Worksheet 9: the Riemann Mapping Theorem.

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1 Introduction

Suppose Ω is the interior of a simple closed curve. We will construct a bijective, holomorphic map $F : \Omega \to \mathbb{D}_1$ to the unit disk, proving that they are conformally equivalent. Of course this implies that any two such domains are conformally equivalent: if Ω, Ω' are two domains with $F : \Omega \to \mathbb{D}, G : \Omega \to \mathbb{D}$ the relevant maps then $G^{-1} \circ F : \Omega \to \Omega'$ is also a conformal equivalence.

Remark 1. *In fact, this statement can be made stronger. It turns out that a domain $\Omega \subset \mathbb{C}$ is conformally equivalent to the disk $\mathbb{D}$ so long as it is*

- *Simply connected (every loop can be contracted without passing through a "hole" of Ω) and*
- *A proper open subset: i.e., not $\mathbb{C}$ itself.*

We do not use this level of generality here, though this is what the book does.

Why is this so strong? We have seen earlier that $\mathbb{C}$ is not *conformally equivalent to $\mathbb{D}$: indeed, any holomorphic map $\mathbb{C} \to \mathbb{D}$ is constant by Liouville, so why does properness and simple connectedness guarantee what we need?*

Intuitively, if $\Omega \subset \mathbb{C}$ is a proper subset then there is some point z_0 in $\mathbb{C}$ which is not in Ω. Now just removing z_0 or a bounded set containing z_0 will result in a non-simply connected region (there will be a large loop with z_0 in its interior, hence non-contractible). Therefore "at minimum" one needs to remove z_0 and at least one point of every circle containing z_0: a "minimal" such set to remove is an infinite ray from z_0 in some complex direction ("going towards infinity"). Now, the complement of a ray is actually equivalent to the upper half-plane (take a branch of the square root function, after an appropriate shift.) The upper half-plane is then conformally equivalent to the disk, as we've seen. A procedure like this (involving a logarithmic derivative) works to convert any proper simply connected domain into something bounded and simply connected. The astute reader will notice that our proof for interioros of curves in fact works for anything bounded and simply connected. (See book for details.)

We have the following facts at our disposal, from the last two lectures.

- For any sequence $F_1, F_2, \dots$ with $F_k : \Omega \to \mathbb{C}$ holomorphic maps, if the F_k are uniformly bounded then there exist integers $n_1, n_2, \dots,$ such that the sequence $F_{n(1)}, F_{n(2)}, \dots$ converges locally uniformly to a limit F.
- Moreover F is holomorphic with derivative equal to the limit of derivatives of the F_n.
- If the F_n are injective and F is non-constant then F is also injective.

We also have the following fact from the last worksheet: if $F : \mathbb{D} \to \mathbb{D}$ is a function from the disk to itself then $|F'(0)| \leq 1$, and in case of equality $|F'(0)| = 1$ then F is a bijection. This suggests that in order to find a bijection $F : \Omega \to \mathbb{D}$, we should look for the map with maximal derivative at some interior point.

So fix a point $z_0 \in \Omega$. We will look for maps $\Omega \to \mathbb{D}$ which take z_0 to 0. This may seem like an restriction: surely if z_0 is not "in the middle" of Ω in some sense it should not go to the center of the disk under a conformal equivalence. But it turns out that if any conformal equivalence exists, there is one that takes z_0 to 0, as can be seen by composing with a "standard" conformal equivalence between $\mathbb{D}$ and itself.

Question 1. *Recall that for any α, the function $f_\alpha : z \mapsto \frac{z-\alpha}{\bar{\alpha}z-1}$ is an equivalence from $\mathbb{D}$ to itself. Deduce that if there is an equivalence $F_0 : \Omega \to \mathbb{D}$ sending z_0 to λ, it can be composed with some f_α to get a conformal equivalence $F : \Omega \to \mathbb{D}$ sending z_0 to 0.*

2 Constructing a sequence of embeddings

We will construct a sequence of holomorphic injective maps $F_n : \Omega \to \mathbb{D}$ which send z_0 to 0 and "fill out more and more space" in the circle. The best stand-in for "amount of space" in a conformal context will be the absolute value of the derivative.

We begin with a linear map F_0 defined as follows.

Question 2. *Show that there is a real number $b > 0$ such that the map $F_0 : z \mapsto b(z - z_0)$ takes every point of Ω to a point of $\mathbb{D}$. Show that F_0 is an injection and takes z_0 to 0.*

Intuitively F_0 shifts Ω, then shrinks it to fit inside the unit circle. The derivative $|F_0'(z_0)|$ is of course b (same as its derivative at any other point).

Now let $\mathcal{S}$ be the set of injective holomorphic functions which take z_0 to 0, so

$$\mathcal{S} = \{f : \Omega \to \mathbb{D} \mid f \text{ holomorphic, injective and } f(z_0) = 0.$$

By our construction of F_0 above, $\mathcal{S}$ is nonempty. We have a function $D : \mathcal{S} \to \mathbb{R}$ given by $D(f) := |f'(z_0)|$. The function D is bounded by Cauchy's inequality.

Question 3. *If you are not comfortable with Cauchy's inequality, show using a contour integral that for f a holomorphic function on Ω, we have $|f'(z_0)| \leq C max_{\zeta \in C_r(z_0)}|f(\zeta)|$, for C_r a fixed circle around z_0 fully contained in Ω and $C > 0$ some constant. Deduce that if the image of f is in the unit disk $\mathbb{D}_1$, then $|f'(z_0)| \leq \frac{1}{C}$.*

Write the set of absolute derivatives $\mathcal{D} := \{D(f) \mid f \in \mathcal{S}\} \subset \mathbb{R}$ is an upper-bounded, nonempty set of positive real numbers with some nonzero $b = D(F_0) > 0$ in $\mathcal{D}$. Knowing what we know about real numbers, we see that there is a supremum $d = \mathrm{Sup}\mathcal{D}$ such that $0 < d < \infty$.

Now our strategy of proof is as follows.

Lemma 1. *Prove that $\mathcal{D}$ contains its supremum, i.e., there is some function $F \in \mathcal{S}$ such that $|F'(0)|$ is maximal possible.*

Lemma 2. *Prove that if $F : \Omega \to \mathbb{D}$ is a function in $\mathcal{S}$ (so: holomorphic, injective, with $F(z_0) = 0$) whose derivative at z_0 is the maximal possible, then F is bijective, hence the desired conformal mapping.*

Proof of Lemma 1. Let $d_1, d_2, \dots$ be a set of numbers in $\mathcal{D}$ whose limit is d. By definition of $\mathcal{D}$, there are functions G_n in $\mathcal{S}$ with $|G'_n(0)| = d_n$.

Now the analysis tools we've developed give us precisely what we need:

Question 4. *Show that there exists a subsequence $F_1, F_2, \dots$ of the G_n which converges locally uniformly to a function F, and F is in $\mathcal{S}$, with $F'(z_0) = d$. (Hint: first show that $F(z_0) = 0$ and $F'(z_0) = r$. Now remember that the limit of a sequence of injective holomorphic functions is either injective or constant. Why is F not constant?)*

This concludes the proof of Lemma 1.

Now we prove Lemma 2. We want to show that F is a holomorphi bijection. We already know F is holomorphic and injective, so it remains to prove *surjectivity*, i.e. that for every point $w \in \mathbb{D}$ of the disk, there is some $z \in \Omega$ with $F(z) = w$.

We will use a neat argument to show that if F is not surjective then we can modify it to another function $G : \Omega \to \mathbb{D}$ which has larger absolute value of derivative at 0.

Indeed, assume for the sake of contradiction $w \in \mathbb{D}$ be a point that is "missed" by F.

To construct G we use the following procedure.

Question 5.

1. *First, compose $F : \Omega \to \mathbb{D}$ with a conformal automorphism $f_\alpha : \mathbb{D} \to \mathbb{D}$ such that 0 is not in the image of $F_1 := f_\alpha \circ F$. (Hint: move w to 0).*
2. *Since Ω is simply connected and F_1 is holomorphic and nonzero, convince yourself that we can find a holomorphic function $\log F_1 : \Omega \to \mathbb{C}$ satisfying $\exp(\log F_1)(z) = F_1(z)$ for all z.*

We have seen one way to do this: take F_1 to be an antiderivative of the logarithmic derivative, $\frac{F_1'}{F_1}$. Use Cauchy's theorem on antiderivatives to see this antiderivative exists. This is where we're using that Ω "has no holes" (otherwise an antiderivative might not exist.)

3. *Define a new function $F_2 := \exp(\frac{\log F_1}{2})$, which satisfies $F_2^2(z) = F_1$ for all z.*

4. *Now compose with another fractional linear automorphism f_β from $\mathbb{D}$ to itself to get a function $G = f_\beta \circ F_2$ satisfying $G(z_0) = 0$.*

5. *At the end of the day, we have made sense of G as $G(z) = f_\beta \circ \sqrt{F \circ f_\alpha}$, as a well-defined function $\Omega \to F$. Show that G is injective (hint: use that if $F_2(z) = F_2(z')$ then $F_2^2(z) = F_2^2(z')$.)*

6. *Verify the relationship*
$$G = \Phi \circ F$$
where $\Phi(z) = f_\beta^{-1} \circ sq \circ f_\beta^{-1}$, with $sq(z) := z^2$ is the squaring function.

   Check that the function Φ is a non-injective function from $\mathbb{D}$ to $\mathbb{D}$ with $\Phi(0) = 0$. We deduce that $|\Phi'(0)| < 1$.

7. *Finally, apply the chain rule at 0 the formula $G = \Phi \circ F$ to deduce that $|G'(0)| > |F'(0)|$, contraduction.*

If Ω is a domain (or a more general surface), a *conformal automorphism* of Ω is a (bijective) conformal mapping $\alpha : \Omega \to \Omega$. We will start this section by constructing a family of conformal automorphisms of the unit disk, starting from the family of fractional linear automorphisms of the upper half-plane constructed in the last official worksheet. Then we will show that there are no others.

Recall that the disk is conformally equivalent to the upper half-plane: let

$$\mathbb{D} := \{z \mid |z| < 1\}$$

be the open unit disk as usual and

$$\mathbb{H} := \{z \mid \mathrm{Im}(z) > 0\}$$

be the open upper half-plane. Then in the previous worksheet we have constructed a map $F : \mathbb{H} \to \mathbb{D}$ given by $F : z \mapsto \frac{-z+i}{z+i}$. Its inverse is the map $F : \mathbb{D} \to \mathbb{H}$ given by $w \mapsto i\frac{1-w}{1+w}$.

Question 6. *Show that if $f : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of the disk then $G \circ f \circ F$ is a conformal automorphism of the plane. And conversely, if $g : \mathbb{H} \to \mathbb{H}$ is a conformal automorphism of the plane then $f := F \circ g \circ G$ is a conformal automorphism of the disk. Therefore conformal automorphisms of the disk and of the plane are determined by the same data.*

Recall that the fractional linear transformation $g : z \mapsto \frac{az+b}{cz+d}$ is a conformal automorphism of $\mathbb{H}$ if and only if a, b, c, d are real and the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$ is positive. (Recall also that two fractional linear transformations are equal if and only if the resulting matrices differ by a nonzero scalar).

Question 7. *Show that if $g(z) := \frac{az+b}{cz+d}$ is a fractional linear automorphism of $\mathbb{H}$ as above, then the automorphism $f := F \circ g \circ G$ of the disk can be rewritten as $e^{i\theta} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}$, for some $\theta \in \mathbb{R}, \alpha \in \mathbb{C}$ (depending on a, b, c, d).*

Now there is no obvious reason why there cannot be other conformal automorphisms of the disk to itself. However, it turns out that every automorphism takes the above form (this implies that every automorphism of the upper half plane is of the form specified... why is this implication true?)

To see this, we need the following result.

Question 8. *(a) Show that every map $f : \mathbb{D} \to \mathbb{D}$ satisfies $|f'(0)| \leq 1$, and if in addition $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some θ.*

Hint. *Note that this follows from one of the problems on the Riemann inequalities: namely, $f'(0)$ can be computed as a Riemann integral over a unit circle, and is therefore bounded by $\max_{z \in C_r} \frac{|f(z)|}{r}$. Equality is attained if and only if the integral computing $f'(0)$ integrates a constant function. On the other hand, $|f(z)| < 1$ for all z. You should be able to finish this by taking the $r \to 1$*

limit.

(b) Show that every automorphism $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ *satisfies* $|f'(0)| = 1$. *Hint: use that* f *has a* compositional *inverse* f^{-1} *and the chain rule gives* $\left(f^{-1}\right)'(0) = \frac{1}{f'(0)}$.

Together the two parts of the last problem imply that every automorphism of the disk *which takes* 0 *to* 0 is given by $f(z) = e^{i\theta}z$, for some θ.

Now we can start with any automorphism of the disk and compose with one of our fractional linear transformations to get a new automorphism of the disk, as follows.

Question 9. *Suppose* $f : \mathbb{D} \to \mathbb{D}$ *is an automorphism of the disk. Show that* $f \circ f_\alpha$ *takes* 0 *to* 0*, where* $f_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$*, and* $\alpha = f^{-1}(0)$ *(using compositional inverse for* f^{-1} *this exists since* f *is a bijection).*

Question 10. *(a) Deduce that every automorphism of the disk is given by* $e^{i\theta} \cdot f_\beta$*, for* $f_\beta : z \mapsto \frac{\beta - z}{1 - \bar{\beta} z}$*, similar to the above.*

(b) Deduce also that every automorphism of the upper half-plane $\mathbb{H}$ *is given by a fractional linear transformation* $\frac{az+b}{cz+d}$ *(with real coefficients and positive determinant), as above. Hint: use* F, G *from the first question.*