Math 104 Reading and exercises

February 8, 2019

Please read chapters 1-8 of the book by Thursday 2/7 (p. 1-32).

Reading Exercises due Thursday, 1/24. p. 10: without looking ahead, take a few minutes to think about how you would prove Example 2. Reformulate the statement "$\sqrt{2}$ is irrational" as a logical statement. Try to formulate it using only integers. Hint: the statement should start "there does not exist . . .".

Due Tuesday, 1/26. p. 13-14: remember that axioms are supposed to hold for *any* choice of variables. In the case of axioms A1-A4, M1-M4, and DL, the variables are rational numbers. Plug in some random simple rational values (like 2/3) for the variables a, b, c in each part and check that each axiom holds. Remember that since you're looking at this like a logic machine, you need to evaluate each expression in parentheses first (a lot of these calculations will be pretty stupid: don't worry).

p. 14 For axiom O5, what are the two conditions $a \leq b$ and $0 \leq c$ for? Try an example where first the first, then the second condition doesn't hold and see if it makes the conclusion false. Also: is this result true for any a, b, c if you strengthen the $\leq$ signs to strict $<$ signs?

Due Thursday 1/31 p. 22-23 Work out in a little more detail at least one part each of Example 3 and Example 4.

p. 23-24 What is going on in the proof of corollary 4.5? Plug in some bounded set S and see what happens when you take the supremum of $-S$ (for example the set of inverses, Inv). What do you need to prove to show that it is the negative of the infimum of S (and thus an infimum exists)?

Due Thursday, 2/7 Chapter 4: Look at 4.7 (p. 25). If you've already taken notes on it, don't worry about this part. If you are taking notes, figure out how the statement of 4.7 follows from equation (1) at the beginning of p. 25. The proof uses the "Archimedean property", a property of both the rational real numbers) which is so obvious that you wouldn't think to name it, but which is important when building up the theory of real numbers from scratch. Give an example n for the archimedean property (4.6) when $a = .001$ and $b = 10^{10}$.

Chapter 7: p. 37. (optional but useful:) use a calculator or a math programming tool (or a common-sense argument) to find a step N at which the sequences (a), (c), (d), (e) become closer than .01 to the designated limit. Check that the next couple of values are still closer to .01. Of course in order for *all* values of the sequence after the Nth to be closer than (i.e. for N to validate the limit condition for $\epsilon = .01$) you need to check all the infinitely many values $N, N+1, N+2$, etc., or alternatively give a proof that all of these will be closer than .01 from the limit.

p. 37, bottom: Where is the triangle inequality being used in the proof that limits are unique?

Tuesday, 2/12

p. 39. Try to repeat the "Formal proof" in Example 1 with $N = 1/\epsilon$. Working out the proof you'll see that it *almost* works: indeed, you're done so long as you have the inequality $\epsilon \geq 1/N^2 = \epsilon^2$, which is true so long as $\epsilon \leq 1$. (Why is $\epsilon \geq \epsilon^2$ for $0 < \epsilon \leq 1$?) Now convince yourself that if you take

$$N = \begin{cases} 1 & \epsilon > 1 \\ 1/\epsilon & \epsilon \leq 1 \end{cases}$$

and the proof still works. This is a useful technique in general: if a limit proof works for ϵ less than some small value (say, for $\epsilon < 1/1000$) then you can just "replace" all larger ϵ by $1/1000$ and use the N corresponding to this value.

p. 42. Carefully read example 5. Consider the sequence

$$s_n = \frac{n-1}{n} = (0, 1/2, 2/3, 3/4, \ldots).$$

This sequence converges to 1, and you can take $N = 1/\epsilon$ in the convergence proof (you don't need to re-derive this fact here). Now (using a simple calculator if necessary) try to run the convergence proof on pages 42-43 (Case I) for $\epsilon = 1/200$ — i.e., plug in the appropriate values for s_n, s, etc., and find the N prescribed by this proof. Either prove for all $n \geq N$ or check specifically for $n = \lceil N \rceil, n = \lceil N+1 \rceil$ and $n = 2\lceil N \rceil$ (or some values close to these) that N "works" (i.e. $|s_n - 1| < \epsilon$).

p. 48. Look at 9.7 example (a). Try to give a different proof from the one in the book, using Theorem 9.4, the fact that $1/n$ converges to 0 (you don't have to prove this) and the fact that

$$\frac{1}{n^p} = (1/n) \cdot (1/n) \cdots (1/n)$$

(with p factors).