Math 104 (Vaintrob) Midterm 2 April 9, 2019

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
Unless stated otherwise, you may use a result without proving it if it was shown in one of the lectures or readings.

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Question	Points	Score
1	25	
2	30	
3	30	
4	20	
5	25	
Total:	130	

1. (25 points) For each of the following series determine, with proof, whether or not it converges.

(a) (5 points) $\sum(-1)^n \log(n)$.

Does not converge by Cauchy criterion: difference even between neighboring terms does not go to 0 (since $\log(n) \to \infty$ as $n \to \infty$).

(b) (5 points) $\sum \frac{\sin(n/3)}{n^2}$

Converges, comparison with

$$\sum \frac{1}{n^2}$$

(c) (5 points) $\sum \frac{n^n}{3^n}$

Does not converge by root test:

$$\limsup \left| \sqrt[n]{\frac{n^n}{3^n}} \right| = \limsup \frac{n}{3} = \infty$$

(d) (5 points) $\sum \frac{3^n}{n^n}$

Converges by root test:

$$\limsup \sqrt[n]{\left|\frac{3^n}{n^n}\right|} = \limsup \left|\frac{3}{n}\right| = 0 < 1.$$

(e) (5 points) $\sum_{n=2}^{\infty} \frac{1}{\log(n)}$

diverges by comparison with

$$\sum_{n=2}^{\infty} \frac{1}{n}, \text{ which diverges}$$

(since $\log(n) < n \implies \frac{1}{\log(n)} > \frac{1}{n}$ for $n > 1$)

2. (30 points) For each of the following functions, prove whether or not it is continuous at the point specified. You may assume the functions $e^x, \log(x), \sin(x)$, etc. are continuous on their domains and use any result from the book or lecture on continuity.

(a) (10 points) $f(x) = x^{x+1}$, at $x_0 = 1$ (hint: $x^x = e^{x\log(x)}$.)

Yes. $x^x = e^{x \log x}$ is defined on $[0, \infty)$,

as $\log x$ is defined on $[0, \infty)$.

$\log x$ continuous at 1 $\Rightarrow$ $x \log x$ continuous at 1.

e^x continuous at 0 $(= 1 \cdot \log 1)$ $\Rightarrow$ $e^{x \log x}$ continuous at 1.

(b) (10 points) $f(x) = \begin{cases} e^{1/x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, at $x_0 = 0$

No. Consider $x_1 = 1$, $x_2 = 1/2$, $\dots$, $x_n = 1/n$.

Then $\lim x_n \to 0$ but

$\lim e^{1/x_n} = \lim e^n = \infty$.

(c) (10 points) $f(x) = \begin{cases} e^{-|1/x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$, at $x_0 = 0$

~~A~~ $\boxed{\text{Proof \#1}}$ $\forall \varepsilon > 0$, let $\delta = \begin{cases} -1/\log \varepsilon, & \varepsilon < 1 \\ 1, & \text{else} \end{cases}$. Then for $x \neq 0$, if $|x - 0| < \delta$ then $e^{-|1/x|} < e^{-1/\delta} < \min(\varepsilon, 1) \leq \varepsilon$

Proof #2. Yes. Enough to check

$\lim_{x \to 0^+} f(x) = f(0)$ and $\lim_{x \to 0^-} f(x) = f(0)$.

If $(x_1, x_2, \dots) \to 0$ and x_n are positive

then $\lim 1/|x_n| = \infty$ and $\lim e^{-|1/x_n|} = \lim_{y \to \infty} e^{-y} = 0$

This implies $\lim_{x \to 0^+} f(x) = 0 = f(0)$.

Similarly, if $(x_1, x_2, \dots) \to 0$ and x_n negative

then $\lim 1/|x_n| = \infty$ and $\lim e^{-|1/x_n|} = \lim_{y \to \infty} e^{-y} = 0$.

3. (30 points) Compute (with proof) the following limits, or show they do not exist.

(a) (10 points) $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$. limit exists, equal to 7

For $x \neq 3$:

$$\frac{x^2+x-12}{x-3} = x+4.$$

Since $x+4$ is continuous at $x=3$,

$$\lim_{x\to 3} \frac{x^2+x-12}{x-3} = \lim_{x\to 3} x+4 = 3+4 = \underline{7}.$$

(b) (10 points) $\lim_{x\to 1} \frac{x^2+x-12}{|x-3|}$. limit does not exist.

For $x \neq 3$: $$\frac{x^2+x-12}{|x-3|} = \frac{x^2+x-12}{x-3} \cdot \frac{x-3}{|x-3|}$$

$$= \begin{cases} x+4, & x>3 \\ -(x+4), & x<3 \end{cases}.$$

So: $$\lim_{x\to 3^+} \frac{x^2+x-12}{|x-3|} = \lim_{x\to 3^+} x+4 = 7$$

$$\lim_{x\to 3^-} \frac{x^2+x-12}{|x-3|} = \lim_{x\to 3^+} -(x+4) = -7.$$ These are different.

(c) (10 points) $\lim_{x\to 0^-} e^{1/x}$

limit exists

limit equal to 0.

$$\lim_{x\to 0^-} e^{1/x} = \lim_{y\to -\infty} e^y = 0$$

(Alternatively, see proof in 3(c))

4. (20 points)

(a) (5 points) Show that if f is continuous and $f(-4) = 11$ and $f(4) = 0$ then there is some x_0 with $f(x_0) = \pi$.

Apply IVT to $f(x)$ on

$[-4, 4]$.

since $f(-4) > \pi > f(4)$,

$\exists\ x_0 \in [-4, 4]$ with

$f(x_0) = \pi$.

(b) (15 points) Assume $f(x)$ is a continuous function and $f(n) = n$ for all integers n. Show that $f(x)$ takes all real values, i.e. for every y there exists some x_0 with $f(x_0) = y$.

Let $y \in \mathbb{R}$ be a real number.

Then there exists an integer $m \in \mathbb{Z}$ with

$m < y$ and another integer $n > y$

(can take $m = \lfloor y - 1 \rfloor$, $n = \lceil y + 1 \rceil$.)

Now $f(m) = m$, $f(n) = n$ by assumption,

so IVT $\Rightarrow$ $\exists\ x_0 \in [m, n]$ with

$f(x_0) = y$

(as $y \in [f(m), f(n)]$).

5. (25 points) True/False. Give a short (~ 1 – 2 sentence) justification or cite a result (for true statements)/give a counterexample (for false).

(a) (5 points) A continuous function $f(x)$ on $[a, b]$ is uniformly continuous on (a, b).

True: by class/book, it is unif. continuous on $[a,b] \Rightarrow$ also on (a,b).

(b) (5 points) If a function $f(x)$ is defined on $\mathbb{R} \setminus a$ (the complement to a) and $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist then $\lim_{x\to a} f(x)$ exists.

False: these two limits also need to be the same for this to be true

(example: $f(x) = \frac{x}{|x|}$ for $x \in \mathbb{R} \setminus 0$)

(c) (5 points) If $f(0) = 0$ and $g(0) = 0$ and f, g are continuous then $\lim_{x\to 0^+} f(x)/g(x)$ exists.

False. Consider any function $h(x)$ with no limit at 0 which is bounded, for example $h(x) = \sin(1/x)$. Take $f(x) = x \cdot h(x)$, $g(x) = x$. Then $f(x)/g(x) = h(x)$ is a counterex.

Alternative: $f(x) = x$, $g(x) = x^2$.

(d) (5 points) If $f(0) = -1$ and $f(1) = 1$ and $f(x)$ is continuous then $|f(x_0)| = 1/2$ for some value $x_0 \in (-1, 1)$.

T. In fact, by IVT, $\exists x_0$ with $f(x_0) = 1/2$

(e) (5 points) Assume $f_n(x) = (f_1(x), f_2(x), \ldots)$ is a sequence of continuous functions and $g(x)$ is another function such that for each point $x \in \mathbb{R}$, the limit $\lim f_n(x)$ exists and equals $g(x)$. Then $g(x)$ is continuous.

False: $(f_n(x))$ must also be uniformly continuous

(see counter example in class:

$f(x) = \max(1 - n|x|, 0)$.)