjectures. Arguments in this section are sketches rather than full proofs.

5.1 Integral splitting

Our spaces FLD_n^{log}, the composition maps $\circ_i$ and the Σ_n actions involved in the operad structure are all defined and (idealized) smooth over $\mathbb{Z}$. Let A_n^* be the complex

$$A_n^* = \left(H^0\Omega^0_{\mathbb{Z}}(FLD_n^{log}) \to H^0\Omega^1_{\mathbb{Z}}(FLD_n^{log}) \to \cdots \to H^0\Omega^{2n-1}(FLD_n^{log})\right).$$

Smoothness implies in particular that this is a complex of free $\mathbb{Z}$-modules which are submodules in the $\mathbb{Q}$-basechange; our formality result over $k = \mathbb{Q}$ then implies that the complexes A_n^* are formal. By standard functoriality and symmetric monoidicity of H^0 (for connective complexes), we obtain a cooperad of graded spaces A^*, which is an integral lattice in $H^*(FLD^{log})$. The "truncation" maps $H^0(\Omega^k) \to C^*(\Omega^k)$ then define a map of co-operads of $\mathbb{Z}$-complexes

$$A^* \to C^*_{dR,\mathbb{Z}}(FLD^{log}).$$

Dualizing, we obtain a map of *operads*

$$C_*^{dR}(FLD^{log}, \mathbb{Z}) := \left(C^*_{dR}(FLD^{log}, \mathbb{Z})\right)^\vee \to (A^*)^\vee;$$

this implies that given any algebra over the formal $\mathbb{Z}$ *operad* $(A^*)^\vee$ one obtains canonically an operad over the "integral de Rham chains" of FLD^{log}, an integral version of de Rham splitting. It is natural to ask two questions.

Question. *Is the "formality map" $A^* \to C^*_{dR,\mathbb{Z}}(FLD^{log})$ a quasiisomorphism?*

Question. *Is the co-operad A^* isomorphic (non-canonically) to the integral cohomology co-operad $H^*(FLD)$, i.e., dual to the integral BV operad?*

The results of this paper imply that both of these are true after extending coefficients to $\mathbb{Q}$, but they may well have torsion obstructions integrally. Note that, as $C^*_{dR,\mathbb{Z}}(FLD^{log})$ might not be quasiisomorphic to $C^*(FLD, \mathbb{Z})$ (though see the next question), both of these may be true without implying that the topological chains operad $C_*(FLD, \mathbb{Z})$ is formal.

5.2 Prismatic cohomology

For a smooth and proper scheme X over $\mathbb{Z}_p$, there is a *Prismatic cohomology* theory ([8], [7]) with complex of cochains $C^*_{prism}(X, A_{inf})$ with coefficients in Fontaine's period ring A_{inf} which interpolates at different points of $\mathrm{Spec}(A_{inf})$ between characteristic-p de Rham cohomology $C^*_{dR,\overline{\mathbb{F}}_p}(X)$ and étale cohomology $C^*_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{F}}_p)$ (equivalent to Betti cohomology with $\overline{\mathbb{F}}_p$-coefficients).

Question. *Does there exist a Prismatic cohomology theory $C^*_{prism}(FLD^{log}_{\mathbb{Z}_p})$ over A_{inf} which interpolates between étale and de Rham cohomology in an analogous way?*

This would follow from a sufficiently powerful *logarithmic* p-adic Hodge theory for idealized smooth log varieties. Some steps towards such a theory are taken by T. Koshikawa and K. Cesnavicius in [33] and [13]; see also [54].

5.3 Comparison with other rational de Rham theories

The canonical splittings for $C_*(LD, \mathbb{C})$ and $C_*(FLD, \mathbb{C})$ constructed here are particularly interesting compared to previously known splittings, because they are the first explicitly constructed splittings which are compatible with a rational structure on $C_*(LD, \mathbb{C})$; namely, the de Rham rational structure. This structure (or rather its dual, C^*_{dR}) was first constructed as part of a mixed Hodge structure in the paper [12], where it was observed to follow from the Grothendieck-Teichmüller action discovered in [49] (the corresponding action on the framed operad FLD follows from [45]). Another, log geometric, interpretation for this rational structure was given in [55]. We give here without proof two other places where the rational lattice $C^*_{dR}(LD, \mathbb{Q})$ should appear in a canonical way.

5.3.1 FLD, homotopy pushout and moduli of nodal curves

In the paper [14] (see also [15]), Drummond-Cole shows that the topological Deligne-Mumford-Knudson operad DMK (with $DMK_n = \overline{\mathcal{M}}_{0,n+1}(\mathbb{C})$) is homotopy equivalent to the "homotopy trivialization" of the sub-operad of $1 \to 1$ operations in FLD, i.e. to the homotopy pushout (in a standard model structure on operads) of the diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & FLD \\ \mathrm{pt}, & \end{tikzcd}$$

where in the left hand column for G a group (more generally, a monoid), we interpret G as an operad by taking

$$G_{n\to 1} := \begin{cases} G, & n = 1 \\ \emptyset, & \text{else.} \end{cases}$$

This implies an analogous pushout diagram on the level of chains. Since the operad DMK is algebro-geometric and defined over $\mathbb{Q}$ (indeed, also over $\mathbb{Z}$), there is an evident de Rham lattice $C_*^{dR}(DMK)$ (dual to rational de Rham cochains) in $C_*(DMK, \mathbb{C})$. Techniques of [14] and [38] can be extended to a log geometric context to show that $C_*^{dR}(DMK, \mathbb{Q})$ in fact fits in a canonical pushout diagram

$$\begin{tikzcd} C_*^{dR}(\mathbb{G}_m, \mathbb{Q}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_*^{dR}(FLD^{log}, \mathbb{Q}) \arrow[d] \\ \mathbb{Q} \arrow[r] & C_*^{dR}(DMK) \end{tikzcd}$$

There is an explicit combinatorial computation of pushouts of derived operads (using the Boardman-Vogt resolution), and this seems to be the first explicit construction of a quasiisomorphism between $C_*^{dR}(DMK)$ and an explicitly constructed combinatorial operad. The dual of the resulting quasiisomorphism is compatible with product structure, and it seems to be a new result to give an explicit combinatorial dg model (i.e., quasiisomorphic via explicit maps) for the cdga's $C^*(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q})$ which is directly compatible with all boundary maps.

5.3.2 Derived vertex algebras

There is another algebro-geometric model for the chain operad of unframed little disks $C_*(LD)$, via derived vertex algebras. Namely, via work of Francis-Gaitsgory, [17] (extending work of Beilinson-Drinfeld, [4]), the category of $E2$ dg algebras (equivalently, $C_*(LD)$-algebras) is equivalent to the category of locally constant translation-equivariant (derived) factorization algebras over the line (equivalently, locally constand derived vertex algebras). We make the following conjecture.

Conjecture 1. *There is a canonical equivalence of ∞-categories between locally constant derived vertex algebras (as defined above) and representations of the operad $C_*^{dR}(LD, \mathbb{Q})$.*

The formality constructed here then implies a canonical equivalence of ∞-categories between locally constant derived factorization algebras and rational dg Gerstenhaber algebras.

It would be very interesting to find an extension of such a result to conformal vertex algebras (i.e., vertex operator algebras).

6 Appendix

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