14.3 a ratio test: $\left|\frac{1}{\sqrt{n!}}\right| \Big/ \left|\frac{1}{\sqrt{(n-1)!}}\right| = \frac{1}{\sqrt{n}} \longrightarrow 0$

$\Rightarrow$ converges.

b comparison test:

$$\left|\frac{2+\cos n}{3^n}\right| \leq \frac{3}{3^n} \text{ converges}$$

(as $1/3^n$ converges & this is $\underset{\text{scalar}}{3} \cdot \frac{1}{3^n}$.) converges.

c $\frac{1}{2^n+n}$ comparison w/ $\frac{1}{2^n}$. converges.

d converges. Comparison with $\underbrace{(1/2^n)}_{\text{converges}} \cdot \underbrace{52}_{\text{scalar}}$

e diverges, as summand attains value $\sin(1/9) > 0$ infinitely many times, so fails Cauchy criterion (see corollary 14.5 on p. 98).

f converges. Ratio test:

$$a_{n+1}/a_n = \left(\frac{100^n}{n!} \Big/ \frac{100^{n+1}}{(n+1)!}\right)^{-1} = \frac{100}{n+1} \to 0$$

[Bonus: what does it converge to? Hint: power series.]

14.6 (a) Assume $\Sigma |a_n|$ converges and b_n bounded, i.e. $|b_n| < M$ for some constant M.

Then $|a_n b_n| < |a_n| \cdot M$. Now $\Sigma |a_n| \cdot M$ converges since it has partial sums $s_n \cdot M$ (here $s_n = \sum_{k=1}^{n} a_n$) and s_n converges. So $\Sigma a_n b_n$ converges by comparison test.

14.6(a) alternate proof. Say $\sum a_n$ converges. Equivalently, $\sum_{k=m}^{n} a_k$ satisfies the Cauchy criterion:

$\forall \varepsilon' \; \exists N'$ s.t. $m,n > N' \Rightarrow \sum_{k=m}^{n} a_k < \varepsilon'$. Then by triangle inequality, $\sum_{k=m}^{n} a_k \cdot b_k \leq M \cdot \sum_{k=m}^{n} a_k \leq M\varepsilon'$.

Plugging in $\varepsilon' = \varepsilon / (M+1)$, the corresponding N verifies Cauchy criterion for $\sum a_k b_k$.

(Here $M = \text{Max}(|b_k|)$).

(b) Corollary 14.7 follows by taking all $b_n = 1$.

14.13 ~~The~~ One strategy for all of these is to prove by induction that the n'th partial sum S_n is given by some closed expression, then take the limit of the closed expression.

(a) Lemma: [illegible]

For any $x \in \mathbb{R}$,

$$\sum_{k=1}^{n} x^k = \frac{x - x^{k+1}}{1-x}$$

Proof Base case: $\sum_{k=1}^{1} x^k = x = \frac{x - x^2}{1-x}$ ✓

Induction step:

$$\left(\frac{x - x^{k+1}}{1-x}\right) - \left(\frac{x - x^k}{1-x}\right)$$

$$= \frac{x^k - x^{k+1}}{1-x} = x^k. \quad \square$$

so the partial sums for (a) are:

$S_n := \sum_{k=1}^{n} \left(\frac{2}{3}\right)^k$ given by

$$S_n = \frac{\frac{2}{3} - \left(\frac{2}{3}\right)^{k+1}}{1 - 2/3} = \underset{\substack{\uparrow \\ \text{cost}}}{2} - \underset{\substack{\uparrow \\ \text{conv. to } 0}}{3\left(\frac{2}{3}\right)^n}$$

so $\lim S_n = 2$.

For b:

$t_n := \sum_{k=1}^{n} \left(\frac{-2}{3}\right)^k$ given by

$$\frac{-\frac{2}{3} - \left(-\frac{2}{3}\right)^{k+1}}{1-(-2/3)} = \frac{3}{5}\left(-\frac{2}{3} - \left(-\frac{2}{3}\right)^{k+1}\right)$$

conv to 0

so t_n converges to $\frac{3}{5} \cdot \frac{-2}{3} = \frac{-2}{5}$.

b Follow the hint, ~~use~~ induction.

~~follow tele hint by first proving~~
~~lemma~~ $\cancel{\sum_{k=1} \frac{n-1}{2^{n+1}} = \frac{1}{2}}$

A convenient way to re-package the induction argument: "Telescoping"

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{k(k+1)}$$

$$= \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \dots + \left(\tfrac{1}{k} - \tfrac{1}{k+1}\right)$$

all terms cancel except $1 - \frac{1}{k+1}$,

so the partial sum $S_k = 1 - \frac{1}{k+1}$.

$\underline{c}$ $\quad s_n = \frac{0}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \dots + \frac{n-1}{2^{n+1}}$

$= \underbrace{(1/2 - 2/4) + (2/4 - 3/8)} + \underbrace{(3/8 - 4/16) + \dots +} \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right)$

all terms cancel except

$1/2 - \frac{n+1}{2^{n+1}}$.

$\lim \frac{n+1}{2^{n+1}} = 0$ so

$\sum \frac{n-1}{2^{n+1}} = \lim s_n = 1/2$.

$\underline{d}$ $\quad \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum 2\left(\frac{n-1}{2^{n+1}}\right) + \frac{1}{2^n}$

$= 2 \sum \frac{(n-1)}{2^{n+1}} + \sum \frac{1}{2^n}$

$= 2 \cdot 1/2 + 1 = 2$.

15.4 (a) diverges.

Solution 1:

Lemma $2\sqrt{n} \geq \log n$ for $n \geq 1$

Proof for $x \geq 1$ we have $\frac{1}{x} \leq \frac{1}{\sqrt{x}}$ so

$$\log n = \int_1^n \frac{1}{x} \leq \int_1^n \frac{1}{\sqrt{x}} = 2\sqrt{x} - 2 < 2\sqrt{x}. \quad \square$$

So: $$\sum_{n=2}^{\infty} \frac{1}{(\log n)\sqrt{x}} \geq \sum_{n=2}^{\infty} \frac{1}{2\sqrt{n}\cdot\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n} = +\infty. \quad \square$$

Solution 2 integral test:

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)\sqrt{n}} \geq \int_{x=2}^{\infty} \frac{1}{(\log x)\sqrt{x}} dx$$ set $x = e^u$, then

$$= \int_{u=\log 2}^{\infty} \frac{1}{u\sqrt{e^u}} de^u = \int_{u=\log 2}^{\infty} \frac{e^u}{u\sqrt{e^u}} du$$

$$= \int_{u=\log 2}^{\infty} \frac{e^{u/2}}{u} = \infty.$$ integrand approaches ∞!

[note we never had to fully compute the integral: just reduce it to a divergent expression]

(b) $\frac{\log n}{n} \geq \frac{\log 2}{n}$ for $n \geq 2$, so

$\sum \frac{\log n}{n} \geq \log 2 \sum \frac{1}{n}$ diverges.

(c) integral test: $\int_4^t \frac{1}{x \log x \log\log x} = \log\log\log t - \log\log\log 4$

~~which app~~ and $\log\log\log x \to \infty$ as $x \to \infty$.

(d) converges. By Lemma above, $\frac{\log n}{n^2} \leq \frac{2\sqrt{n}}{n^2}$

and $\sum_{n=2}^{\infty} \frac{2\sqrt{n}}{n^2} = 2\sum_{n=2}^{\infty} 1/n^{1.5}$ converges by Theorem 15.1

17.8 (a)

$h(x) := \left[\frac{1}{2}(f+g-|f-g|)\right](x) = \begin{cases} \frac{1}{2}(f(x)+g(x)-(f(x)-g(x))) & \text{if } f(x) \geq g(x) \\ \frac{1}{2}(f(x)+g(x)+(f(x)-g(x))) & \text{if } f(x) \leq g(x) \end{cases}$

Now $\frac{1}{2}\left(f(x)+g(x)-(f(x)-g(x))\right)$

$= \frac{1}{2}(2g(x)) = g(x)$ (for $f(x) \geq g(x)$)

so $h(x) = g(x)$ if $f(x) \geq g(x)$

and ~~otherwise~~ if $g(x) \geq f(x)$

$h(x) = \frac{1}{2}\left(f(x)+g(x)+(f(x)-g(x))\right)$

$= \frac{1}{2}\,\cancel{g(x)}\,2f(x) = f(x)$

So $h(x) = \cancel{\max(f(x), g(x))}\ \min(f(x), g(x))$

$\square$

(b) for any x, $\min(f(x), g(x))$

$= -\max(-f(x), -g(x))$

(c) $|x|$ is continuous and $f(x) - g(x)$ is continuous (by Thm 17.4, using $f(x) - g(x) = f(x) + (-1)\cdot g(x)$)

So by Thm 17.5, $|f-g|$ is continuous

Again applying Thm 17.4,

$\frac{1}{2}(f+g-|f-g|)$ is continuous.

Done by (a)

17.9 (a) for $\varepsilon > 0$

set $\delta = \min(\varepsilon/5, 1)$.

Assume $|x-2| < \delta$. Write

$x = 2 + d$ (so $|d| < \delta$).

Then $|x^2 - 2^2| = |4d + d^2| \leq |4d| + |d^2|$

$< 4\delta + \delta^2 \leq 5\delta \leq \varepsilon$ $\square$

since $\delta \leq 1$ since $\delta \leq \frac{\varepsilon}{5}$

(b) for $\varepsilon > 0$ set $\delta = \varepsilon^2$.

Note that $\mathrm{Dom}(\sqrt{x}) = \{x \in \mathbb{R} \mid x \geq 0\}$.

So if $x \in \mathrm{Dom}(\sqrt{x})$ and $|(x - 0)| < \delta = \varepsilon^2$

then $0 \leq x < \delta$ and

$0 \leq |0 - \sqrt{x}| < \sqrt{\delta} = \varepsilon$. $\square$

(c) done in class

(d) Lemma assume $|x - x_0| < 1$. Then

$x^2 + x_0 x + x_0^2 < 3(|x_0| + 1)^2$.

Proof $|x| < |x_0| + 1$ by triangle ineq

and $|x_0| < |x_0| + 1$ clearly. So

all three terms x^2, $x_0 x$, x_0^2 are $< (|x_0| + 1)^2$

$\square$

Now for $\varepsilon > 0$ set $\delta = \min\left(1, \dfrac{\varepsilon}{3(|x_0|+1)^2}\right)$.

Now apply argument similar to (a) using hint.

18.5 (a) Let $h(x) = f(x) - g(x)$.
The conditions of the problem imply
$h(a) \geq 0$, $h(b) \leq 0$. so by
IVT (18.2), $\exists x_0 \in [a,b]$
such that $h(x_0) = 0$, so
$f(x_0) - g(x_0) = h(x_0) = 0$ $\square$

(b) Take $g(x) = x$, on $[a,b] = [0,1]$
Then $f(0) \geq 0$ and $f(1) \leq 1 \Rightarrow$
$f(0) \geq g(0)$ and $f(1) \geq g(1)$, so
for some $x_0 \in [0,1]$ have
$f(x_0) = g(x_0) = x_0$.

18.7 $0 \cdot e^0 = 0 < 0$
$1 \cdot e^1 = e > 2$.
x and e^x both continuous $\Rightarrow$
$x \cdot e^x$ continuous, so by EVT
$x e^x = 2$ for some $0 \leq x \leq 1$.
Since $0 \cdot e^0 \neq 2$ and $1 \cdot e^1 \neq 2$,
must have $0 < x < 1$. $\square$

19.2 (a) If $\delta < \frac{\varepsilon}{3}$ ~~then~~ (for any $\varepsilon > 0$) then

$$|y-x| < \delta \Rightarrow \left|(3y+11) - (3x+11)\right| = 3|y-x| < 3\delta = \varepsilon \quad \square$$

(b) for $\varepsilon > 0$ set $\delta = \frac{\varepsilon}{100}$.

Say $x, y \in [0,3]$ and $|y-x| < \delta$.

Then
$$\begin{aligned}|y^2 - x^2| &= |y-x|\,|y+x| \\ &< \delta\,|y+x| \leq \delta(|y|+|x|) \\ &\leq \delta \cdot (3+3) = 6\delta = \frac{6\varepsilon}{100} \\ &< \varepsilon.\end{aligned}$$

(c) Set $\delta = \varepsilon/4$ (for $\varepsilon > 0$).

Then $|y-x| < \delta$ and $x, y \geq 1/2$

$$\Rightarrow \left|\frac{1}{y} - \frac{1}{x}\right| = \left|\frac{x-y}{xy}\right| = \frac{|y-x|}{|x|\,|y|}$$

$$< \frac{\delta}{|x|\,|y|} \leq \frac{\delta}{1/2 \cdot 1/2} = 4\delta = \varepsilon.$$

(d)

19.8 < skipping this one: not ~~used~~ needed for exam >

20.4 (a) $\frac{x^2-a^2}{x-a} = x+a$ (for $x \neq a$).

For any sequence $s_n \to a$ (and $s_n \in \mathrm{Dom}(f)$, so $s_n \neq a$ $\forall n$)

have $\lim \frac{s_n^2-a^2}{s_n-a} = \lim s_n+a$
$= a+a = 2a$ □

(b)

Similarly to (a), use that

for $x \neq b$ have $\frac{\sqrt{x}-\sqrt{b}}{x-b} = \frac{\sqrt{x}-\sqrt{b}}{(\sqrt{x}-\sqrt{b})(\sqrt{x}+\sqrt{b})}$

$= \frac{1}{\sqrt{x}+\sqrt{b}}$. Since

$\frac{1}{\sqrt{x}}$ is continuous at $b>0$,

if $s_n \to b$ then

$\lim \frac{1}{\sqrt{s_n}+\sqrt{b}} = \frac{1}{\sqrt{\lim s_n}+\sqrt{b}} = \frac{1}{2\sqrt{b}}$. □

(c) For $x \neq a$,

$$\frac{x^3-a^3}{x-a} = x^2+ax+a^2$$

For any sequence $s_n \to a$, $s_n \neq a$

standard manipulations of limits

[or continuity of $f(x) := x^2+ax+a^2$]

implies

$$\lim \frac{s_n^3-a^3}{s_n-a} = \lim (s_n^2+as_n+a^2$$

$$= a^2 + a \cdot a + a^2 = 3a^2.$$ □

20.17

Assume $s_n > a$ and $s_n \to a$.

Then $\lim f_1(s_n) = L$ and

$\lim f_3(s_n) = L$ and

$f_1(s_n) \leq f_2(s_n) \leq f_3(s_n)$,

~~so lim sup $f_2 \geq$ lim sup f_1~~

Now compare:

$\limsup f_2(s_n) \leq \limsup f_3(s_n) = L$

$\liminf f_2(s_n) \geq \liminf f_1(s_n) = L$

and ~~since lim~~ so $L \geq \limsup \geq \liminf \geq L$.

Thus $\liminf f_2(s_n) = \limsup f_2(s_n) = L$]

~~and~~ so $f_2(s_n)$ has limit L.

Since s_n was any sequence approaching a from above, we have shown

$\lim_{x \to a^+} f_2 = L$. $\square$

23.1

(a) recall: $\limsup (n^2)^{1/n} = \lim \frac{(n+1)^2}{n^2}$

(when limit exists) $= \lim \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

$= 1.$

So $\beta = 1$, $r = 1/\beta = 1$

at $x = 1$, $\sum n^2$ diverges

$\sum (n^2)(-1)^n$ diverges

as the terms do not approach 0

(so the Cauchy criterion cannot be satisfied).

(c) $\beta = \lim \text{supp} \left(\frac{2^n}{n^2}\right)^{1/n} =$

$\lim \frac{2^{n+1}/(n+1)^2}{2^n / n^2}$. Now $\lim \frac{2^{n+1}}{2^n} = 2$

$\lim \frac{(n+1)^2}{n^2} = 1$ [so exists]

by above,

so ~~this = lim sup~~ $\beta = 2$

$r = 1/2$.

(e) $\beta = \lim \text{supp} \left(\frac{2^n}{n!}\right)^{1/n} =$

$\lim \frac{2^{n+1}}{(n+1)!} \Big/ \frac{2^n}{n!}$ [when exists]

$= \lim \frac{2}{n+1} = 0$. [so lim exists]

$r = \frac{1}{\beta} = \infty$, so converges everywhere

(g) $\beta = \lim \text{supp} \left(\frac{3^n}{n \cdot 4^n}\right)^{1/n} = \lim \frac{3^{n+1}}{(n+1)4^{n+1}} \Big/ \frac{3^n}{n \cdot 4^n}$

$= \lim \frac{3}{4} \cdot \frac{n}{n+1} = 3/4$ [so lim exists]

and $r = (3/4)^{-1} = 4/3$.