Math 113 Homework 3, due 2/12/2019

1. Book exercises 3.2, 3.3, 3.4.

Make sure you are using the 7th edition of Abstract algebra by Fraleigh – if you do the wrong problems, you won't get points!

2. Book exercises 4.1-4.7

3. (a) Show that "two-sided cancellation" holds in groups. I.e., if $a, b, \tilde{b}, c$ are elements of a group G with operation $\cdot$, and $a \cdot b \cdot c = a \cdot \tilde{b} \cdot c$ then $b = \tilde{b}$.

(b) If G is a group and $g \in G$ is a fixed element, show that the function i_g with $i_g(x) = g{\cdot}x{\cdot}g^{-1}$ is an isomorphism of G with itself.

Quiz prep questions. Your % score on the following problems will be added to your quiz score percentage on Quiz 2, up to 100%.

i. Let $U_4 := (\{1, i, -1, -i\}, \cdot)$ be the binary structure with four elements, and with operation multiplication (as we shall see in class, this is a group). Let $\equiv_4$ be the "mod 4" equivalence relation on $\mathbb{Z}$ with $a \equiv_4 b$ if and only if $4 \mid b - a$. Remember the notation $\mathbb{Z}/\equiv_4$ for the quotient set: this is the set of classes $[a]$ like $[-1]$, $[0]$ or $[5]$, where $[a]$ and $[b]$ are identified iff $a \equiv_4 b$ (so $[1] = [5]$ but $[1] \neq [3]$). Suppose $[a] \in \mathbb{Z}/\equiv_4$ (here $[a]$ is the class corresponding to some integer $a \in \mathbb{Z}$). Now define $f([a]) := i^a$ (here i is the square root of -1). Show that this determines a *well-defined* function

$$f : \mathbb{Z}/\equiv_4 \to U_4,$$

i.e. if $a \equiv_4 b$ then $f(a) = f(b)$.

ii. Let $(\mathbb{C}^*, \cdot)$ be the group of complex numbers with multiplication. Let $U \subset \mathbb{C}^*$ be the subgroup of all elements which have absolute value equal to 1.

Define an equivalence relation $\sim$ on $\mathbb{C}^*$ by $z \sim z'$ if and only if $z/z' \in U$, i.e. if the quotient has absolute value 1.

(a) Check that $\mathbb{C}^*$ is closed under multiplication (so the operation makes sense).

(b) Check that $z \sim z'$ if and only if $|z| = |z'|$.

(c) Check that $\sim$ is in fact an equivalence relation (i.e. it satisfies r, s, t).

(d) Check that multiplication on $\mathbb{C}^*/\sim$ is well-defined, i.e. if $a \sim a'$ and $b \sim b'$ then $ab \sim a'b'$.

(e) Is addition well-defined? Prove this or give a counterexample.

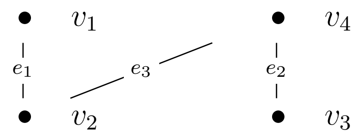
(f) Let $(\mathbb{R}_{>0}, \cdot)$ be the group of (strictly) positive real numbers under multiplication. Construct an isomorphism $\varphi : \mathbb{R}_{>0} \to \mathbb{C}^*/\sim$. Hint: try $\varphi(r) = [r]$ for $r \in \mathbb{R}_{>0}$. Now check that multiplication goes to multiplication and that the map is a bijection.

iii A (simple) *graph* Γ is a finite set of vertices $V = \{v_1, v_2, v_3, \ldots, v_n\}$ together with a set of *edges* $E = \{e_1, e_2, \ldots, e_k\}$ where each edge is an un-ordered pair $\{v_i, v_j\}$ of two distinct vertices. Here is an example of a graph:

Vertices: v_1, v_2, v_3, v_4.

Edges: $e_1 = \{v_1, v_2\}, e_2 = \{v_3, v_4\}, e_3 = \{v_2, v_4\}$:

(1)



(graphs are drawn by drawing the vertices as points and the edges as lines between pairs of points). Now define a relation $\sim_\Gamma$ on the set of points: v_i is related to v_j if one of the following conditions hold:

$$v_i \sim_\Gamma v_j \text{ if } \begin{cases} \text{The vertices are the same, i.e. } v_i = v_j \\ \text{There is an edge } \{v_i, v_j\} \in E. \end{cases}$$

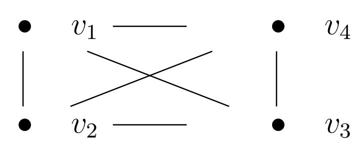
(a) Show that $\sim_\Gamma$ for Γ as above is not an equivalence relation.

(b) Show that the following two graphs do define equivalence relations:

(2)

and

(3)



(for (3) it doesn't matter how we label the edges: in this graph every pair of vertices is connected).

Show that in graph (2), there are two equivalence classes and in graph (3) there is one equivalence class.

(c) Can you explain (not necessarily rigorously) what it means for the graph Γ that it defines an equivalence relation? Draw a graph on the vertices $v_1, \ldots, v_4$ which has two equivalence classes, one of which is just $\{v_1\}$ and the other is $\{v_2, v_3, v_4\}$. (No proofs necessary for this part.)