Math 113 Homework 6, due 3/7/2019

1. This is an important exercise, defining the group D_n.

Let $C_n \subset S_n$ be the subset consisting of the following permutations:

$$\{\rho_k = (1, 2, 3, \ldots, n)^k, k = 0, \ldots, n-1\}.$$

Here remember that $(1, 2, 3, \ldots, n)$ is the cyclic notation for the permutation

$$\begin{pmatrix} 1, & 2, & 3, & \ldots, & n-2, & n-1, & n \\ 2, & 3, & 4, & \ldots, & n-1, & n, & 1 \end{pmatrix}.$$

(a) Show that C_n is a subgroup (hint: it's cyclic!)

(b) Show that, if each $1, \ldots, n$ is relabeled by its corresponding class in $\mathbb{Z}_n$ (remember that $[n] = [0]$), then C_n consists of all functions of the type $\rho_{[k]} : \mathbb{Z}_n \to \mathbb{Z}_n$ for $[k] \in \mathbb{Z}_n$, with

$$\rho_{[k]}([x]) = [k] + [x]$$

defined to be the function that adds $[k]$ to any remainder modulo n.

(c) Let $\lambda \in S_n$ be the permutation given by $\lambda(k) = n + 1 - k$, so

$$\lambda = \begin{pmatrix} 1, & 2, & 3, & \ldots, & n-2, & n-1, & n \\ n, & n-1, & n-2, & \ldots, & 3, & 2, & 1 \end{pmatrix}$$

in permutation notation.

Show that $\lambda^2 = e$.

(d) Define $\lambda_k = \rho_{k-1} \circ \lambda$. Show that if the indices $1, \ldots, n$ are viewed as remainders modulo n then $\lambda_{[k]}([x]) = [k] - [x]$.

(e) Give formulas for the four possible compositions, $\rho_{[k]} \circ \rho_{[k']}$, $\rho_{[k]} \circ \lambda_{[k']}$, $\lambda_{[k]} \circ \rho_{[k']}$ and $\lambda_{[k]} \circ \lambda_{[k']}$ (it is easier to do this by thinking of ρ's and λ's as functions on remainders, but computing these directly in S_n is also legitimate). Observe in particulat that $\lambda_{[k]}^2 = e$ for any k.

(f) Deduce that the subset $D_n \subset S_n$ consisting of ρ_k and λ_k is a subgroup.

Note: D_4 is another way to interpret the group in example 8.10 of symmetries of the square: keep in mind that in this example, the ρ_k are the same as our ρ_k, but what we call $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, respectively, the book calls $\mu_2, \delta_2, \mu_1, \delta_1$, respectively (our notation is more consistent: to see what one of the λ's is called, just apply it to the rightmost index, which in the case of D_4 is 4).

2. Book exercises 6.17-6.19

3. Book exercise 6.32

4. Book exercise 6.44

5. Book exercises 8.1, 8.4, 8.5 (hint: to invert a permutation in permutation notation, just flip the top and bottom rows. So in the exercise

$$\tau^{-1} = \begin{pmatrix} 2 & 4 & 1 & 3 & 6 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

You can then put it in more standard notation by rearranging the columns (which doesn't change what the function does), so

$$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix}.$$

6. Book exercises 10.1, 10.6, 10.7

Quiz Prep questions 10.2-10.4, 10.20-10.24, 10.25, 10.26