On coherent-constructible correspondences and incomplete topologies

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1 Introduction

Recall that homological mirror symmetry, as conjectured by Kontsevich [K], relates the derived category of coherent sheaves on a (nice) algebraic variety to the Fukaya category of the *mirror* symplectic manifold. Since Kontsevich's formulation, many variations on the theme of homological mirror symmetry have emerged, in which the categories compared keep track of additional structures such as equivariance, potential functionals, or boundary conditions on one or both of the manifolds in the mirror pair. The sequence of papers [NZ], [N], [FLTZ1], [FLTZ2], [FLTZ3] and [T] works out such a (modified) mirror symmetry statement for toric varieties. The correspondence is broken up into two parts. The first step is Nadler's proof in [N] (based on work with Zaslow [NZ]) that, for S a topological space, the (triangulated envelope of the) Fukaya category on the cotangent space T^*S is equivalent to the derived category of complexes of (nice) constructible topological sheaves on S. The second part is Treumann's [T] proof (building up from [FLTZ1] and [FLTZ2], with Fang, Liu and Zaslow) that, for X an n-dimensional toric variety, the derived category of perfect coherent sheaves on X embeds into the derived category of constructible sheaves on the topological torus $(S^1)^n$. These two results then compose to give an embedding of triangulated categories $D^b\,\mathrm{Perf}(X) \to D\mathrm{Fuk}((S^1)^n)$. Treumann's analysis further shows that the functor $D^b\,\mathrm{Perf}(X) \to D^b\,\mathrm{Constr}(X)$ factors through the full (triangulated) subcategory $D^b\,\mathrm{Constr}_\Lambda(X)$ of constructible complexes with a certain *singular support* condition (determined by the fan of X). On the Fukaya side, this corresponds (via the equivalence of [N]) to a certain asymptotic condition on the branes in the Fukaya category. All of this is surveyed (with another construction of the direct functor $D^b\,\mathrm{Perf}(X) \to D\mathrm{Fuk}((S^1)^n)$) in [FLTZ3]. The functor $D^b\,\mathrm{Perf}(X) \to D^b\,\mathrm{Constr}_\Lambda((S^1)^n)$ is called the *coherent-constructible correspondence* by Treumann et al, and it is conjectured e.g. in [FLTZ1] that it is an isomorphism: they prove that a related constructible model for *torus-equivariant* sheaves on X is equivalent to the category of perfect sheaves on X. The paper [SS] by Sibilla and Scherotzke proves such a statement for certain special classes of toric varieties (including all Fano varieties). In the present paper we prove the coherent-constructible correspondence for arbitrary smooth toric varieties. We further generalize results of [FLTZ1] and [FLTZ2] to categories of (lower-bounded) complexes of sheaves without a finite-dimensionality condition. This implies an equivalence of categories between the category $D^b \operatorname{Perf}(X)$ and a certain explicit Fukaya category, completing a mirror symmetry statement for such X.

In the course of this writing, Nicolò Sibilla has informed the author that in an upcoming paper with Sarah Scherotzke they have strengthened techniques in [SS] to a proof for all toric varieties (without a smoothness condition) of the fact that the nonequivariant coherent-constructible correspondence is an equivalence.

Although the original motivation for the coherent-constructible correspondence comes from mirror symmetry, the fact that it is an equivalence is also interesting from a couple of alternative points of view. Firstly, note that for the toric variety $X = \mathbb{P}^1$, the singular support condition in the target of the coherent-constructible correspondence simply picks out all complexes of sheaves on S^1 with cohomology constructible with respect to the stratification $S^1 = (0,1) \sqcup \{\infty\}$. Now note that such a sheaf $\mathcal{V}$ on S^1 is fully determined by the two cospecialization maps between fibers $\mathcal{V}_{\{\infty\}} \to \mathcal{V}_{(0,1)}$ corresponding to deforming the singular fiber to the left or to the right. From this point of view, the fact that the coherent-constructible correspondence is an equivalence implies an equivalence between the category $D^b \operatorname{Constr}_\Lambda(S^1)$ and the derived category of representations of the *Kronecker* quiver $\bullet \rightrightarrows \bullet$. This provides an alternative ("topological") explanation (and a generalization to all smooth toric varieties) of Beilinson's derived equivalence between $\operatorname{coh}(\mathbb{P}^1)$ and $\operatorname{Rep}(\bullet \rightrightarrows \bullet)$ which arguably kicked off the serious study of derived categories. The generalization (also due to Beilinson [B]) which gives quiver-like models for $\mathbb{P}^n$ can also be reconstructed from this singular-support model. Note that these results follow also from [SS].

Another point of view on this result (and the one that led the author to it) comes from the theory of p-adic representations. Namely, in [BK], Bezrukavnikov and Kazhdan construct a certain category $\overline{\operatorname{Rep}}_G$ associated to any (split, semisimple, n-dimensional) p-adic group which *compactifies* (in a category-theoretic sense). There is a functor from the category of equivariant sheaves on the Bruhat-Tits building for G to $\overline{\operatorname{Rep}}_G$ which is analogous to the Beilinson-Bernstein localization functor for category O, and it is an interesting problem to identify the image of this functor. Now the category $\overline{\operatorname{Rep}}$ has a component compactifying the category of Iwahori-biinvariant representations which is a deformation of category of Weyl-group equivariant sheaves on a toric variety X_G defined by the root data. The analysis in the present paper implies that the derived category of such representations is equivalent to the category of $\Lambda \ltimes W$-equivariant complexes of constructible sheaves on $\mathbb{R}^n$ which satisfy a certain singular support condition. Now $\mathbb{R}^n$ is naturally identified with an "apartment" in the Bruhat-Tits building $\mathbb{B}^n$, and using this identification one can deduce a localization result for the Iwahori-biinvariant part of the compactified category. This theory is related also to Theorem 16 in section 11.3.

2 Acknowledgements

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3 Statement of results

Let T be an algebraic torus, $\Lambda = X^*(T)$ its character lattice and $N = X_*(T)$ the cocharacter lattice. Write $\Lambda_{\mathbb{R}}, N_{\mathbb{R}}$ for the $\mathbb{R}$-vector spaces spanned by Λ and N. We will often use alternative notation $\mathbb{A} := \Lambda_{\mathbb{R}}$ when thinking of $\Lambda_{\mathbb{R}}$ as a Λ-equivariant real affine space. Let Σ be a toric fan in $N_{\mathbb{R}}$, and X the corresponding toric variety. Recall that every complex of constructible sheaves $\mathcal{V}$ (constructible with respect to a Whitney stratification) on a smooth topological manifold X has a coisotropic singular support (singular) submanifold $\mathrm{Supp}(\mathcal{V}) \subset T^*X$. It is natural to consider the full (DG) subcategory $\mathrm{Constr}_{(X,L)}$ of complexes of sheaves with singular support contained in some fixed subspace $L \subset T^*X$: this category is in particular stable under formation of finite limits and directed colimits. The most well-studied example of such categories occurs when L is the union of (closures of) normal varieties of strata of a Whitney stratification S of X: in this case, the DG category $\mathrm{Constr}_{(X,L)}$ is equivalent to the category of constructible complexes whose cohomology is constructible with respect to the stratification S. This category is equivalent to the derived category of complexes of constructible sheaves along S if the Whitney stratification is polyhedral, although in general categories with singular support conditions may not have a natural t-structure.

In [FLTZ1], Fang, Liu, Treumann and Zaslow relate the derived category of sheaves on an n-dimensional toric variety X to a category of sheaves on $\mathbb{A}/\Lambda$ with singular support contained in a certain polyhedral submanifold L_Σ/Λ defined in terms of the toric fan. Namely, define the following singular Lagrangian manifold [FLTZ1].

Definition 1. *Let $L(= L_\Sigma) \subset T^*(\mathbb{A}) \cong \Lambda_{\mathbb{R}} \oplus \Lambda_{\mathbb{R}}^\vee$ be the union*

$$L := \bigcup_{\sigma\in\Sigma,\lambda\in\Lambda} (\sigma^\vee + \lambda) \times \sigma.$$

Define also

$$-L := \bigcup_{\sigma\in\Sigma,\lambda\in\Lambda} (\sigma^\vee + \lambda) \times (-1)\cdot\sigma$$

for the "antipodal" variety whose fiber $(-L)_x$ over $x \in \mathbb{A}$ is the antipode of L_x.

Then [FLTZ1] proves the following result.

Theorem (Compactly-supported equivariant coherent-constructible equivalence, [FLTZ1])**.** *Let X be an arbitrary toric variety. Then there is an equivalence of categories $\kappa_{cs} : \mathrm{Perf}^T(X) \to D^b\,\mathrm{Constr}^{cs,fin}_{-L_\Sigma}(\mathbb{A})$ from perfect equivariant complexes on X to compactly-supported constructible sheaves with finite-dimensional stalks on the affine space $\mathbb{A}$ with singular support in $-L_\Sigma$.*

It is conjectured in [FLTZ1] that the *non-equivariant* category $\mathrm{Perf}(X)$ is equivalent to a singular support subcategory of derived constructible sheaves. In [T], Treumann constructs a non-equivariant version of this functor, $\overline{\kappa}$: $\mathrm{Perf}(X) \to D^b\,\mathrm{Constr}^{fin}_{(-L_\Sigma)/\Lambda}(\mathbb{A}/\Lambda)$ to the category of complexes of sheaves with finite-dimensional stalks and singular support in the quotient coisotropic variety, L/Λ, which he shows is fully faithful. He conjectures

Conjecture 1 (Non-equivariant coherent-constructible equivalence)**.** *The functor $\overline{\kappa}$ is an equivalence.*

He proves this conjecture in certain special cases. The paper [SS] proves it for a more general class of toric varieties generalizing Fano varieties. The paper [Ku] treats the case of smooth toric surfaces. In this paper, we will prove this conjecture for all smooth toric varieties. Note that (see Section 10) the category $D^b\,\mathrm{Qcoh}(X) \cong (D^b\,\mathrm{Qcoh}^T(X))^\Lambda$, where the character group $\Lambda = X^*(T)$ acts on the category $D^b\,\mathrm{Qcoh}(X)$ by functors, with λ twisting by the character $\mathbb{C}_\lambda$ of T. From this it follows that in order to prove Conjecture 1 for smooth varieties, it is sufficient to strengthen Theorem 3 in a way that treats equivariant complexes of quasicoherent sheaves on X which are not necessarily coherent, which correspond to complexes of sheaves on $\mathbb{A}$ which are not necessarily finite-dimensional or compactly-supported. Thus we reduce to the following conjecture (indeed, stronger then Conjecture 1).

Conjecture 2 (General equivariant coherent-constructible equivalence)**.** *There is an equivalence of derived categories, $\kappa : D^b\,\mathrm{Qcoh}^T(X) \to D^b\,\mathrm{Constr}_{-L}(\mathbb{A})$ between such that the action of Λ by characters on the left is intertwined with the action by shifts of $\mathbb{A}$ on the right (both viewed as genuine group actions on categories).*

It turns out that all of the above results are implied from a "cosheaf" version of Conjecture 2 above, which we will spend the bulk of this short paper proving. Define the abelian category of *cosheaves* on $\mathbb{A}$ to be the opposite category of shaves on $\mathcal{A}$ with values in $\mathrm{Vect}^{\mathrm{op}}$. We use the notation:

$$\mathcal{A}_{\mathbb{A}} := \left(\mathrm{Sh}_X^{\mathrm{Vect}^{\mathrm{op}}}\right)^{\mathrm{op}}.$$

Intuitively, the notion of cosheaf is what happens if we invert all arrows in the definition of a sheaf, and replace colimits with limits in the glueing axioms. The category of cosheaves has a good derived category, which we will denote

$$\mathcal{C}_{\mathbb{A}} := D^b\mathcal{A}_{\mathbb{A}},$$

and a good notion of (combinatorial) singular support (see Definition 2).

Now write $\mathcal{C}^{\mathcal{A}}_{(X,L)}$ for the full subcategory of complexes of cosheaves with singular support in L. Then we will prove the following result.

Theorem 1. Suppose that X is proper. *The category $\mathcal{C}^{\vee}_{(\mathbb{A},\widetilde{L})}$ of complexes of constructible* co*sheaves on the universal cover $\mathbb{A}$ of $\mathbb{A}/\Lambda$ with singular support contained in the lift $\widetilde{L}$ is equivalent to the category of complexes of T^*-equivariant quasicoherent sheaves on X. Further, this equivalence takes derived global sections over $\mathbb{A}$ to (derived) global sections over $T \subset X$, and action of Λ on the category $\mathcal{C}_{(\mathbb{A},\widetilde{L})}$ coincides with action on $D^b \operatorname{coh}(X)^T$ of the character lattice of the torus.*

From this we will deduce in section 10 our main result:

Theorem 2. *Conjectures 1 and 2 hold.*

In particular, in that section we will see how results for proper smooth varieties imply results for arbitrary smooth varieties.

4 Constructible sheaves and cosheaves

We will collect here for use in future sections some technical properties of constructible cosheaves and their singular support. The section has no deep content.

We make the following definition.

Definition 2. *We say that a vector $\chi \in T^*_x(\mathbb{A})$ is* not *in the singular support subvariety of a complex of sheaves $\mathcal{V}$ if, for any Morse function f on $\mathbb{A}$ with $f(x) = 0$ and $d_x\chi = v$, there is a small ball $B \ni x$ such that the cospecialization map*

$$R\Gamma\left(B \cap f^{-1}\left((-\infty;0)\right), \mathcal{V}\right) \to \Gamma\left(B \cap f^{-1}\left((-\infty;\epsilon)\right), \mathcal{V}\right)$$

is a quasiisomorphism for $\epsilon > 0$ sufficiently small.

When we are studying a polyhedral stratification of an affine space $\mathbb{A}$, it is sufficient to only consider linear f.

It will be more convenient for us to work with the DG category of complexes of cosheaves, but it turns out that the derived categories of sheaves and cosheaves with a support condition can be identified using the following result.

Lemma 3 (Covariant Verdier correspondence, essentially [KS] Prop. 9.4.4)**.** *The derived categories $\mathcal{C}_{(\mathbb{A},strat)}$ of cosheaves stratified with respect to the S_i and the category $\mathcal{C}^{\vee}_{(\mathbb{A},strat)}$ of sheaves stratified with respect to the S_i are equivalent, with the equivalence taking a sheaf $\mathcal{V}$ to the cosheaf $U \mapsto \Gamma_c(U, \mathcal{V})$ of compactly supported sections. Moreover, this "Verdier equivalence" takes the singular support manifold L to the fiberwise antipodal manifold $-L$ (with fiber $(-1)\cdot L_x$ over any $x \in \mathbb{A}$).*

Proof. The covariant Verdier correspondence takes a sheaf $\mathcal{V}$ to its cosheaf of derived sections with compact support (see [D], chapter 3 for a definition, and [C] shows this is an equivalence in the cellular case.) For bounded complexes of sheaves with finite-dimensional fibers, the correspondence takes a sheaf to the *stalkwise* dual cosheaf of the Verdier dual *sheaf*, given by $V_{cov} : \mathcal{V} \mapsto (D\mathcal{V})^\vee$. Now it is clear that taking a cellular constructible sheaf to its stalkwise dual cosheaf $\mathcal{V}^\vee$ does not change singular support (defined in Definition 2), as a corestriction map $R\Gamma(D, \mathcal{V}^\vee) \to R\Gamma(D', \mathcal{V}^\vee)$ on bounded open subsets is dual to the restriction map.

Now we have ([KS], Prop. 9.4.4) that Verdier duality D takes a bounded complex of sheaves with finite-dimensional fibers with singular support L to a complex of sheaves with singular support $-L$. This completes the lemma in the case when $\mathcal{V}$ is a bounded complex of sheaves with finite-dimensional stalks. We deduce the result for general cosheaves by noting that, as the question is local, we can assume that the base is compact, and any bounded complex of cosheaves on a compact cellular manifold is a colimit of complexes with finite fibers. $\square$

We will use the following consequence of the singular support definition above.

Proposition 4. *Suppose $\mathcal{V}$ is a constructible (dg) cosheaf on $\mathbb{A}$ with respect to a polyhedral stratification, with singular support $L \subset T^*\mathbb{A}$. Let Θ be some open affine cone in $\mathbb{A}$ (or, more generally, the intersection of any finite collection of half-spaces). Suppose $v \in \Lambda_{\mathbb{R}}$ is a nonzero vector such that all fibers of L over $\mathbb{A}$ are contained the perpendicular half-space $v^\perp \subset (\Lambda_{\mathbb{R}})^*$. Then the sheaf of sections $R\Gamma(\mathcal{V}, \Theta + \epsilon v)$ (for $\epsilon \in \mathbb{R}^+$ small and nonnegative) is constant.*

Proof. For sheaves, this is a standard consequence of the combinatorial definition of singular support, see e.g. [NY], Lecture 13. For cosheaves, dual arguments apply. $\square$

5 A combinatorial model for equivariant quasicoherent sheaves

We use a combinatorial model for the category of T-equivariant quasicoherent sheaves on X which is closely related to representations of the category $\langle\Theta\rangle$, constructed in [FLTZ1], and which we will call J in this paper. However, we end up having to impose a descent condition on the category of representations of $J \cong \langle\Theta\rangle$, which we will later think of in analogy with a sheaf condition in topology.

Consider the category of T-equivariant coherent sheaves on our smooth, compact toric variety X. Readers familiar with the combinatorics of fans will recall that X is covered by a collection of T-equivariant affine subspaces X_σ where σ varies over the toric fan, that $X_\sigma \cap X_\tau = X_{\sigma\cap\tau}$, and that the ring of functions $\mathbb{O}_\sigma$ on X_σ is the semigroup ring spanned by t^λ for λ varying over the integral points $\Lambda \cap \sigma^\vee$. Torus action just establishes a Λ-grading on all the rings and spaces involved, with the unsurprising rule that t^λ has weight λ (in every equivariant affine). Now suppose $\mathcal{F}$ is an equivariant coherent sheaf on X. It is determined by the collection of its restrictions $\mathcal{F}_\sigma$ which have to be $\mathbb{O}_\lambda$ modules, along with restriction maps $r_{\sigma\tau} : \mathcal{F}_\sigma \to F_\tau \mid \tau \subset \sigma$, and a torus action, which imposes compatible gradings on the $\mathcal{F}_\sigma$. Write InQcoh^T for the category of "incoherent equivariant sheaves", i.e. compatible collections of λ-graded $\mathbb{O}_\sigma$ modules, $(\mathcal{F}_\sigma, r_{\sigma\tau})$ (as $\sigma \supset \tau$ vary), without the "quasicoherence" condition that $\mathcal{F}_\sigma \otimes_{\mathbb{O}_\sigma} \mathbb{O}_\tau \cong \mathcal{F}_\tau$.

Remark 1. *One can show that the data of an object of InQcoh^T defines a T-equivariant sheaf of $\mathbb{O}$-modules over X which is* not necessarily quasicoherent *(hence the terminology), and it is straightforward to check that InQcoh^T is in fact a full subcategory of T-equivariant sheaves of $\mathbb{O}$-modules. With a little more work, one can show that this Abelian category is naturally identified with the category of modules with a certain stratified version of quasicoherence. Namely, define the category of* stratified coherent *sheaves (with respect to the stratification of X by torus orbits) to be the full subcategory of sheaves $\mathcal{F}$ of $\mathbb{O}$-modules on X in the Zariski topology with the following condition.*

- *For any pair of open sets $V \subset U$ such that $U \setminus V$ does not intersect the generic point of any torus orbit $T \cdot x$, we require that $\mathcal{F}_V \cong \mathcal{F}_U \otimes_{\mathbb{O}_U} \mathbb{O}_V$.*

Then the category InQcoh is equivalent to the category of T-equivariant stratified coherent sheaves. Moreover, Λ-equivariant representations of the poset J is equivalent (via arguments similar to Lemma 14) to the category of (not necessarily equivariant) sheaves on X with the analogous stratified coherence condition. See [T] for another model for the category of Λ-equivariant representations of J. We will not use these facts here.

If we write more precisely $\mathcal{F}_\sigma\langle\lambda\rangle$ for the λ-graded component of $\mathcal{F}_\sigma$, the data of a collection $(\mathcal{F}_\sigma) \in \mathrm{InQcoh}$ amounts to the following:

- Morphisms $t^\mu_{(\sigma)} : \mathcal{F}_\sigma\langle\lambda\rangle \to \mathcal{F}_{\sigma+\lambda}\langle\lambda+\mu\rangle$ for $\mu \in \Lambda \cap \Sigma^\vee$;
- Restriction maps $r_{\sigma\tau} : \mathcal{F}_\sigma\langle\lambda\rangle \to \mathcal{F}_\tau\langle\lambda\rangle$.

These have to satisfy the conditions that

- $r_{\sigma\tau}$ is compatible with $\Lambda \cap \sigma^\vee$-action
- $t^\mu_{(\sigma)}$ commute and
- $r_{\tau\upsilon} r_{\sigma\tau} = r_{\sigma\upsilon}$.

Note that both the generators and the relations above have no addition or multiplication. This means that the category InQcoh is in fact the category of representations in Vect of some (nonadditive) index category, with objects $\bullet_\sigma\langle\lambda\rangle$ indexed by pairs (σ, λ) and morphisms spanned by the $t^\mu_{(\sigma)}$ and $r_{\sigma\tau}$, satisfying the relations above. Call this category J. Now it can be checked combinatorially that J is equivalent to the following poset category which we call J (called $\langle\Theta\rangle$ in [FLTZ1]).

Definition 3. *Write $\Theta(\lambda,\sigma)\subset\mathbb{A}$ for the open subset $\sigma^\vee+\lambda$. We call such subsets "integral affine cones".*

Now we define J:

- Objects of J are the integral affine cones.
- J has a single map from $\Theta(\lambda,\sigma)$ to $\Theta(\lambda',\sigma')$ if and only if $\Theta(\lambda,\sigma)\subset\Theta(\lambda',\sigma')$.

The condition $\Theta(\lambda,\sigma)\subset\Theta(\lambda',\sigma')$ occurs whenever $\sigma\supset\sigma'$ and $\lambda-\lambda'\in(\sigma')^\vee$. Note that there is some ambiguity in the definition since $\Theta(\lambda,\sigma)=\Theta(\lambda',\sigma)$ when $\lambda-\lambda'\in\sigma^\perp$. But this is resolved by observing that the category with objects parametrized by pairs (λ,σ) and morphisms as above is equivalent to the poset of the distinct integral cones ordered by inclusion (one is the isomorphism-contracted version of the other).

Now taking $r_{\sigma\tau}$ to the containment $\Theta(\sigma,\lambda)\subset\Theta(\tau,\lambda)$ and $t^{\mu}_{(\sigma)}$ to the inclusion $\Theta(\sigma,\lambda)\to\Theta(\sigma,\lambda-\mu)$ (when $\mu\in\sigma^\vee$) gives us a functor of categories from the index category of the $\bullet_{\sigma,\lambda}$ with J, and this can be seen to be an equivalence of index categories. This gives us an interpretation of $\mathrm{InQcoh}^T(X)$ as representations of J.

Now the category $\mathrm{coh}(X)$ is the full subcategory of $\mathrm{Semicoh}(X)$ of collections $\mathcal{F}_\sigma$ satisfying the localization condition $\mathcal{F}_\tau=\mathcal{F}_\sigma\otimes_{\mathbb{O}_\sigma}\mathbb{O}_\tau$. Localization is a direct limit. Because the $\mathbb{O}_\sigma$ are semigroups algebras, we can interpret the operation $-\otimes_{\mathbb{O}_\sigma}\mathbb{O}_\tau$ as the direct limit over an explicit diagram.

Namely, choose any collection $\lambda_1,\lambda_2,\lambda_3,\ldots\in\tau^\vee\cap\Lambda$ such that $\lambda_i-\lambda_{i+1}\in\sigma^\vee$ (they become "more negative" with respect to σ), and such that for any point $m\in\sigma\setminus\tau$ of the complement of the two cones in the fan, the dot products $\langle m,\lambda_i\rangle$ go to $-\infty$ (i.e. the λ_i eventually become "as negative as possible" with respect to σ while staying in $\tau^\vee$.) Given such a collection, we get maps from F_σ to itself which we can index by the diagram: $t^{\lambda_1}\mathcal{F}_\sigma\to t^{\lambda_2}\mathcal{F}_\sigma\to t^{\lambda_3}\mathcal{F}_\sigma\to\ldots$. (Note that λ_i may not be in $\mathbb{O}_\sigma$, but we still make sense of this formally since $t^{(\lambda_i-\lambda_{i+1})}$ is indeed in $\mathbb{O}_\sigma$). Now

$$\mathcal{F}_\sigma\otimes_{\mathbb{O}_\sigma}\mathbb{O}_\tau=\operatorname{colim} t^{\lambda_i}\mathcal{F}_\sigma.$$

Note that the condition on the λ_i above is precisely the condition that the sets $\Theta(\sigma,\lambda_i)$ satisfy $\Theta(\sigma,\lambda_1)\subset\Theta(\sigma,\lambda_2)\subset\ldots$, and the union $\bigcup_i\Theta(\tau)$. From this we can see that in terms of the language of representations of J, the localization condition correpsonds to the following descent condition.

Theorem 5. *The category of T-equivariant quasicoherent sheaves on X is equivalent to the category of representations of J with the following condition.*

(∗) *The evident map $\operatorname{colim}_i\mathcal{F}(\Theta(\sigma,\lambda+\lambda_i))\to\mathcal{F}(\Theta(\tau,\lambda))$ must be an isomorphism for $\sigma\subset\tau$, the λ_i as above and $\lambda\in\Lambda$ arbitrary (corresponding to choice of graded component).*

$\square$

This looks suspiciously like a cosheaf condition. The crucial insight we gain from this geometric encoding of our index categories is the following:

Observation 1. *If $\mathcal{V}$ is a cosheaf on $\mathbb{A}$, then we have an "evaluation at cones" functor from the category of cosheaves on $\mathbb{A}$ to* $\mathrm{Rep}(J)$*, given by*

$$\Theta(\sigma, \lambda) \mapsto \Gamma(\mathcal{V}, \Theta(\sigma, \lambda)).$$

As we've encoded condition $(*)$ above as an infinite coglueing condition (which any cosheaf on $\mathbb{A}$ must satisfy), we see that the image of a cosheaf on $\mathbb{A}$ under this functor satisfies condition $(*)$, hence gives an equivariant coherent sheaf on X.

In the next sections we will study the derived functor of this evaluation functor, which we call Π_*, as well as its right adjoint.

6 Cosheaves, Copresheaves, and change of space functors

We will view the conditions $(*)$ as a purely combinatorial condition. But it is helpful for intuition to think of the $\Theta(\sigma, \lambda)$ as "sort of" opens of a topological space $\mathcal{J}$, which we call the "Morelli topology", which has open covers in the form of the diagram $(*)$, and to think of quasicoherent sheaves on X as corresponding to sheaves on $\mathcal{J}$. The notation we use will be suggestive of this point of view.

Starting from this section, we will mostly be working in the world of DG categories and DG functors between them. Whenever we write $\mathrm{Hom}, \lim, \mathrm{colim}$, etc. in such a category, we will always mean the DG version (e.g. Hom will take values in $D^b\,\mathrm{Vect}$). We will use the following notation.

Notation.

- *Write $\mathcal{O}pen_{\mathbb{A}}$ for the (ordinary) category of open subsets of $\mathbb{A}$.*
- *Write $\mathcal{P}^{pre}_{\mathbb{A}}$ for the triangulated category of complexes of (covariant) representations of $\mathcal{O}pen_{\mathbb{A}}$ in $D^b\,\mathrm{Vect}$. This is the derived category of the category of copresheaves on $\mathbb{A}$.*
- *Write $\mathcal{A}_{\mathbb{A}}$ for the category of cosheaves on $\mathcal{A}$ and*
- *$\mathcal{C}_{\mathbb{A}} = D^b\mathcal{A}_{\mathbb{A}}$ for its derived category.*
- *Write $\mathcal{P}^{pre}_{\mathcal{J}}$ for the category of representations of J into $D^b\,\mathrm{Vect}$.*
- *Write $\mathcal{A}_{\mathcal{J}}$ for the category of representations of J satisfying the condition $(*)$ above (in* Vect*) and*
- *$\mathcal{C}_{\mathcal{J}}$ for its derived category.*

Before proceeding, we need to make a technical point. Let $\mathcal{C}_{\mathcal{J}}^{\text{true}}$ be the full subcategory in $\mathcal{P}_{\mathcal{J}}$ of functors $\mathcal{C} \to D^b\,\mathrm{Vect}$ which satisfy condition $(*)$ for the usual notion of colimit *in the derived category.* It will be more convenient for us to think about objects of $\mathcal{C}_{\mathcal{J}}^{\text{true}}$ than complexes of objects in $\mathcal{C}_{\mathcal{J}} = D^b\mathcal{A}_{\mathcal{J}}$, but for this we need to be sure that the two categories are equivalent. There is an obvious functor $F_{\text{true}} : \mathcal{C}_{\mathcal{J}} \to \mathcal{C}_{\mathcal{J}}^{\text{true}}$.

Lemma 6 (technical)**.** *The functor F_{true} above is an equivalence of DG categories.*

This lemma follows from the following classical result from the theory of DG categories relating the DG notion of a quasicoherent sheaf with the derived category of the abelian category of quasicoherent sheaves.

Lemma 7 (classical)**.** *Let X be a smooth, finite-type algebraic variety with an affine Zariski cover $U_i \subset X$, closed under intersection. Then the bounded derived category of sheaves on X is equivalent to the category of collections $\mathcal{F}_i$ of complexes of sheaves on U_i together with restriction maps $\mathcal{F}_i \to \mathcal{F}_j$ for $\mathcal{U}_i \supset U_j$, strictly compatible with composition, under the condition that all adjoint maps $\mathcal{F}_i \otimes_{\mathbb{O}_i} \mathbb{O}_j \to \mathcal{F}_j$ are quasiisomorphisms.*

This follows from [TT], Theorem 2.4.3.

6.1 The functors $\Pi^* : \mathcal{C}_{\mathcal{J}} \to \mathcal{C}_{\mathbb{A}}$, $\Pi_* : \mathcal{C}_{\mathbb{A}} \to \mathcal{C}_{\mathcal{J}}$

Define a "pullback of opens" functor,

$$\Pi^{-1} : J \to \mathcal{O}\mathrm{pen}_{\mathbb{A}}$$

which can be thought of as a map of topologies, since it takes diagrams of the form $(*)$ above to infinite open covers in $\mathbb{A}$ (note that the notation is motivated by topology: Π^{-1} is not invertible). Write $\Pi_*^{\text{pre}} : \mathcal{P}_{\mathbb{A}} \to \mathcal{P}_J$ for the restriction functor on representation categories associated to Π^{-1}. From Observation 1, we see that when applied to a real cosheaf, the functor Π_*^{pre} satisfies our descent condition and takes values in $\mathcal{C}_{\mathcal{J}}$. This means that we have a functor $\Pi_* : \mathcal{C}_{\mathcal{X}} \to \mathcal{C}_{\mathcal{J}}$, with $\Pi_*(\mathcal{V})(\Theta(\Pi, \lambda)) = \Gamma(\mathcal{V}, \Theta(\Pi, \lambda))$, which fits into the following diagram.

$$\begin{tikzcd} \mathcal{C}_{\mathbb{A}} \arrow[r, "\Pi_*"] \arrow[d, "\phi"'] & \mathcal{C}_{\mathcal{J}} \arrow[d, "\phi"] \\ \mathcal{P}_{\mathbb{A}} \arrow[r, "\Pi_*^{\text{pre}}"] & \mathcal{P}_{\mathcal{J}}. \end{tikzcd}$$

Since the functor Π_*^{pre}, is a pullback functor of representation categories, it has both right and left adjoints. We call its right adjoint functor Π^*_{pre} (the notation having opposite handedness from what we are used to for sheaves and presheaves, since we're working with cosheaf categories). The functor Π^*_{pre}

applied to an object $\mathcal{V}$ of $\mathcal{C}_{\mathcal{J}}$ takes $U \in \mathcal{O}\mathrm{pen}(\mathbb{A})$ to the inverse limit of complexes of vector spaces $\lim_{(\sigma,\lambda)|U\subset\Theta(\sigma,\lambda)} \mathcal{V}(\Theta(\sigma,\lambda))$. The functor Π^*_{pre} does not preserve the property of being a cosheaf, so in order to get a functor to $\mathcal{C}_{\mathbb{A}}$ we need to compose with the right adjoint to the forgetful functor $\phi : \mathcal{C}_{\mathbb{A}} \to \mathcal{P}_{\mathbb{A}}$, which (with opposite handedness to the situation for sheaves) is the (derived) cosheafification functor, $\mathrm{shff}_{\mathcal{X}} : \mathcal{P}_{\mathcal{X}} \to \mathcal{C}_{\mathcal{X}}$. This gives us the definition: $\Pi^* := \mathrm{shff}_{\mathcal{X}} \circ \Pi^*_{\mathrm{pre}}$. We check that this is indeed a right adjoint

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{A}}(\mathcal{F}, \Pi^*\mathcal{G}) \cong \quad & \mathrm{Hom}_{\mathbb{A}}(\mathcal{F}, \mathrm{shff}\Pi^*_{\mathrm{pre}}\phi\mathcal{G}) \cong \mathrm{Hom}_{\mathbb{A},\mathrm{pre}}(\phi\mathcal{F}, \Pi^*_{\mathrm{pre}}\phi\mathcal{G}) && (1)\\
\cong \quad & \mathrm{Hom}_{\mathcal{J},\mathrm{pre}}(\Pi^{\mathrm{pre}}_*\phi\mathcal{F}, \phi\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{J}}(\phi\Pi_*\mathcal{F}, \phi\mathcal{G}) && (2)\\
\cong \quad & \mathrm{Hom}_{\mathcal{J}}(\Pi_*\mathcal{F}, \mathcal{G}), && (3)
\end{aligned}$$

(where in the second-to-last equality we are using the commutative diagram above, and in the last equality, we are using full faithfulness of the forgetful functor ϕ).

7 Line bundles and local behavior of Π^*, Π_*

Here we will briefly recall the classification of equivariant line bundles on X, and express them in terms of our Morelli topology model. Namely, let Σ^1 be the collection of rays in Σ. For each $\eta \in \Sigma^1$, we parametrize the dual line $\eta^* := \Lambda_{\mathbb{R}}/\eta^\perp$ by $\mathbb{R}$ in such a way that points of η^* parametrized by positive numbers pair with positive points of η, and such that the lattice $\Lambda/(\Lambda\cap\eta^\perp) \subset \eta^*$ is identified with $\mathbb{Z} \subset \mathbb{R}$. Write $\Omega_{\mathbb{R}} := \prod_{\eta\in\Sigma^1} \eta^*$ and $\Omega \subset \Omega_{\mathbb{R}}$ the lattice in $\Omega_{\mathbb{R}} \cong \mathbb{R}^{|\Sigma^1|}$ consisting of tuples of integers (according to the parametrization above). Then it's a classical result that the equivariant Picard group of T-equivariant line bundles on X is canonically Ω. For $w \in \Omega$, write $\mathbb{O}(w)$ for the corresponding bundle. We have a linear evaluation map $\omega : \mathbb{A} \to \Omega_{\mathbb{R}}$, sending $x \mapsto (x \mod \eta_1^\perp, x \mod \eta_2^\perp \ldots)$, which maps Λ into Ω. The image $\Lambda \subset \Omega$ under this evaluation map consists of those line bundles that are twists of the trivial bundle by torus characters.

Remark 2. *There is a common reformulation of these spaces and maps in terms of piecewise linear functions. Namely, $\Omega_{\mathbb{R}}$ is isomorphic to the vector space of continuous functions f on $N_{\mathbb{R}}$ which are linear on each cone. A function f is in Ω if it takes integral values on N and the embedding ω of Λ in Ω is the embedding of globally linear functions in piecewise linear ones. We choose not to use this (admittedly, elegant) language in favor of the more explicit coordinates as above.*

Now for each $w \in \Omega$ and $\eta \in \Sigma^1$, write $\Theta_w(\eta)$ for the half-space $\{x \in \mathbb{A} \mid x \mod \eta^\perp \geq w_\eta\}$, and for $\sigma \in \Sigma$ an arbitrary cone, write $\Theta_w(\sigma)$ for the affine dual cone $\bigcap_{\eta\subset\sigma} \Theta_w(\eta)$. When $w \in \Omega$, the $\Theta_w(\sigma)$ are all objects of the Morelli category J, and admit maps $\Theta_w(\sigma) \to \Theta_w(\tau)$ whenever $\tau \subset \sigma$. For $\Theta \in J$ an object, define $\mathbb{C}_\Theta \in \mathcal{C}_{\mathcal{J}}$ to be the right Yoneda functor

$$\mathbb{C}_\Theta(\Theta') = \begin{cases} \mathbb{C}, & \Theta' \supset \Theta \\ 0, & \text{else.} \end{cases}$$

For $w \in \Omega$, define $\mathbb{C}_w = \operatorname{colim}_\sigma \mathbb{C}_{\Theta_w(\sigma)} \in \mathcal{C}_{\mathcal{J}}$, where the limit is taken over cones in the fan ordered by reverse containment. Then it's easy to see that the line bundle $\mathbb{O}(w)$ corresponds to $\mathbb{C}_w$ under the correspondence of Theorem 5.

7.1 Local behavior of Π_* and Π^* on $\mathbb{A}$

Suppose $x \in \mathbb{A}$ is a point. Let δ_x be the skyscraper sheaf at x. Then we compute

$$\Pi_*(\delta_x)(\Theta(\sigma, \lambda)) = \begin{cases} \mathbb{C}, & x \in \Theta(\Sigma, \lambda) \\ 0, & \text{else} \end{cases}$$

Write $\omega(x)^- \in \Omega$ for the largest integral point $w \in \Omega$ which is strictly smaller (coordinatewise) than $\omega(x)$. Then it's clear that $\Theta(\sigma, \lambda) \ni x$ if and only if $\Theta(\sigma, \lambda) \supset \Theta_{\omega(x)^-}(\sigma)$ in terms of the notation above. This gives us the end result:

Lemma 8. *We have*

$$\Pi_*(\delta_x) \cong \mathbb{C}_{\omega(x)^-}.$$

Suppose now that $\mathcal{F}$ is an arbitrary sheaf on $\mathcal{J}$, and we want to find the costalk $\Pi^*_x(\mathcal{F})$ at $x \in \mathbb{A}^1$ of $\Pi^*(\mathcal{F})$. Since the costalk is $\operatorname{Hom}(\delta_x, \Pi^*\mathcal{F})$, adjunction gives us $\Pi^*_x \cong \operatorname{Hom}(\Pi^*\delta_x, \Pi^*\mathcal{F})$, which, from the above Lemma, implies

Lemma 9.

$$\Pi^*_x(\mathcal{F}) \cong \operatorname{Hom}_{\mathcal{C}_{\mathcal{J}}}(\mathbb{C}_{\omega(x)^-}, \mathcal{F}).$$

This result implies that our functor Π^* agrees (up to the Verdier correspondence, 3 and a twist by an equivariant line bundle) with the functor κ in [FLTZ1].

8 Faithfulness of Π^*

In this section we use a category-theoretic trick (explained to the author by Roman Bezrukavnikov) to prove the following Lemma.

Lemma 10. *The functor $\Pi^* : \mathcal{C}_{\mathcal{J}} \to \mathcal{C}_{\mathbb{A}}$ is fully faithful (as a functor of DG categories).*

In this section, we will use for the first time our hypothesis that X is proper. Recall that an object X of a derived category $\mathcal{C}$ is *compact* if for any diagram of objects $D \to \mathcal{C}$, taking Hom with X preserves colimit along D. Specifically:

$$\forall D : I \to \mathcal{C} \text{ admitting a direct limit, we have}$$
$$\operatorname{Hom}(X, \operatorname{colim}_I(D)) = \operatorname{colim}_{Y \in I}(\operatorname{Hom}(X, D(Y))).$$

Recall also that a collection of objects X_i *colimit generate* a DG category $\mathcal{C}$ if every object of $\mathcal{C}$ can be expressed as a colimit of the X_i. (Note that there are different notions of DG generation and this is the strongest). Then we have the following categorical proposition.

Proposition 11. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor of DG categories that commutes with colimits and X_i is a collection of compact objects that colimit generate $\mathcal{C}$, such that $F(X_i)$ are also compact. Then F is fully faithful if and only if $F_{ij} : \mathrm{Hom}_{\mathcal{C}}(X_i, X_j) \to \mathrm{Hom}_{\mathcal{D}}(F(X_i), F(X_j))$ is a quasiisomorphism for all pairs i, j.*

Proof. For any pair of objects $X, Y \in \mathcal{C}$, we can express $X = \lim_I X_i$ and $Y = \lim_J X_j$. Then by compactness, $\mathrm{Hom}(F(X), F(Y))$

$$\cong \mathrm{colim}_{I^{\mathrm{op}}} \mathrm{Hom}(F(X_i), F(Y)) \text{ (by colimit compatibility)} \tag{4}$$
$$\cong \mathrm{colim}_{I^{\mathrm{op}}} \mathrm{colim}_J \mathrm{Hom}(F(X_i), F(Y_j)) \text{ (by compactness of } F(X_i)\text{)} \tag{5}$$
$$\cong \mathrm{colim}_{I^{\mathrm{op}} \times J} \mathrm{colim} \mathrm{Hom}(X_i, Y_j) \text{ (by faithfulness on generators)} \tag{6}$$
$$\cong \mathrm{Hom}(X, Y) \text{ (by reverse arguments in } \mathcal{C}\text{).} \tag{7}$$

□

We apply this proposition with F the functor Π^* (which commutes with colimits because it is given by a finite limit on the level of stalks, and colimits commute with finite limits in pretriangulated DG categories) and the collection of generators consisting of the objects $\mathbb{C}_w$, corresponding to equivariant line bundles $\mathbb{O}(w)$ via the correspondence of section 5. Since our fan is complete, the cosheaves $\Pi^*(\mathbb{C}_w)$ are compactly supported with finite-dimensional fibers, hence compact as objects of the category of cosheaves. That Π^* is fully faithful on the compact objects $\mathbb{C}_w$ follows from the result of [FLTZ1]. So in order to apply the proposition and prove the lemma, it suffices to prove the following.

Proposition 12. *The objects $\mathbb{C}_w$ colimit generate the DG category $\mathcal{C}_{\mathcal{J}}$.*

This is equivalent to the statement that equivariant line bundles colimit-generate $D^b \mathrm{coh}(X)^T$. For $\sigma \in \Sigma$, let $\mathbb{O}_\sigma\langle\lambda\rangle$ (corresponding to $\mathbb{C}_{\sigma,\lambda}$ in notation above) be the twist of the pushforward of the constant coherent sheaf $\mathbb{O}_{X_\sigma}$ to X. (This quasicoherent sheaf is no longer coherent.) It is easy to see that these sheaves DG generate $\mathrm{Qcoh}(X)$. (In the Morelli topology picture, they already generate the presheaf category: this follows from the standard result that every representation of a category J into Vect is a colimit of Yoneda representations $\mathbb{C}_X : Y \mapsto \mathbb{C}^{\oplus \mathrm{Hom}(X,Y)}$.) It remains to exhibit every pushforward bundle $\mathbb{O}_{X_\sigma}$ as a limit of equivariant line bundles. It's easy to see that a line bundle $\mathbb{O}(w)$ maps to $\mathbb{O}_{X_\sigma}$ if and only if the coordinates w_η corresponding to rays $\eta \subset \sigma$ are all ≤ 0. It turns out we can simply take the colimit over *all* $\mathbb{O}(w)$ such that $w_\eta = 0$ for $\eta \subset \sigma$: this can be shown either in the realm of geometry, or combinatorially (note that such a statement would be false in the category Rep_J: we need the sheaf condition for it to be true). A sketch of the combinatorial proof is as follows. Note that $\mathbb{O}_\sigma$ is the sheaf $\mathbb{C}_0(\sigma)$ (defined above). We need to show that this is the direct limit of $\mathbb{C}_w$ where $w \in \Omega$ varies over $\Omega_{0/\sigma}$ of all tuples such that $w_i = 0$ when $\eta_i \in \sigma$. Recall that $\mathbb{C}_w$ is a limit of $\mathbb{C}_w(\sigma)$. We can commute the finite limit and the colimit past each other by first taking colimits of the $\mathbb{C}_w(\tau)$ components. Each of these is easily seen (using our descent condition) to return $\mathbb{C}_{\sigma\cap\tau}$. At the end of the day we are left with with $\lim_{\tau\in\Sigma} \mathbb{C}_{\sigma\cap\tau}$ for τ ordered by opposite inclusion. Now we can further split this limit into pieces along $\Sigma_{\sigma,\sigma'} := \{\tau \in \Sigma | \tau \cap \sigma = \sigma'\}$ for σ' varying over faces of σ. It's not hard to see that the cohomology of the diagram, $\lim_{\tau\in\Sigma_{\sigma,\sigma'}} \mathbb{C} = \mathbb{C}$, and so we end up with a limit over faces of σ ordered by opposite inclusion, which (as there is a terminal object) returns $\mathbb{C}_\sigma$, as desired. □

This completes the proof of faithfulness of Π^*. Note that, since Π^* is a right adjoint, its full faithfulness implies that $\Pi_*\Pi^* \cong \mathbb{I}\mathrm{d}$. (As we can write $\mathrm{Hom}(\Pi_*\Pi^*(X), Y) \cong \mathrm{Hom}(\Pi^*(X), \Pi^*(Y)) \cong \mathrm{Hom}(X, Y)$ by full faitfhulness, and we are finished by Yoneda).

9 The composition $\Pi^*\Pi_*$

The paper [FLTZ1] proves that the extension by zero sheaves $(j_{\Theta(\sigma,\lambda)})_!(\mathbb{C}_{\Theta(\sigma,\lambda)})$ have singular support in L_Σ. Dual arguments imply that the cosheaves $\mathbb{C}_{\sigma,\lambda}$ we defined above have singular support contained in L_Σ.

Definition 4. *Write $\mathcal{C}_{(\mathbb{A},L)}$ for the category of bounded complexes of cosheaves with singular support contained in L.*

Since (as we've shown in the previous section), the $\mathbb{C}_\Theta$ generate the image of the functor Π^*, the functor Π^* factors through a functor

$$\kappa^* : D^b\,\mathrm{Qcoh}(X)^T \to \mathcal{C}_{(\mathbb{A},L)}.$$

(Here we are identifying $\mathcal{C}_{\mathcal{J}}$ with $D^b\,\mathrm{Qcoh}(X)^T$ by Theorem 5). Write similarly

$$\kappa_* : \mathcal{C}_{(\mathbb{A},L)} \to D^b\,\mathrm{Qcoh}(X)^T$$

for the (left) adjoint, given by restricting Π_* to $\mathcal{C}_{(\mathbb{A},L)} \subset \mathcal{C}_\mathbb{A}$. It is clear that $\kappa_*\kappa^* \cong \Pi_*\Pi^* \cong \mathbb{I}\mathrm{d}_{D^b\,\mathrm{Qcoh}(X)^T}$. To show that the other composition $\kappa^*\kappa_*$ is also $\mathbb{I}\mathrm{d}_{\mathcal{C}_{(\mathbb{A},L)}}$, it remains to show that the composition $\Pi^*\Pi_*$ is equivalent to the identity functor when restricted to cosheaves with singular support contained in L.

Write $M : \mathcal{C}_\mathbb{A} \to \mathcal{C}_\mathbb{A}$ (M for monad) for the functor $M := \Pi^*\Pi_*$. Let $\alpha : M \to \mathbb{I}\mathrm{d}$ be the adjunction map, and let $E := \mathrm{Cone}(\alpha)$ be the functor $\mathcal{C}_\mathbb{A} \to \mathcal{C}_\mathbb{A}$ measuring the "error of Π^* being the left inverse to Π_*". We want to show that E is the trivial functor when restricted to $\mathcal{C}_{(\mathbb{A},L)}$. It suffices to show that its fibers $E_x = \mathrm{Cone}(M_x \to \delta_x) : \mathcal{C}_\mathbb{A} \to \mathrm{Vect}$ are trivial on $\mathcal{C}_{(\mathbb{A},L)}$. We can compute the fibers M_x using 7.1. We get that $M_x(\mathcal{V}) \cong \mathrm{colim}_{\sigma\in\Sigma}\,\Gamma(\Theta_{\omega(x)^-}(\sigma), \mathcal{V})$. Now we make the following claim.

Lemma 13. *The functor* $\mathrm{Hom}(\mathbb{C}_{\omega(x)^-}, -)$ *on the category* $\mathcal{C}_{(\mathbb{A},L)}$ *is equivalent to the functor* $\mathrm{Hom}(\mathbb{C}_{\omega(x)-\epsilon}, -)$ *for any sufficiently small, positive (coordinatewise)* $\epsilon \in \Omega$.

Proof. We use the following proposition.

Recall (Definition 1) that the fiber of L over a point $x \in \mathbb{A}$ is the union of cones $\sigma \subset \Lambda^\vee_{\mathbb{R}}$ in the fan such that $x \mod \sigma^\perp$ is integral in the lattice $\Lambda/(\Lambda \cap \sigma^\perp)$. From this and the proposition we see that, for V with singular support contained in L, the sections on the affine cone $\Gamma(\Theta(\sigma, x + \epsilon v), \mathcal{V})$ do not change for small ϵ unless there is a ray of the fan $\eta \subset \sigma$ such that the coordinate $x \mod \eta^\perp$ is integral and v pairs positively with η. The lemma follows by decomposing $\mathbb{C}_{\omega(x)^-}$ into constant sheaves on cones, $\mathbb{C}_{\Theta\omega(x)^-(\sigma)}$ and observing that while moving $\Theta_{\omega(x)^-}(\sigma)$ to $\Theta_{\omega(x)-\epsilon}(\sigma)$ linearly, the sections stay constant. ☐

Now let $\mathcal{C}_{(\mathbb{A},\mathrm{strat})}$ be the category of cosheaves stratified with respect to the polyhedral Whitney stratification given by unions of the affine planes $\sigma^\perp + \lambda$. The functor $\iota : \mathcal{C}_{(\mathbb{A},\mathrm{strat})} \to \mathcal{C}_{\mathbb{A}}$ has a right adjoint functor $\pi_{\mathrm{strat}} : \mathcal{C}_{\mathbb{A}} \to \mathcal{C}_{(\mathbb{A},\mathrm{strat})}$. It is easy to see that the functor Π_*, hence also the functor M, factors as $M = M\pi_{\mathrm{strat}}$. Hence we are reduced to proving that, for $\mathcal{V} \in \mathcal{C}_{(\mathbb{A},\mathrm{strat})}$, we have an isomorphism of the fiber $\mathcal{V}_x \cong \operatorname{colim} \Gamma(\Theta_{\omega(x)-\epsilon}(\sigma), \mathcal{V})$. Since the colimit cosheaf $\operatorname{colim}_\sigma \mathbb{C}_{\Theta_{\omega(x)-\epsilon}(\sigma)}$ is supported on a small ball B^ϵ_x around x, we can replace the homotopy colimit with $\operatorname{colim} \Gamma(\Theta_{\omega(x)-\epsilon}(\sigma) \cap B^\epsilon_x, \mathcal{V})$. But it's clear that, for ϵ sufficiently small and for $\mathcal{V} \in \mathcal{C}_{(\mathbb{A},\mathrm{strat})}$, the sections $\Gamma(\Theta_{\omega(x)-\epsilon}(\sigma) \cap B^\epsilon_x, \mathcal{V}) \cong \mathcal{V}_x$ on these small convex open sets converge to the stalk. Thus we end up with

$$E(\mathcal{V})_x \cong \mathrm{Cone}\left((\operatorname{colim}_{\sigma\in\Sigma} \mathcal{V}_x) \to \mathcal{V}_x\right).$$

Now since our fan is complete, we have $\operatorname{colim}_{\sigma\in\Sigma} \mathbb{C} \cong \mathbb{C}$ and so $E(\mathcal{V})_x \cong \mathcal{V}_x \otimes \mathrm{Cone}(\mathbb{C} \xrightarrow{1} \mathbb{C}) \cong 0$. This implies that the functors κ_*, κ^* defined above between $D^b \operatorname{Qcoh}^T(X)$ and $\mathcal{C}_{\mathbb{A},L}$ are in fact inverse equivalences of categories, completing the proof of Theorem 1.

10 Corollaries of Theorem 1

We will now use Theorem 1 to prove Theorem 2 and deduce Conjectures 1 and 2. Note that by Lemma 3, the category $\mathcal{C}_{(\mathbb{A},L)} := D^b\mathcal{A}_{\mathbb{A},L}$ of complexes of constructible cosheaves on $\mathbb{A}$ with singular support in L is equivalent to the category $D^b\mathrm{Constr}_{-L_\Sigma}$ of complexes of constructible *sheaves* with antipodal singular support. This equivalence respects properties of compact support and finite-dimensionality of stalks, and is compatible with the Λ-action. In particular, Conjectures 1 and 2 follow from the analogous statements about cosheaves, which we prove in the remainder of this section.

We begin by getting rid of the properness condition on X. For this, note that any smooth toric variety X has a smooth toric compactification $\overline{X}$. Since open pushforward is fully faithful (and this is true on the derived and equivariant levels as well), we have $j_* : D^b \operatorname{Qcoh}(X) \to D^b \operatorname{Qcoh}(\overline{X})$ a fully faithful functor of DG categories. Now let Σ be the toric fan of X and $\overline{\Sigma}$ that of $\overline{X}$. Write $L = \cup_{\sigma\in\Sigma,\lambda\in\Lambda}(\sigma^\vee + \lambda)\times\sigma \subset T^*\mathbb{A}$, and write $\overline{L} = \cup_{\sigma\in\overline{\Sigma},\lambda\in\Lambda}(\sigma^\vee + \lambda)\times\sigma \subset T^*\mathbb{A}$. Then $L \subset \overline{L}$, so the condition of singular support being contained in Σ is a stronger condition than being contained in the larger Lagrangian variety $\overline{\Sigma}$, and since both are full subcategories of $D^b\,\mathrm{Constr}(\mathbb{A})$, we see that the functor $\mathcal{C}_L \to \mathcal{C}_{\overline{L}}$ is a fully faithful functor of DG categories. Now it is clear that the diagrams of functors

$$\begin{tikzcd}
\mathcal{C}_{(\mathbb{A},L)} \arrow[r, "\kappa_*"] \arrow[d] & D^b\,\mathrm{Qcoh}(X) \arrow[d] \\
\mathcal{C}_{(\mathbb{A},\overline{L})} \arrow[r, "\overline{\kappa}_*"] & D^b\,\mathrm{Qcoh}(\overline{X}),
\end{tikzcd}
\qquad
\begin{tikzcd}
D^b\,\mathrm{Qcoh}(X) \arrow[r, "\kappa^*"] \arrow[d] & \mathcal{C}_{(\mathbb{A},L)} \arrow[d] \\
D^b\,\mathrm{Qcoh}(\overline{X}) \arrow[r, "\overline{\kappa}^*"] & \mathcal{C}_{(\mathbb{A},\overline{L})}
\end{tikzcd}$$

are commutative. Since the vertical arrows are fully faithful embeddings and the bottom arrows are mutually inverse functors, it follows that the top arrows are mutually inverse as well. This concludes the proof of Conjecture 2.

To get Conjecture 1, we use the following result. Suppose X is a variety of finite type with T-action. Then the category of equivariant quasicoherent sheaves $\mathrm{Qcoh}(X)^T$ has (genuine) action by the group $\Lambda := X^*(T)$, given by twisting by characters of the torus.

Lemma 14. *1. The category of Λ-equivariant objects in the abelian category $\left(\mathrm{Qcoh}(X)^T\right)^\Lambda$ is equivalent to $\mathrm{Qcoh}(X)$ and*

2. If X is smooth, the category of Λ-equivariant objects in the derived category $\left(D^b\,\mathrm{Qcoh}(X)^T\right)^\Lambda$ is equivalent to $D^b\,\mathrm{Qcoh}(X)$.

Proof. We have a forgetful functor $\left(\mathrm{Qcoh}(X)^T\right)^\Lambda \to \mathrm{Qcoh}(X)^\Lambda$, where the action of Λ is trivial. Now the category $\mathrm{Qcoh}(X)^\Lambda$ is the tensor-product category $\mathrm{Qcoh}(X)\otimes\mathrm{Rep}(\mathbb{C}[\Lambda]) \cong \mathrm{Qcoh}(X\times T)$. Let $\mathcal{F}$ be an object of $\mathrm{Qcoh}(X)^\Lambda \cong \mathrm{Qcoh}(X\times T)$, one checks combinatorially that of $\mathcal{F}$ to $\left(\mathrm{Qcoh}(X)^T\right)^\Lambda$ is equivalent to a T-equivariant structure on $\mathcal{F}$, where T acts on $T\times X$ diagonally. Now since T acts without fixed points on $T\times X$, a T-equivariant sheaf on $X\times T$ is equivalent to a sheaf over $\frac{X\times T}{T,\Delta}$ on the quotient by the diagonal action, which is isomorphic to X. This completes the proof on the level of Abelian categories. Applying the same arguments to the abelian category of complexes of sheaves on X and checking the resulting functor respects quasiisomorphisms, we deduce the analogous fact for derived categories. $\square$

It is clear from our construction that, under the functors Π^*, Π_*, twisting an object of $\mathrm{Qcoh}(X)^T$ by a character λ is intertwined with the "shift-by-λ" functor on $\mathcal{C}_{\mathbb{A}}$. Thus applying this to Conjecture 2, we see that

$$D^b\,\mathrm{Qcoh}(X) \cong \left(D^b\,\mathrm{Qcoh}(X)^T\right)^\Lambda \cong D^b\,\mathrm{Sh}^\Lambda_{\mathbb{A},L}\,.$$

But since the Λ action on $\mathbb{A}$ is free, this is equivalent to the category $D^b \operatorname{Sh}_{\mathbb{A}/\Lambda, L/\Lambda}$. This completes the proof of conjecture 2.

11 Concluding remarks and generalizations

Because of the formal nature of constructions in this paper, we see that they easily generalize in certain natural ways.

11.1 $\mathbb{Z}$ coefficients

In our generator and relation presentation for $\operatorname{Qcoh}(X)^T$, the relations had no addition or scaling. This means that we can define the functors Π^*, Π_* with coefficients in $\mathbb{Z}$, and the results of this paper will go through verbatim. This gives us the following result.

Theorem 15. *Conjectures 1 and 2 hold with coefficients in $\mathbb{Z}$ (hence over an arbitrary ring).*

Remark 3. *In fact, it seems that with a little bit of bookkeeping, we can make this result hold over the sphere spectrum, $\underline{\mathbb{S}}$, where we define the category $\operatorname{Qcoh}_{\underline{\mathbb{S}}}$ as collections of topological spectra $V_{\lambda,\sigma}$ satisfying the relations of Theorem 5, and the category $\mathcal{C}_{(\mathbb{A},L,\underline{\mathbb{S}})}$ is defined as the category of cosheaves of spectra on $\mathbb{A}$ whose sections satisfy the condition of Definition 1. We will not give a proof here.*

11.2 Finiteness and compact support

Suppose that X is smooth and compact. Then we observe that the equivalence constructed between $\mathcal{C}_{(\mathbb{A},L)}$ and $D^b \operatorname{Qcoh}^T(X)$ sends objects of $\mathcal{C}_{(\mathbb{A},L)}$ whose cohomology sheaves are compactly-supported and have finite-dimensional stalks to complexes of T-equivariant sheaves on X with coherent cohomology, and vice versa, and thus defines an equivalence of the corresponding categories. Similarly, the non-equivariant equivalence $\mathcal{C}^{\Lambda}_{(\mathbb{A},L)} \cong D^b \operatorname{Qcoh}(X)$ sends complexes of sheaves with finite-dimensional fibers to complexes of coherent sheaves, and vice versa, and thus gives a constructible model for the category $\operatorname{Perf}(X) \cong D^b \operatorname{coh}(X)$.

11.3 Additional symmetries of the fan

Suppose that we have an additional finite group of symmetries W acting on the lattice $\Lambda^\vee$, in a way which permutes cones of the fan Σ. Then we have an induced (genuine) action of W on the category $\operatorname{coh}(X)$. We have the following generalization of Conjecture 1.

Theorem 16. *We have an equivalence of derived categories between the category $D^b \operatorname{Qcoh}(X)^W$ of W-equivariant complexes of sheaves on X and the category*

$\mathcal{C}^{\Lambda\ltimes W}_{(\mathbb{A},L)}$ *of* $\Lambda \ltimes W$*-equivariant complexes of constructible cosheaves on* $\mathbb{A}$ *with singular support contained in* $\mathbb{A}$ *(and an analogous result holds for sheaves).*

This follows from modifying Lemma 14 to get a derived equivalence between W-equivariant complexes of sheaves on X and the category $D^b\left(\mathrm{Qcoh}(X)^T\right)^{\Lambda\ltimes W}$. From this we get the following remarkable corollary.

Corollary 17. *Let* W *be a finite group acting on a lattice* Λ *(possibly with stabilizer at every point). Write* $\widetilde{W}$ *for the semidirect product* $\Lambda \ltimes W$*. For any point* $x \in \Lambda \otimes \mathbb{R}$*, write* W_x *for the stabilizer of* x *under the action of* $\widetilde{W}$ *(for example* $W_0 \cong W$*). Then any finitely-generated representation of* $\widetilde{W}$ *has a finite resolution by direct sums of representations of the form* $\mathrm{Ind}_{W_x}^{\widetilde{W}} V$ *for* V *a finite-dimensional representation of* W_x*.*

Proof. Let T be the torus $\mathrm{Spec}(\mathbb{C}(\Lambda))$, viewed as a smooth toric variety with W-action, and choose a smooth W-equivariant compactification X of T. Let V be a finitely-generated representation of $\widetilde{W}$. The category of finitely-generated sheaves on $\widetilde{W}$ is equivalent to the category of W-equivariant coherent sheaves on T. It is possible to extend any W-equivariant coherent sheaf on T to a coherent sheaf on X in a W-equivariant way. Call $\mathcal{F}$ such an extension of the coherent sheaf corresponding to V. Then from Theorem 16 and the finite-dimensionality statement of section 11.2, we see that $\mathcal{F}$ is represented by a finite $\widetilde{W}$-equivariant complex of sheaves $\mathcal{V} := \kappa_{\mathbb{A}}^*(\mathcal{F})$ on $\mathbb{A}/\Lambda$ with finite-dimensional stalks, and constructible with respect to a polyhedral stratification (with convex polyhedral cells) of $\mathbb{A}/\Lambda$ which is compatible with $\widetilde{W}$-action. Now tracing through our arguments, the representation V (corresponding to the restriction of $\mathcal{F}$ to T, and viewed as a complex in degree 0), is given by $R\Gamma_{\mathrm{cosheaf}}(\mathbb{A}, \mathcal{V})$ (with induced $\widetilde{W}$-action). Now note that there are finitely many $\widetilde{W}$-orbits of convex k-dimensional cells $D_i^{(k)}$ in $\mathbb{A}$. Pick a point $x_i^{(k)} \in D_i^{(d)}$. Let $V_i^{(k)}$ be the stalk of $\mathcal{V}$ at $x_i^{(k)}$, viewed as a complex of representations of $W_{x_i^{(k)}}$. Then we have coboundary maps

$$d_{k,k+1} : \bigoplus_i \mathrm{Ind}_{W_{x_i^{(k)}}}^{\widetilde{W}} V_i^{(d)} \to \bigoplus_j \mathrm{Ind}_{W_{x_i^{(k+1)}}}^{\widetilde{W}} V_i^{(d+1)}$$

(induced by cospecialization of stalks). These form a bicomplex of $\widetilde{W}$-modules with global sections V, completing our proof. $\square$

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