Math 104 Homework 12 (Vaintrob)

Due Tuesday, 4/30

1 Finishing proof of l'Hôpital's theorem

Suppose $f(x), g(x)$ are functions defined on some interval to the right of $x = a$. Suppose $f(a) = g(a) = 0$ and

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$$

(converges to a finite value). In particular, to make the limit make sense, we assume $g(x) \neq 0$ in some interval $[a, b]$. Use the following results from class to complete the proof.

First, we proved that $g(x)$ is strictly monotone on $[a, b]$, i.e. $f(x) \neq f(y)$ for any $x < y \in [a, b]$. Next, we proved the following lemma.

Lemma 1. *If g is a differentiable function on $[x, y]$ and g' is non-zero on all of $[x, y]$, then there exists $z \in [x, y]$ such that*

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(z)}{g'(z)}$$

for some $z \in (x, y)$.

Now use the following arguments to complete the proof.

(a) Fix $y \in (a, b)$. Show that for any ϵ, there exists a $z \in (a, y)$ such that

$$\left|\frac{f'(z)}{g'(z)} - \frac{f(y)}{g(y)}\right| < \epsilon.$$

(Hint: using the lemma and the fact that $f(a) = g(a) = 0$, this is just a normal manipulation with limits). Note that this expression no longer involves x.

(b) Now let $(y_1, y_2, \dots)$ be a sequence converging to a from above. Prove that there is a sequence $z_n \in (a, y_n)$ with $\lim \frac{f(y_n)}{g(y_n)} = \lim \frac{f'(z_n)}{g'(z_n)}$ by using the previous part to bound the difference.

(c) Prove that $\lim \frac{f'(z_n)}{g'(z_n)} = L$ (hint: first prove $\lim z_n = a$).

(d) Explain why this concludes the proof (hint: it is important that the $y_n \in (a, b)$ were picked arbitrarily subject to $\lim y_n = a$).

(e) For bonus points: try to understand the proof given in the book (pages 242-244), especially the part on page 244. Without rewriting it here, can you roughly put its steps in correspondence with the proof we did here? How are the proofs different? Which do you prefer? (Something to keep in mind: the book uses a what is essentially a "lim sup" argument but for functions instead of sequences, where it proves that $\frac{f(y)}{g(y)}$ is bounded above by $L + \epsilon$ if $|y - a|$ is sufficiently small: together with a symmetric lim inf argument, this implies what is needed – can you see why?)

2 30.1

3 30.3

4 30.5

Hint: see examples 6, 7, 8 (p. 246-47).

5 32.1

6 32.2

7 32.4

8 Extra credit: 32.6