# Math 104 Homework 1 (Vaintrob)

Due Thursday, 1/31, reading exercises due earlier

**Reading Exercises due Thursday, 1/24.** p. 10: without looking ahead, take a few minutes to think about how you would prove Example 2. Reformulate the statement "$\sqrt{2}$ is irrational" as a logical statement. Try to formulate it using only integers. Hint: the statement should start "there does not exist ...".

**Due Tuesday, 1/26.** p. 13-14: remember that axioms are supposed to hold for *any* choice of variables. In the case of axioms A1-A4, M1-M4, and DL, the variables are rational numbers. Plug in some random simple rational values (like 2/3) for the variables $a, b, c$ in each part and check that each axiom holds. Remember that since you're looking at this like a logic machine, you need to evaluate each expression in parentheses first (a lot of these calculations will be pretty stupid: don't worry).

p. 14 For axiom O5, what are the two conditions $a \leq b$ and $0 \leq c$ for? Try an example where first the first, then the second condition doesn't hold and see if it makes the conclusion false. Also: is this result true for any $a, b, c$ if you strengthen the $\leq$ signs to strict $<$ signs?

## 1 Exercise 2.8.

Find all rational solutions of the equation $x^8 + 4x^5 + 13x^3 - 7x + 1 = 0$.

## 2 Exercise 3.1

## 3 Exercise 3.3

## 4 Exercise 3.4

## 5 Exercise 3.6

## 6 Exercise 3.7

## 7

Write down the Rational Zeroes Theorem in the book and deduce the *monic* version from the first class, which is Corollary 2.3 ("corollary" means "consequence"). See if you can do this without looking at the proof in the book.

## 8

This problem is worth two regular problems. Say a number $x$ satisfies a polynomial equation none of whose coefficients are rational. Can $x$ be algebraic? Can it be rational? (Give examples.) Can you find a monic polynomial with some transcendental (i.e. non-algebraic) coefficients which has an algebraic solution? Hint: $\pi, \pi^2$ are both transcendental numbers.

## 9 "Complex rational numbers".

This problem is worth two regular problems: Let $\mathbb{C}_{rat} := \{a + bi \mid a, b \in \mathbb{Q}\}$ be the set of complex rational numbers whose coordinates are rational. Write down how to multiply and add two elements $a + bi$ and $a' + b'i$ of $\mathbb{C}_{rat}$. I claim this is a field (i.e. satisfies axiosm A, M, DL on pages 13-14). You're welcome to check all of them, but for this problem check just axioms M1, M4 and DL.

## 10 Challenge problem

(you can choose to do this one instead of five regular problems of your choice. This problem is harder but a useful problem to do if you are an advanced student) Define $\mathbb{H}_{rat}$ to be the "rational quaternions", symbols of the form $a * e + bi + cj + dk$ for $a, b, c$ rational numbers. These elements are added like vectors with basis $\widehat{1}, i, j, k$ (similar to how complex numbers are added like vectors with basis $1, i$ – I'm using $\widehat{1}$ instead of 1 to avoid confusion of vectors with scalars). Now define multiplication on the basis vectors as follows: $\widehat{1} * x = x$ (for $x$ any basis vector); $i^2 = j^2 = k^2 = -\widehat{1}$ (we define $-\widehat{1} := (-1) * \widehat{1}$) and $ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$. Use axiom DL to guess a multiplication law on any pair of vectors $(a\widehat{1}+bi+cj+dk)(a'\widehat{1}+b'i+c'j+d'k)$. Check that this satisfies the associativity axiom M1. Check that $(a\widehat{1}+bi+cj+dk)(a\widehat{1}-bi-cj-dk) = (a^2+b^2+c^2+d^2)\widehat{1}$ (note that in the second factor, we just changed the signs of all coordinates but the "ones" coordinate. Compare to multiplying a complex number by its conjugate). Deduce that axiom M4 holds, i.e. for each $a \neq (0, 0, 0, 0) \in \mathbb{H}_{rat}$, there is an element $a^{-1}$ such that $aa^{-1} = \widehat{1}$. In fact, all the field axioms hold except for the *commutativity* axiom M2, and number systems like $\mathbb{H}_{rat}$ are called *skew-fields*.