Math 113 Quiz 4 April 2, 2019

Additional sheets available (write your name on any additional sheets!!)

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1. In $\mathbb{Z}_2 \times \mathbb{Z}_2$, let $e = (0,0), a = (0,1), b = (1,0), c = (1,1)$. Write down a table for this group (in terms of the letters a, b, c, e).

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

2. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$. Let $H = \langle (1,2) \rangle \leq G$.
   (a) Write down all elements of H and all cosets of H (with all elements of each). Label them by letters a, etc.

$H = \{(0,0), (1,2), (2,1)\}$

$e = e + H = \{(0,0), (1,2), (2,1)\}$

$a = (1,0) + H = \{(1,0), (2,2), (0,1)\}$

$b = (2,0) + H = \{(2,0), (0,2), (1,1)\}$.

   (b) Explain (in one sentence using a keyword) why addition is well-defined on G/H. Write down a table for G/H. What standard group is it isomorphic to? (Give a one-sentence explanation.)

Addition well-defined because $H \leq G$ normal (any subgroup of a commutative group is normal)

Table:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

, iso to $\mathbb{Z}_3$

   (c) For what elements $g \in G$ is the coset $g + H$ equal to the trivial coset $e + H$?

$g \in H$, so

$(0,0)$, $(1,2)$, $(2,1)$.

3. In $G = S_6$, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 3 & 4 & 5 \end{pmatrix}$$

and

$$\tau = (1,4,5)(2,3).$$

(a) Write σ in cycle notation and τ in permutation notation.

$\sigma = (1\,2)(3\,6\,5\,4)$

$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 5 & 1 & 6 \end{pmatrix}$

(b) Write down the composition $\sigma \circ \tau$ (any notation ok).

$\sigma \circ \tau$:

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 4 & 2 & 5 \end{pmatrix}$

(c) What are the signs (parities) of σ, τ, and $\sigma \circ \tau$?

$\operatorname{sgn}(\sigma) = (-1)^3 \cdot (-1)^5 = 1$

$\operatorname{sgn}(\tau) = (-1)^4 \cdot (-1)^3 = -1$

$\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}\sigma \cdot \operatorname{sgn}\tau = -1.$

(d) What is the order of σ? What is the order of τ?

$\operatorname{ord}(\sigma) = \operatorname{lcm}(2,4) = 4$

$\operatorname{ord}(\tau) = \operatorname{lcm}(3,2) = 6.$

(e) Let $H = \langle \sigma \rangle$. What is the number of cosets $(S_6 : H)$? What is the order of H?

$|H| = |\langle \sigma \rangle| = \operatorname{ord}\sigma = 4.$

By Lagrange: $(S_6 : H) = \frac{|S_6|}{4} = \frac{6!}{4} = \frac{720}{4}$

$= 180$

4. Which of the following subsets are subgroups? If not, which condition for being a subgroup do they fail?

(a) $\mathbb{Z} \subseteq \mathbb{R}$

subgroup $(+)$

(b) $\mathbb{Z}^* \subseteq \mathbb{R}^*$

operation: multiplication. $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Not subgroup : inverse

$(2^{-1} \notin \mathbb{Z}^*$ but 2 is$)$

(c) $\mathbb{R}^* \subseteq \mathbb{C}^*$

subgroup $(\cdot)$

(d) $\mathbb{N} \subseteq \mathbb{Z}$

not subgroup $(+)$: fails inverse

$(2^{-1} = -2 \notin \mathbb{N})$.

(e) $\{0, 1, \ldots, n-1\} \subseteq \mathbb{Z}$

not subgroup $(+)$: not closed

$((n-1)+(n-1) \geq n$ for $n \geq 2)$.

5. True/False (give a short argument, about one sentence):

(a) Every group of order 8 is cyclic.

F, consider

D_4 or $\mathbb{Z}_2 \times \mathbb{Z}_4$

(b) Every abelian group is cyclic

F, consider

$\mathbb{Z}_2 \times \mathbb{Z}_2$

(c) Every cyclic group is abelian

T ($x^a \cdot x^b = x^{a+b}$ and

$x^b \cdot x^a = x^{a+b}$)

(d) Every cyclic group is equal to $\mathbb{Z}_n$ for some $n \geq 1$.

False for two reasons:

* $\{\pm 1, \cdot\} \cong (\mathbb{Z}_2, +)$ is isomorphic to $\mathbb{Z}_2$ but not equal
* the infinite cyclic group $\mathbb{Z}$ is not even isomorphic to $\mathbb{Z}_n$ for any n.

(e) There is a group of order n for every positive number n.

T: consider $\mathbb{Z}_n$.