Math 113 Homework 9, due Tuesday, 4/2/19 (no late submissions)

This homework does not contribute to ordinary HW scores and contains only quiz prep.

1. Extra credit

problem 15.39 (prove that the group A_n is simple for $n \geq 5$). *Warning: this problem is challenging.* If you do this problem, only do numbers 1-12 of the prep problems (and you get an automatic 8/22 on the rest). Alternatively, you can count this as an additional extra credit problem in homework 8 (please indicate if you want to use this option instead).

2. Midterm quiz prep questions:

2.1. 15.19, a-e, j

2.2. 15.20-15.24

2.3. 15.24

2.4. 15.25

2.5. 15.26

2.6. 15.27 (hint: you may use a past homework)

2.7. 15.35-36

2.8. (a) Define the relation $\sim$ on $\mathbb{R}^*$ with $x \sim y$ if $x - y \in \mathbb{R}_+$. Is $\sim$ an equivalence relation? If it is, how many equivalence classes does it have and what are its equivalence classes?

(b) Define the relation on $\mathbb{R}^*$ with $x \sim y$ if $x/y \in \mathbb{R}_+$. Is $\sim$ an equivalence relation? If so, how many equivalence classes does it have and what are its equivalence classes?

(c) Define the equivalence relation $\sim$ on $\mathbb{R}$ with $x \sim y$ if $\exists r \in \mathbb{R}_+$ such that $xr = y$. Is $\sim$ an equivalence relation? If so, how many equivalence classes does it have and what are its equivalence classes?

2.9. What is the image of the function $\exp : \mathbb{R} \to \mathbb{R}^*$? Is it injective? Show that it is a homomorphism where $\mathbb{R}$ on the left is endowed with group structure under the operation $+$ of addition and $\mathbb{R}^*$ on the right is endowed with group structure under the operation $\cdot$ of multiplication.

2.10. Let $\mathbb{R}_+ \subset \mathbb{R}^*$ be the subset of all positive elements. Show that this is a subgroup (where both sides are groups with operation $\cdot$). How many elements are there in $[\mathbb{R}^* : \mathbb{R}_+]$? Can you describe these as left cosets (i.e. subsets of $\mathbb{R}^*$)? Can you describe the right cosets?

2.11. Let $n : \mathbb{C}^* \to \mathbb{R}^*$ be the norm function, $n(z) := |z|$. Show that n is a homomorphism, where both sides are endowed with the multiplication operation. What is its kernel? What is its image? If its kernel is K, can you write down the canonical map $\mathbb{C}^*/K \to \mathbb{R}^*$?

2.12. Show that the elements $\sigma_1 = (12)$ and $\sigma_2 = (12)(34) \in S_4$ (written in cycle notation) have order two. Show that every element of order two in S_4 is conjugate to either σ_1 or to σ_2, but not both.

2.13. Describe the cyclic subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$ generated by $(1,1)$. Describe the cyclic subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_4$ generated by $(1,0)$. Are either of the groups $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_4$ cyclic?

2.14. (a) Write the following permutation in cycle notation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 5 & 3 \end{pmatrix}$

(b) Write the following permutation in permutation form: $(4,5,2)(3,6) \in S_6$

(c) Write down the subgroup generated by the permutation in part (b) above.

(d) Write down all orbits of the two permutations above (by listing elements of each orbit, in any order).

(e) What are the signs of the two permutations in parts (a), (b)?

2.15. G is some non-abelian group of order 60. Show that this group does not have an element of order 60.

2.16. True or false:

(a) If X is any finite set then $|S(X)| = |X|!$ (here the ! sign means "factorial").

(b) For any n, the order of the alternating group A_n is 2.

(c) Let m be a residue modulo n. Then the subgroup $\langle m \rangle \subset \mathbb{Z}_n$ has order equal to the gcd of m, n.

(d) All groups of order 8 are abelian.

2.17. Let $G = S_4$ be the permutation group, and let H be the cyclic subgroup $\langle \sigma \rangle$ for $\sigma = (1,2,3)$, or equivalently, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

(a) Show that for any element α of H, the conjugation $i_\alpha(\sigma) = \sigma$. Show that in fact if $\alpha \in S_4$ is any element such that $i_\alpha(\sigma) = \sigma$, then α is in H. Hint: show that i_α would have to take the ordered triple $(1,2,3)$ (viewed as a tuple, not a permutation!) to a tuple consisting of a cyclic permutation of the same elements, and therefore would have to take the element 4 to itself.

(b) What is the index $G : H$?

(c) Show that if $\alpha, \beta \in S_4$ are permutations and αH and βH have an element in common then $\alpha H = \beta H$ as subsets of S_4.

(d) Give an example of two elements α, β such that αH and $H\beta$ have an element in common but are not equal as subsets. Write out all elements in the cosets αH and in $H\beta$.

2.18. Let $H \subset S_5$ be the set of elements which have a one-element orbit of the form $\{1\}$. Show that this is a subgroup. Show that it is not a normal subgroup.

2.19. Show that any subgroup of an abelian group is normal.

2.20. What is the subgroup of $\mathbb{Z}$ (under addition) generated by the elements 12 and 15?

2.21. Is the quotient $\mathbb{Z}_4 \times \mathbb{Z}_4/\langle(1,1)\rangle$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or to $\mathbb{Z}_4$?

2.22. (a) Prove rigorously that an isomorphism $\varphi : G_1 \to G_2$ takes elements of order two to elements of order two.

(b) If G is a group of odd order, show that $\mathbb{Z}_2 \times G$ has exactly one element of order 2.

(c) Deduce that any abelian group of order $2n$ where n is odd has exactly one element of order 2.