Worksheet 8: Conformal equivalences from the disk to itself.

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1 Introduction

Starting today, our lectures will follow the book's chapter 8 (on conformal mappings). The main result will be the following:

The interior of any simple closed curve is conformally equivalent to the unit disk.

We will prove this in a later lecture. Today, we will study conformal equivalences between the disk and itself, called conformal *automorphisms* of the unit disk. We will show (using a version of the Cauchy inequalities) that every conformal equivalence of the disk with itself is given by a fractional linear transformation of a particular form.

2 Constructing a family of equivalences

If Ω is a domain (or a more general surface), a *conformal automorphism* of Ω is a (bijective) conformal mapping $\alpha : \Omega \to \Omega$. We will start this section by constructing a family of conformal automorphisms of the unit disk, starting from the family of fractional linear automorphisms of the upper half-plane constructed in the last official worksheet. Then we will show that there are no others.

Recall that the disk is conformally equivalent to the upper half-plane: let

$$\mathbb{D} := \{z \mid |z| < 1\}$$

be the open unit disk as usual and

$$\mathbb{H} := \{z \mid \mathrm{Im}(z) > 0\}$$

be the open upper half-plane. Then in the previous worksheet we have constructed a map $F : \mathbb{H} \to \mathbb{D}$ given by $F : z \mapsto \frac{-z+i}{z+i}$. Its inverse is the map $F : \mathbb{D} \to \mathbb{H}$ given by $w \mapsto i\frac{1-w}{1+w}$.

Question 1. *Show that if* $f : \mathbb{D} \to \mathbb{D}$ *is a conformal automorphism of the disk then* $G \circ f \circ F$ *is a conformal automorphism of the plane. And conversely, if* $g : \mathbb{H} \to \mathbb{H}$ *is a conformal automorphism of the plane then* $f := F \circ g \circ G$ *is a conformal automorphism of the disk. Therefore conformal automorphisms of the disk and of the plane are determined by the same data.*

Recall that the fractional linear transformation $g : z \mapsto \frac{az+b}{cz+d}$ is a conformal automorphism of $\mathbb{H}$ if and only if a, b, c, d are real and the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$ is positive. (Recall also that two fractional linear transformations are equal if and only if the resulting matrices differ by a nonzero scalar).

Question 2. *Show that if* $g(z) := \frac{az+b}{cz+d}$ *is a fractional linear automorphism of*

$\mathbb{H}$ as above, then the automorphism $f := F \circ g \circ G$ of the disk can be rewritten as $e^{i\theta} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}$, for some $\theta \in \mathbb{R}, \alpha \in \mathbb{C}$ (depending on a, b, c, d).

Now there is no obvious reason why there cannot be other conformal automorphisms of the disk to itself. However, it turns out that every automorphism takes the above form (this implies that every automorphism of the upper half plane is of the form specified... why is this implication true?)

To see this, we need the following result.

Question 3. *(a) Show that every map $f : \mathbb{D} \to \mathbb{D}$ satisfies $|f'(0)| \leq 1$, and if in addition $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some θ.*

Hint. *Note that this follows from one of the problems on the Riemann inequalities: namely, $f'(0)$ can be computed as a Riemann integral over a unit* *circle, and is therefore bounded by* $\max_{z \in C_r} \frac{|f(z)|}{r}$*. Equality is attained if and only if the integral computing* $f'(0)$ *integrates a constant function. On the other hand,* $|f(z)| < 1$ *for all* z*. You should be able to finish this by taking the* $r \to 1$ *limit.*

(b) Show that every automorphism $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ *satisfies* $|f'(0)| = 1$*. Hint: use that* f *has a* compositional *inverse* f^{-1} *and the chain rule gives* $\left(f^{-1}\right)'(0) = \frac{1}{f'(0)}$.

Together the two parts of the last problem imply that every automorphism of the disk *which takes* 0 *to* 0 is given by $f(z) = e^{i\theta}z$, for some θ.

Now we can start with any automorphism of the disk and compose with one of our fractional linear transformations to get a new automorphism of the disk, as follows.

Question 4. *Suppose $f : \mathbb{D} \to \mathbb{D}$ is an automorphism of the disk. Show that $f \circ f_\alpha$ takes 0 to 0, where $f_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$, and $\alpha = f^{-1}(0)$ (using compositional inverse for f^{-1} this exists since f is a bijection).*

Question 5. *(a) Deduce that every automorphism of the disk is given by $e^{i\theta} \cdot f_\beta$, for $f_\beta : z \mapsto \frac{\beta - z}{1 - \bar{\beta}z}$, similar to the above.*

(b) Deduce also that every automorphism of the upper half-plane $\mathbb{H}$ is given by a fractional linear transformation $\frac{az+b}{cz+d}$ (with real coefficients and positive determinant), as above. Hint: use F, G from the first question.