Math 104 Homework 6 (Vaintrob)

Due Tuesday, 3/5

1 Reading Exercises (to be submitted with regular homework)

1. Read section 10.3 (p. 58-59). Prove using induction that $\sum_{k=1}^{n} \frac{9}{10^k} = 1 - 1/10^n$. Deduce directly that $\lim \sum_{k=1}^{n} \frac{9}{10^k} = 1$. What familiar decimal identity does this rigorously prove?

2. Read Theorem 10.4 (p. 59) and prove part (ii).

3. Read Theorem 11.2 (page 68). It's very important to understand the proof of this theorem. In your own words give a self-contained proof for part (ii) of the theorem (be sure to show you understand the main steps of the proof, though it's ok if you don't check obvious details).

4. Read example 3 of chapter 11 (p. 70). It is a remarkable fact that there is a sequence that contains every rational number! Use theorem 11.2 and density of the rational numbers to prove that π is a subsequential limit of this sequence. (Is it a value of this sequence?)

5. Remember that $\limsup s_n$ is the largest subsequential limit of a sequence s_n. Give a one-line (not necessarily rigorous) explanation of theorem 12.1 using this fact.

6. Look at theorem 13.10. Give an example of a series of intervals $I_1 = [a_1, b_1] \supseteq I_2 = [a_2, b_2] \supset I_3 = [a_3, b_3] \supseteq \dots$ with each $a_i < b_i$, such that their limit is a single point. This means that although the theorem implies any such intersection has at least one point, this point can be unique!

2 Book exercises

1. **9.9 (a), 9.10 (a, c)**
2. **9.12, 9.13**
3. **11.10**
4. **12.3**
5. **12.4**
6. **12.13**
7. Extra credit 1: First do 13.8(a). Now suppose $U \subseteq \mathbb{R}$ is an *open* set, i.e. for each $x \in \mathbb{R}$ there exists some $\epsilon > 0$ such that $|y - x| < \epsilon$ implies $y \in U$. Prove that
   - for this epsilon, $(x - \epsilon, x + \epsilon) \subseteq U$
   - We've just shown that if U is open then for every $x \in \mathbb{R}$, we can choose an $\epsilon(x)$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. Prove that U is a union of open intervals, $U = \cup_{x \in U}(x - \epsilon, x + \epsilon)$. This proves Exercise 13.7.
8. Extra credit 2: **13.9**