Math 113 Homework 11, due 4/16/2019

1. Find all x with $x^2 - 4 = 0$ in $\mathbb{Z}_5 \times \mathbb{Z}_7$. Now find all x with $x^2 - 4 = 0$ in $\mathbb{Z}_{35}$ (by the Chinese remainder theorem, the two sets of solutions have the same cardinality and get sent to one another under the isomorphism that takes $a \in \mathbb{Z}_{35}$ to $(a \mod 5, a \mod 7)$).

2. Recall that the Gaussian integers $\mathbb{G}$ are complex numbers $a + bi$ with integer a, b.

(a) Check that $\mathbb{G} \subseteq \mathbb{C}$ is a subring.

(b) Let $n \in$ be a positive integer. For $z, z' \in \mathbb{G}$ we say that $z \equiv_n z'$ if $n \mid z' - z$. Recall (from problem set 4) that every class in $\mathbb{G}/\equiv_n$ can be uniquely represented by $a + bi$ for $a, b \in \{0, \ldots, n-1\}$ remainders modulo n. You have checked in earlier homeworks that addition and multiplication are well defined on $\mathbb{G}/\equiv_n$, hence it is also a ring. Check that $\mathbb{G}/\equiv_3$ is a field (by finding an inverse for every element). This is a field with 9 elements — in fact, any other field with 9 elements is isomorphic to this one!

(c) Check that $\mathbb{G}/\equiv_5$ is not a field (hint: find a zero divisor).

3. 20.4, 20.5,

4. 20.8, 20.10 (the function $\varphi(n)$ computes the number of residues mod n which are relatively prime with n, so for example $\varphi(p) = p - 1$ for p prime).

5. 20.23 (notation: "unity" means multiplicative identity element, "unit" means invertible element).

6. Extra credit: 20.27, 20.28