Math 113 Homework 5, due 2/26/2019

1. Say $(G,*)$ is a group and $H \subseteq (G,*)$ is a subset that is closed under the operation $*$. Then whether or not H is a subgroup, we can see that $(H,*)$ (operation inherited from G) is a valid binary structure. Show that if this binary structure is a group, then H is a group in the sense described in class, namely: $e_G \in H$ and if $h \in H$ then $h^{-1_G} \in H$ (you don't have to check closure, since this is one of our assumptions on the subset H). Hint: the group property of H implies there is an identity element $a \in H$. What is $a * a$? Remember that the operation $a * a$ doesn't depend on whether a is understood as an element of H or of G.

(2.-11.) Define $U \subseteq (\mathbb{C}^*, \cdot)$ to be the "unit circle", i.e. the subset $z \in \mathbb{C}^* \mid |z| = 1$. The following problems will have to do with symmetries of the circle U given by rotations and reflections. Don't worry too much here about set-theoretic details and rigor: the idea is to do some computations and get a geometric picture of what are the "symmetry" functions $f : U \to U$ and how they compose.

2. Show that under multiplication, U is a subgroup of the multiplicative group $\mathbb{C}^*, \cdot$.

3. Define the function $e : \mathbb{R} \to U$ with $e(r) := \cos(r) + i \cdot \sin(r)$. Show that e does in fact take values in U, and check that e satisfies the homomorphism property (where $\mathbb{R}$ is viewed as a group with additive structure).

4. We say $r \equiv_{2\pi} s$ if $r - s$ is an integer multiple of 2π. Show that $\equiv_{2\pi}$ is an equivalence relation and that $r \sim r', s \sim s' \implies r + s \sim r + s'$, so the addition operation $[r] + [s] := [r + s]$ is well-defined. Once well-definedness is checked, the group properties for $\mathbb{R}$ imply that $\mathbb{R}/\equiv_{2\pi}$ with the addition operation on classes given above is a group (once you check well-definedness you automatically get that it is a group with identity $[0]$ and $[a]^{-1} = [-a]$, no need to prove this).

For future problems, write Rad $:= \mathbb{R}/\equiv_{2\pi}$ "the group of radians": elements represent angles in radian notation, so that for example $[\pi] = [-\pi] = [3\pi]$ corresponds to the angle $180°$ and $[\pi/2] = [-3\pi/2]$ is $90°$. The operation $+$ defined in problem 4 is "angle addition".

5. Let's define the function $\varphi : \mathrm{Rad} \to U$ (where $\mathrm{Rad} := \mathbb{R}/\equiv_{2\pi}$) by $\varphi([r]) := e(r)$. (The symbol φ is the greek letter Phi.) Show that φ is well-defined. Show that it is an isomorphism (hint: you may use the homomorphism property of the function e.)

6. For $\theta \in \mathrm{Rad}$ (defined in problem 4), define the function $\mathrm{rot}_\theta : U \to U$ by $\mathrm{rot}_\theta(z) := \varphi(\theta) \cdot z$. Check that indeed, $\mathrm{rot}_\theta(z) \in U$ if $z \in U$. Check that, expressing $z = x + y \cdot i$, the operation rot_θ rotates the point (x, y) by θ degrees counterclockwise around the origin, i.e. applies the matrix

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

to the vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

7. Show that the composition $\mathrm{rot}_{[\theta]} \circ \mathrm{rot}_{[\theta']} = \mathrm{rot}_{[\theta+\theta']}$, where all are interpreted as functions from U to itself. (Remember that two functions are the same if all their values are the same and don't try to be fancy or use matrices here.)

8. Define the function $s_\theta : U \to U$ by $s_\theta(u) := \frac{\varphi(\theta)}{u}$ (here "s" stands for symmetry: this is a reflection function). Check that $s_\theta(u) \in U$ if $u \in U$, so s_θ makes sense as a function from U to itself.

9. As an example, draw the four points $1, i, -1, -i$ in U (corresponding to angles $[0], [\pi/2], [\pi], [3\pi/2]$). Draw an arrow from each of these points z to $\mathrm{rot}_{\pi/2}(z)$ (no proof needed: note that $\mathrm{rot}_{\pi/2}(z)$ should once again be one of these four points).

Repeat for $\mathrm{rot}_\pi, s_{\pi/2}$, and s_π. Notice that s_θ is always a *reflection* function (no proofs needed).

10. Here is how you can compute $s_\theta \circ s_{\theta'}$ (composition for two different angles):

$$s_\theta \circ s_{\theta'}(u) = \frac{\varphi(\theta)}{\left(\frac{\varphi(\theta')}{u}\right)} = u \cdot \frac{\varphi(\theta)}{\varphi(\theta')} = u \cdot \varphi(\theta - \theta') = \mathrm{rot}_\theta(u).$$

This shows that $s_\theta \circ s_{\theta'} = \mathrm{rot}_{\theta - \theta'}$. Compute using a similar argument the compositions $\mathrm{rot}_\theta \circ s_{\theta'}$ and $s_{\theta'} \circ \mathrm{rot}_\theta$ (warning: not abelian!). Together with the calculation for $\mathrm{rot}_\theta \circ \mathrm{rot}_{\theta'}$ done above, deduce that composing different functions of the form rot_θ or s_θ once again produces functions either of the form rot_θ or s_θ. In other words, the combined set

$$Sym_U := \{\mathrm{rot}_\theta \mid \theta \in \mathrm{Rad}\} \cup \{s_\theta \mid \theta \in \mathrm{Rad}\}$$

of functions from U to itself is closed under composition. The set of functions Sym_U is the "set of (distance-preserving) symmetries of the circle" $U \subseteq \mathbb{C}$ (any symmetry of a circle that doesn't change distances between points is a rotation or a reflection). This group is also called $O(2)$, the group of orthogonal transformations of a two-dimensional vector space.

11. Show that the set of functions Sym_U with operation $\circ$ (composition) is a group. You may assume composition of functions is associative (so you do not need to prove the assosciativity axiom). Note: don't waste time showing these symmetry functions are bijective (and therefore invertible): the composition rules you've found above will let you quickly find the inverse. Make sure to check, however, that the inverse you're defining is two-sided!

12. Extra credit: An element of a group $g \in (G, \cdot)$ is "central" if $g \cdot x = x \cdot g$ for any other $x \in G$. Show that Sym_U has exactly two central elements, and they form a subgroup.