Math 113 Homework 10, due 4/9/2019

1. (Note: in this exercise, please don't call the additive and multiplicative identity elements 0 and 1 if there is any risk of confusion.)

(a) Let F be the set of all functions $f : \mathbb{R} \to \mathbb{R}$, equipped with the operations of function addition $(f+g)(x) = f(x)+g(x)$ and composition $(f \circ g)(x) = f(g(x))$. Show that $(F, +, \circ)$ satisfies all the axioms of a ring with unity, with just one exception – which one?

(b) Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the set formed by adjoining an element called "∞" to $\mathbb{R}$, and consider the operations $a \oplus b = \min(a, b)$ (= the lesser of a and b; with the convention that ∞ is greater than any real number), and $a \otimes b = a + b$ (with the convention that $a + \infty = \infty + a = \infty$ for all $a \in \mathbb{R}$, and $\infty + \infty = \infty$). Show that $(\bar{\mathbb{R}}, \oplus, \otimes)$ satisfies all the axioms of a field, with just one exception – which one?

(c) Show that $\{a + b\sqrt{5} \,|\, a, b \in \mathbb{Q}\}$ with the usual addition and multiplication is a field.

2. True or false? (As usual, justify your answers)

(a) The set of all pure imaginary complex numbers $\{ai \,|\, a \in \mathbb{R}\}$ with the usual addition and multiplication is a ring.

(b) If R' is a subring of a field K, then R' is also a field.

(c) If K is a field then the equation $x^2 = x$ has exactly two solutions in K.

(d) If K is a field with multiplicative identity 1 and K' is a subfield of K with multiplicative identity $1'$, then $1' = 1$. (Hint: use (c)).

(e) If R is a ring with multiplicative identity $1 \neq 0$ and R' is a subring with multiplicative identity $1' \neq 0$, then $1' = 1$. (Hint: consider a direct product.)

(f) The direct product of two fields is a field.

3. 18.20. The ring $M_2(\mathbb{Z}_2)$ is the ring of matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}_2$ are residues modulo 2. You may use that a matrix M is invertible if and only if $|M| = ad - bc$ is not zero (modulo 2).

4. 19.27 (remember: "unity" means multiplicative identity and "unit" means invertible element).