3.7

(a) "Show $|b| < a$ if and only if $-a < b < a$"

This is an if and only if statement, so two directions.
($\Rightarrow$): assume $|b| < a$. Flipping sigs, $-a < |b|$.
Since $|b| \geq 0$, also $-|b| \leq |b|$. Putting this together:
$-a < -|b| \leq |b| < a$. Since $b = \pm |b|$ and we have shown both $-a < -|b| < a$ and $-a < |b| < a$, we're done.
($\Leftarrow$). Assume $-a < b < a$. Flipping signs:
$a > -b > -a$. So both $-a < b < a$ and $-a < -b < a$
So since $|b| = \pm b$, one of those statements implies
$-a < |b| < a$, and we're done.

"Show $|a-b| < c$ if and only if $b-c < a < b+c$"

(b) By part (a), have $|a-b| < c \Leftrightarrow$ ~~[illegible]~~

$-c < a-b < c \xLeftrightarrow{\text{add } b \text{ to both sides}} b-c < a < b+c$.

"Show $|a-b| \leq c$ if and only if $b-c \leq a \leq b+c$"

(c) I'm going to cheat and use 3.5(a):
the fact that $|b| \leq a \Longleftrightarrow -a \leq b \leq a$.
This is proved exactly analogously to (a).
Now:
$|a-b| \leq c \xLeftrightarrow{3.5(a)} -c \leq a-b \leq c \xLeftrightarrow{\text{add } b} b-c \leq a \leq b+c$.

4.1 - 4.2 - ~~4.4~~ 4.3 - 5.3

(find inf, sup, lower bounds, upper bounds)

(a) inf = 0, sup = 1. Bounded. Three upper bounds: 2, π, 100; any three numbers ≥ 1.
Three lower bounds: 0, -1, -2.

(c) inf = 2 (1, 0 other l. bds) sup = 7 (10, 20 other upper)
bounded

(e) sup = 1 (2, 3 other upper)
inf = 0 (-1, -2 other lower)
bounded.

(g) $[0,1] \cup [2,3] = \{x \mid \text{either } 0 \leq x \leq 1 \text{ or } 2 \leq x \leq 3\}$.
sup = 3 (4, 5 other upper)
inf = 1 (0, -1 other lower)

6c

(i) $\bigcap_{n=1}^{\infty} [-1/n, 1+1/n] = \{x \mid -1/n \leq x \leq 1+1/n \ \forall n \in \mathbb{N}\}$
$= \{x \mid 0 \leq x \leq 1\} = [0,1]$.
So same as [a].

(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\} = \{0, 2+1/2, 3-1/3, 4+1/4, \ldots\}$

inf = 0 (all others positive). Lower bds = (three numbers ≤ 0)
unbounded above, as $n + \frac{(-1)^n}{n} \geq n-1 \ \forall n$,
so sup $= \infty$.

(m) $\{r \in \mathbb{Q} \mid r^2 < 4\} = (-2, 2) \cap \mathbb{Q}$. Since $\mathbb{Q}$ is dense, $a > -2 \Rightarrow$ there is a rational number in $(-2, a)$ so a is not a lower bound.
On the other hand -2 is obviously a lower bound. So it is the greatest lower bound: inf = -2. Also by a similar argument, sup = 2. Bounded.

4.1-4.2-4.3-4.4-5.3

(o) $\sup = 0$, $\inf = -\infty$ (unbounded below)

(q) $\inf = 0$, $\sup = 16$ (bounded)

(s) $\inf = 0$ (as there are infinitely many primes, this can get smaller than any $\varepsilon > 0$: For a formal proof, just pick a prime $> 1/\varepsilon$.)
$\sup = 1/\text{smallest prime} = 1/2$,
bounded.

(u) $\{x^2 \mid x \in \mathbb{R}\} = \{$all nonnegative numbers$\}$.

$\inf = 0$ $\quad$ $\sup = \infty$ (unbounded above)

(w) $\{\sin(\frac{n\pi}{3}) \mid n \in \mathbb{N}\} = \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$

(remember: sets do not care about repetition!) So:
$\sup = \sqrt{3}/2$. $\quad$ $\inf = -\sqrt{3}/2$.
Bounded.

7.1 - 7.2

(a) $s_n = \left(\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{16} \dots\right)$

$\lim = 0$

(b) $b_n = \left(\frac{4}{3}, \frac{7}{7} = 1, \frac{10}{11}, \frac{13}{15}, \frac{16}{19} \dots\right)$

$\lim = 3/4$.

(c) $c_n = \left(\frac{1}{3}, \frac{2}{9}, \frac{3}{27} = \frac{1}{9}, \frac{4}{81}, \frac{5}{243}, \dots\right)$

$\lim = 0$

(d) $\sin\left(\frac{n\pi}{4}\right) = \left(0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, \dots\right)$

no limit.

8.1 limit proofs

(a) $\lim \frac{(-1)^n}{n} = 0.$

proof: For $\varepsilon > 0$, take $N = 1/\varepsilon$.

Then $n > N \Rightarrow \left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \varepsilon$ $\square$.

(b) $\lim 1/n^{1/3} = 0.$

proof. For $\varepsilon > 0$, take $N = 1/\varepsilon^3$.

Then $n > N \Rightarrow |1/n^{1/3} - 0| = 1/n^{1/3} < 1/N^{1/3} = \varepsilon$. $\square$

(c) $\lim \frac{2n-1}{3n+2} = 2/3.$

Proof $\quad 2/3 = \frac{2n + 4/3}{3n+2}$

$$\left|\frac{2n-1}{3n+2} - 2/3\right| = \left|\frac{2n - 1 - (2n + 4/3)}{3n+2}\right|$$

$$= \left|\frac{-7/3}{3n+2}\right| = \frac{7}{9n+6}.$$

8.1 (c) (continued).

Now take $N = \frac{1}{\varepsilon} + 1$ (generous).

For $n > N$, we compute

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| = \frac{7}{9n+6} < \frac{7}{9N+6} = \frac{7}{9(\frac{1}{\varepsilon}+1)+6}$$

$$= \frac{7}{9(\frac{1}{\varepsilon})+15} < \frac{7}{9(1/\varepsilon)+9} = \frac{7}{9}\varepsilon < \varepsilon.$$

8.1(d) $\lim \frac{n+6}{n^2-6} = 0$

$\boxed{\text{⚠ warning: this function is not monotonic, so } n > N \text{ does not necessarily imply } \left|\frac{n+6}{n^2-6}\right| < \left|\frac{N+6}{N^2-6}\right|. \text{ Need to be more careful}}$

For $\varepsilon > 0$, ~~take~~ set $A = \frac{1}{\varepsilon}$.

Set $N = A + 6$. Now for $n > N$ we compute

$$\frac{n+6}{n^2-6} = \frac{n+6}{((n+6)(n-6)+36)-6}$$

$$= \frac{n+6}{(n+6)(n-6)+30} < \frac{n+6}{(n+6)(n-6)} < \frac{n+6}{n+6\cdot(N-6)}$$

$$= \frac{1}{A} = \varepsilon.$$ $\square$

t_n bounded
$\lim s_n = 0$
prove
$\lim t_n s_n = 0$.

8.4 Say M is an upper bound for $|t_n|$, so $|t_n| \le M$. Can take $M > 0$ (for example, set $M = \sup\{|t_n|\} + 1$).

Know: For $\varepsilon' > 0$ $\exists\, N_s(\varepsilon')$ with the ~~limit~~ property:
$n > N_s(\varepsilon') \Rightarrow |s_n| < \varepsilon'$.

For $\varepsilon > 0$, Take $N = N_s(\varepsilon/M)$. Then $n > N \Rightarrow$

$$|s_n t_n| \le |s_n| \cdot M < \varepsilon/M \cdot M = \varepsilon \quad \square.$$

Prove:
If $a_n \le s_n \le b_n$
and $\lim a_n = \lim b_n = s$
then s_n converges to s

8.5 (a) Know For $\varepsilon > 0$ $\exists\, N_a, N_b$ such that $n > N_a \Rightarrow |a_n - s| < \varepsilon$ and

$n > N_b \Rightarrow |b_n - s| < \varepsilon$.

Take $N = \max(N_a, N_b)$.

Then $n > N \Rightarrow$ both $|a_n - s| < \varepsilon$ and $|b_n - s| < \varepsilon$,
so $s - \varepsilon < a_n \le b_n < s + \varepsilon$ (see 3.7 b).

so $s - \varepsilon < a_n \le s_n \le b_n < s + \varepsilon \Rightarrow$

$|s_n - s| < \varepsilon$. $\square$

$|s_n| \le t_n$
$\forall n \in \mathbb{N}$
and $\lim t_n = 0$. Prove
$\lim s_n = 0$

(b) $|s_n| \le t_n \iff -t_n \le s_n \le t_n$.

Since t_n converges to 0, $-t_n$ also converges to 0 ($\forall \varepsilon > 0$ can take the same N), and we use (a) ~~to see~~ (with $a_n = -t_n$, $b_n = t_n$) to see s_n converges to 0.

prove:
$\lim s_n = 0$
$\iff \lim |s_n| = 0$

8.6 (a) It is enough to observe

$|s_n - 0| = ||s_n| - 0|$ so the limit statement

$|s_n - 0| < \varepsilon \quad \forall n > N$ is equivalent to
$||s_n| - 0| < \varepsilon \quad \forall n > N$. $\square$

see that $(-1)^n$ has no limit while $|(-1)^n|$ has limit 1.

(b) A we have seen in class (also in the book),

$(-1)^n$ does not have a limit. Indeed, even for $\varepsilon = 1$ there is no way to make the limit statement true for any potential limit value s.

However $|(-1)^n| = 1$ and the sequence $s_n = 1$ has limit 1.

$s_n \geq a$ for all but finitely many n then $\lim s_n \geq a$

8.9 (a)

Let $E = \{n \mid s_n < a\} \subseteq \mathbb{N}$ be the set of "exceptional" indices, those finitely many n with s_n not $\geq a$. Then there is some value M greater than every index in E (if there are no indices in E, just take $M=1$).

For $n > M$, then, $s_n \geq a$.
Also, $\forall \ \varepsilon$, $\exists N$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$ (for s the limit value).

Rewrite: $-\varepsilon < s_n - s < \varepsilon$. Now for $n \geq \max(M, N)$ (a non-empty set of indices), $a - s \leq s_n - s < \varepsilon$. ~~[illegible]~~
So: $a < \varepsilon + s \Rightarrow s > a - \varepsilon$. This is true for any $\varepsilon > 0$, so $s \geq a$. $\square$

if $s_n \le b$ for all but finitely many n, then $\lim s_n \le b$

(b) Set $t_n = -s_n$, $a = -b$. Then part (a) implies $\lim t_n \geq a$, so $\underline{\lim}\, s_n = \lim -t_n = -\lim t_n \le -a = b$ $\square$.

(c) All but finitely many s_n satisfy both $s_n \geq a$ and $s_n \le b$. So by (a), $\lim s_n \geq a$ and by (b) $\lim s_n \le b$, so $\lim s_n \in [a, b]$. $\square$

<u>11.1</u> (a) $a_n = (1, 5, 1, 5, 1, 5, 1, 5, \dots)$

(b) $a_{2n} = (a_2, a_4, a_6, \dots) = (5, 5, 5, \dots)$

selection fn: $\sigma(n) = 2n$

<u>11.2</u> (i) $a_n = (-1)^n$:

- monotone subsequence: $a_{2n} = (1, 1, 1, \dots)$: $1 \geq 1 \geq 1 \dots$
- subseq limits: $1, -1$
- $\lim\sup = 1$ $\quad \lim\inf = -1$.
- Does not converge, or diverge to $\pm\infty$. Bounded.

(ii) $b_n = \frac{1}{n}$

- monotone: b_n itself (selection: $\sigma(n) = n$).
  $1 \geq 1/2 \geq 1/3 \dots$
- subseq limits: only 0, since ~~a limit exists~~ converges to 0
- $\lim\inf = \lim\sup = 0$ since converges to 0.
- this converges and is bounded.
  (any convergent sequence is bounded)

(iii) $c_n = n^2$. • monotone: c_n itself $(1 \le 4 \le 9 \dots)$

- subseq limits: no finite limits, $+\infty$ only limit.
- $\lim\inf =$ $\quad \lim\sup = \infty$
- diverges to ∞. Bounded below, not above.

11.1 (iv) $a_n = \dfrac{6n+4}{7n-3} = \dfrac{6+\frac{4}{n}}{7-\frac{3}{n}}$

check: • $7-\frac{3}{n}$ never 0 $(\forall n \in \mathbb{N})$

• $\lim 7-\frac{3}{n} = 7 \neq 0$. So

$\lim \dfrac{1}{7-\frac{3}{n}} = \dfrac{1}{7}$ is valid.

Product rule for limits:

$$\lim \frac{6+\frac{4}{n}}{7-\frac{3}{n}} = \left(\lim 6+\frac{4}{n}\right)\left(\lim \frac{1}{7-\frac{3}{n}}\right)$$

$$= 6 \cdot \frac{1}{7} = \frac{6}{7}.$$

So: bounded, lim inf = lim sup = unique subsequential limit $= \frac{6}{7}$.

• Monotone subseq:

write $\dfrac{6n+4}{7n-3} = \dfrac{6}{7} + \dfrac{\cancel{42n}+28-(\cancel{42n}-18)}{7(7n-3)}$

$$= 6/7 + \frac{46}{49n-21}$$

which is itself monotone decreasing. $\square$

$s_1 = 1$
$s_{n+1} = \frac{s_n + 1}{3}$
(recursive sequence).
(a): write $s_1, \dots, s_4$

10.10

(a) $s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$.

$s_3 = \frac{1}{3}\left(\frac{2}{3}+1\right) = \frac{5}{9}$

$s_4 = \frac{1}{3}\left(\frac{5}{9}+1\right) = \frac{14}{27}$

so: $s_n = \left(1, \frac{2}{3}, \frac{5}{9}, \frac{14}{27}, \dots\right)$.

$\approx (1, .6666, .5555, .5185, \dots)$

(b) use induction to show $s_n > 1/2 \ \forall n$

(b) base: $s_1 > 1/2$

induction step: assume $s_n > 1/2$. Then

$s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}\left(\frac{1}{2}+1\right) = 1/2$.

(c) show decreasing

(c) $\underline{s_{n+1}} = \frac{1}{3}(s_n + 1)$

$= \frac{1}{3}\left(s_n + 2 \cdot \frac{1}{2}\right) < \frac{1}{3}(s_n + 2 \cdot s_n)$

$= \frac{1}{3}(3 s_n) = \underline{s_n}$, so

$s_{n+1} < s_n \quad \forall n \Rightarrow$ decreasing.

(d) decreasing and bounded below (by 1/2)

$\Rightarrow$ limit exists.

<u>now</u> $\lim (s_n) = \lim (s_{n+1})$

$= \lim\left(\frac{1}{3}(s_n + 1)\right) = \frac{1}{3}\left(\lim(s_n) + 1\right)$

and so $\frac{2}{3} \lim s_n = \frac{1}{3} \Rightarrow \underline{\lim s_n = \frac{1}{2}}$.

10.12 (a) $|t_{n+1}| = \left(1 - \frac{1}{(n+1)^2}\right) \cdot |t_n| < t_n$.

on the other hand $t_1 > 0$, t_n is a positive multiple of t_{n-1} so each t_n is positive. Thus
$t_n = |t_n| \geq 0$ and so

$t_1 > t_2 > t_3 > \ldots > 0$ is a decreasing sequence, bounded below $\Rightarrow$ convergent.

(c) $t_1 = 1 = \frac{1+1}{2 \cdot 1}$ ✓ base case

$\boxed{\text{Assume } t_n = \frac{n+1}{2n}.}$

$t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) t_n = \left(1 - \frac{1}{(n+1)^2}\right) \frac{n+1}{2n}$

$= \left(\frac{(n+1)^2 - 1}{(n+1)^2}\right) \frac{n+1}{2n} = \frac{n(n+2)}{(n+1)^2} \left(\frac{n+1}{2n}\right)$

$= \frac{n+2}{2(n+1)^2/(n+1)} = \frac{n+2}{2(n+1)}$

✓ induction hypothesis

(d) $\lim \frac{n+1}{2n} = \lim \frac{1}{2} + \frac{1}{2n} = \frac{1}{2}$.